Extensions of Stekloff and Almansi Inequalities to the Complex Integral

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Abstract

In this paper we establish some extensions of Stekloff and Almansi inequalities to the complex integral. Applications for bounding the complex Čebyšev functional are also given.

Keywords: Wirtinger’s inequality, Stekloff inequality, Almansi inequality, Grüss’ inequality.


1. Introduction

It is well known that, see for instance (Diaz & Metcalf, 1967), or (Jaroš, 2011), if \( u \in C^1([a, b], \mathbb{R}) \), namely \( u \) is continuous on \([a, b]\) and has a derivative that is continuous on \((a, b)\) and satisfies \( u(a) = u(b) = 0 \), then the following Wirtinger type inequality is valid

\[
\int_a^b u^2(t) \, dt \leq \frac{(b - a)^2}{\pi^2} \int_a^b [u'(t)]^2 \, dt \tag{1.1}
\]

with the equality holding if and only if \( u(t) = K \sin \left[ \frac{\pi(t-a)}{b-a} \right] \) for some constant \( K \in \mathbb{R} \).

If \( u \in C^1([a, b], \mathbb{R}) \) satisfies the condition \( u(a) = 0 \), then also

\[
\int_a^b u^2(t) \, dt \leq \frac{4(b - a)^2}{\pi^2} \int_a^b [u'(t)]^2 \, dt \tag{1.2}
\]

and the equality holds if and only if \( u(t) = L \sin \left[ \frac{\pi(t-a)}{2(b-a)} \right] \) for some constant \( L \in \mathbb{R} \).

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For some related Wirtinger type integral inequalities see (Almansi, 1905), (Beesack, 1959), (Diaz & Metcalf, 1967) and (Giova, 2008), (Jaroˇs, 2011), (Lupas ¸, 1973), (Ricciardi, 2005).

In 1901, W. Stekloff, (Stekloff, 1901), proved that, if \( u \in C^1 ([a, b], \mathbb{R}) \) and \( \int_a^b u (t) \, dt = 0 \), then

\[
\int_a^b u^2 (x) \, dx \leq \frac{(b - a)^2}{4\pi^2} \int_a^b [u' (x)]^2 \, dx.
\]  
(1.3)

In addition, if \( u (a) = u (b) \), then, as proved by E. Almansi in 1905, (Almansi, 1905), the inequality (1.3) can be improved as follows

\[
\int_a^b u^2 (x) \, dx \leq \frac{(b - a)^2}{4\pi^2} \int_a^b [u' (x)]^2 \, dx.
\]  
(1.4)

We can state the following result for complex functions \( h : [a, b] \to \mathbb{C} \).

**Theorem 1.1.** If \( h \in C^1 ([a, b], \mathbb{C}) \) and \( \int_a^b h (t) \, dt = 0 \), then

\[
\int_a^b |h (x)|^2 \, dx \leq \frac{(b - a)^2}{4\pi^2} \int_a^b |h' (x)|^2 \, dx.
\]  
(1.5)

In addition, if \( h (a) = h (b) \), then

\[
\int_a^b |h (x)|^2 \, dx \leq \frac{(b - a)^2}{4\pi^2} \int_a^b |h' (x)|^2 \, dx.
\]  
(1.6)

The proof follows by (1.3) and (1.4) applied for \( u = \text{Re } h \) and \( u = \text{Im } h \) and by adding the corresponding inequalities.

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose \( \gamma \) is a smooth path parametrized by \( z (t) \), \( t \in [a, b] \) and \( f \) is a complex function which is continuous on \( \gamma \). Put \( z (a) = u \) and \( z (b) = w \) with \( u, w \in \mathbb{C} \). We define the integral of \( f \) on \( \gamma_{u,w} = \gamma \) as

\[
\int_{\gamma} f (z) \, dz = \int_{\gamma_{u,w}} f (z) \, dz := \int_a^b f (z (t)) \, z' (t) \, dt.
\]

We observe that that the actual choice of parametrization of \( \gamma \) does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose \( \gamma \) is parametrized by \( z (t) \), \( t \in [a, b] \), which is differentiable on the intervals \( [a, c] \) and \( [c, b] \), then assuming that \( f \) is continuous on \( \gamma \) we define

\[
\int_{\gamma_{u,w}} f (z) \, dz := \int_{\gamma_{a,v}} f (z) \, dz + \int_{\gamma_{v,w}} f (z) \, dz,
\]

where \( v := z (c) \). This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

\[
\int_{\gamma_{u,w}} f (z) \, |dz| := \int_a^b f (z (t)) \, |z' (t)| \, dt.
\]
and the length of the curve $\gamma$ is then

$$\ell(\gamma) = \int_{\gamma_{uw}} |dz| = \int_{a}^{b} |z'(t)| \, dt.$$ 

Let $f$ and $g$ be holomorphic in $G$, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the integration by parts formula

$$\int_{\gamma} f(z) g'(z) \, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{uw}} f'(z) g(z) \, dz. \quad (1.7)$$

We recall also the triangle inequality for the complex integral, namely

$$\left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f(z)| \, |dz| \leq \| f \|_{\gamma, \infty} \ell(\gamma) \quad (1.8)$$

where $\| f \|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the $p$-norm with $p \geq 1$ by

$$\| f \|_{\gamma, p} := \left( \int_{\gamma} |f(z)|^p \, |dz| \right)^{1/p}.$$ 

For $p = 1$ we have

$$\| f \|_{\gamma, 1} := \int_{\gamma} |f(z)| \, |dz|.$$ 

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder’s inequality we have

$$\| f \|_{\gamma, 1} \leq [\ell(\gamma)]^{1/q} \| f \|_{\gamma, p}.$$ 

In this paper we establish some extensions of Stekloff and Almansi inequalities to the complex integral. Applications for bounding the complex Čebyšev functional are also given.

2. Some Preliminary Facts

We have:

**Theorem 2.1.** Let $g : [a, b] \to [g(a), g(b)]$ be a continuous strictly increasing function that is of class $C^1$ on $(a, b)$.

(i) If $f \in C^1 ([a, b], \mathbb{C})$ with $\frac{\sqrt{f}}{g'}(t) \in L_2 [a, b]$ and $\int_{a}^{b} f(t) g'(t) \, dt = 0$, then

$$\int_{a}^{b} |f(t)|^2 \, g'(t) \, dt \leq \left[ \frac{g(b) - g(a)}{\pi^2} \right]^2 \int_{a}^{b} \left[ \frac{f'(t)}{g'(t)} \right]^2 \, dt. \quad (2.1)$$
(ii) In addition, if \( f(a) = f(b) \), then we have the better inequality
\[
\int_a^b |f(t)|^2 g'(t) \, dt \leq \frac{(g(b) - g(a))^2}{4\pi^2} \int_a^b \frac{|f'(t)|^2}{g'(t)} \, dt. \tag{2.2}
\]

Proof. (i) We write the inequality (1.5) for the function \( f : [g(a), g(b)] \) to get
\[
\int_{g(a)}^{g(b)} \left| (f \circ g^{-1})(z) \right|^2 \, dz \leq \frac{(g(b) - g(a))^2}{\pi^2} \int_{g(a)}^{g(b)} \left| (f \circ g^{-1})'(z) \right|^2 \, dz, \tag{2.3}
\]
provided
\[
\int_{g(a)}^{g(b)} f \circ g^{-1}(z) \, dz = 0.
\]

If \( f : [c, d] \to \mathbb{C} \) is absolutely continuous on \([c, d]\), then \( f \circ g^{-1} : [g(c), g(d)] \to \mathbb{C} \) is absolutely continuous on \([g(c), g(d)]\) and using the chain rule and the derivative of inverse functions we have
\[
(f \circ g^{-1})'(z) = (f' \circ g^{-1})(z)(g^{-1})'(z) = \frac{f' \circ g^{-1}(z)}{(g' \circ g^{-1})(z)} \tag{2.4}
\]
for almost every \((a.e.) z \in [g(c), g(d)]\).

Using the inequality (2.3) we then get
\[
\int_{g(a)}^{g(b)} \left| (f \circ g^{-1})(z) \right|^2 \, dz \leq \frac{(g(b) - g(a))^2}{\pi^2} \int_{g(a)}^{g(b)} \left| \frac{f' \circ g^{-1}(z)}{(g' \circ g^{-1})(z)} \right|^2 \, dz, \tag{2.5}
\]
provided \( \int_{g(a)}^{g(b)} f \circ g^{-1}(z) \, dz = 0. \)

Observe also that, by the change of variable \( t = g^{-1}(z) \), \( z \in [g(a), g(b)] \), we have \( z = g(t) \) that gives \( dz = g'(t) \, dt \),
\[
\int_{g(a)}^{g(b)} f \circ g^{-1}(z) \, dx = \int_a^b f(t) h(t) \, dt,
\]
and
\[
\int_{g(a)}^{g(b)} \left| (f \circ g^{-1})(z) \right|^2 \, dz = \int_a^b |f(t)|^2 \, dt. \tag{2.6}
\]

We also have
\[
\int_{g(a)}^{g(b)} \left| \frac{f' \circ g^{-1}(z)}{(g' \circ g^{-1})(z)} \right|^2 \, dz = \int_a^b \left| \frac{f'(t)}{g'(t)} \right|^2 \, dt = \int_a^b \frac{|f'(t)|^2}{g'(t)} \, dt.
\]

By making use of (2.5) we get (2.1).

(ii) The inequality (2.2) follows by (1.6) in a similar way. \( \square \)

If \( w : [a, b] \to \mathbb{R} \) is continuous and positive on the interval \([a, b]\), then the function \( W : [a, b] \to [0, \infty), W(x) := \int_a^x w(s) \, ds \) is strictly increasing and differentiable on \((a, b)\). We have \( W'(x) = w(x) \) for any \( x \in (a, b) \).
Corollary 2.2. Assume that $w : [a, b] \rightarrow (0, \infty)$ is continuous on $[a, b]$ and $f \in C^1([a, b], \mathbb{C})$.

(i) If $\frac{f}{\sqrt{w}} \in L^2[a, b]$ and $\int_a^b f(t) w(t) \, dt = 0$, then

$$\int_a^b |f(t)|^2 w(t) \, dt \leq \frac{1}{\pi^2} \left( \int_a^b w(s) \, ds \right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)} \, dt. \tag{2.7}$$

(ii) In addition, if $f(a) = f(b)$, then we have the better inequality

$$\int_a^b |f(t)|^2 w(t) \, dt \leq \frac{1}{4\pi^2} \left( \int_a^b w(s) \, ds \right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)} \, dt. \tag{2.8}$$

3. Inequalities for Complex Integral

We have the following extensions of Stekloff and Almanski inequalities to the complex integral:

Theorem 3.1. Let $f$ be analytic in $G$, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a, b)$.

(i) If $\int_\gamma f(z) \, |dz| = 0$, then

$$\int_\gamma |f(z)|^2 \, |dz| \leq \frac{1}{\pi^2} f^2(\gamma) \int_\gamma |f'(z)|^2 \, |dz|. \tag{3.1}$$

(ii) In addition, if $f(a) = f(b) = 0$, then

$$\int_\gamma |f(z)|^2 \, |dz| \leq \frac{4}{\pi^2} f^2(\gamma) \int_\gamma |f'(z)|^2 \, |dz|. \tag{3.2}$$

Proof: (i) Consider the function $h(t) = f(z(t))$ and $w(t) = |z'(t)|$, $t \in [a, b]$. Then $h'(t) = (f(z(t)))' = f'(z(t)) z'(t)$ for $t \in (a, b)$. Also

$$\int_a^b f'(z(t)) |z'(t)| \, dt = \int_\gamma f(z) \, |dz| = 0.$$

By utilising the inequality (2.7) for these choices, we get

$$\int_a^b |f(z(t))|^2 |z'(t)| \, dt \leq \frac{1}{\pi^2} \left( \int_a^b |z'(s)| \, ds \right)^2 \int_a^b \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} \, dt$$

$$= \frac{1}{\pi^2} \left( \int_a^b |z'(s)| \, ds \right)^2 \int_a^b \frac{|f'(z(t))|^2 |z'(t)|^2}{|z'(t)|} \, dt$$

$$= \frac{1}{\pi^2} \left( \int_a^b |z'(s)| \, ds \right)^2 \int_a^b |f'(z(t))|^2 |z'(t)| \, dt,$$

which is equivalent to (3.1).

(ii) Follows by the corresponding result from Corollary 2.2.
We have the following reverses of Schwarz inequality:

**Corollary 3.2.** Let \( h \) be analytic in \( G \), a domain of complex numbers and suppose \( \gamma \subset G \) is a smooth path parametrized by \( z(t) \), \( t \in [a, b] \) from \( z(a) = u \) to \( z(b) = w \) and \( z'(t) \neq 0 \) for \( t \in (a, b) \). Then

\[
0 \leq \frac{1}{\ell(\gamma)} \int_{\gamma} |h(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(z) |dz| \right|^2 \leq \frac{1}{\pi^2} \ell(\gamma) \int_{\gamma} |h'(z)|^2 |dz|. \tag{3.3}
\]

In addition, if \( h(u) = h(w) = 0 \), then

\[
0 \leq \frac{1}{\ell(\gamma)} \int_{\gamma} |h(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(z) |dz| \right|^2 \leq \frac{1}{4\pi^2} \ell(\gamma) \int_{\gamma} |h'(z)|^2 |dz|. \tag{3.4}
\]

**Proof.** First, observe that

\[
\frac{1}{\ell(\gamma)} \int_{\gamma} \left| h(z) - \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2 |dz|
= \frac{1}{\ell(\gamma)} \int_{\gamma} \left| h(z) \right|^2 - 2\text{Re} \left( h(z) \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right) + \left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2 |dz|
= \frac{1}{\ell(\gamma)} \int_{\gamma} \left| h(z) \right|^2 |dz| - 2\text{Re} \left( \frac{1}{\ell(\gamma)} \int_{\gamma} h(z) |dz| \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right) + \left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2 |dz|
= \frac{1}{\ell(\gamma)} \int_{\gamma} \left| h(z) \right|^2 |dz| - 2\left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2 + \left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2
= \frac{1}{\ell(\gamma)} \int_{\gamma} \left| h(z) \right|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2. \tag{3.5}
\]

Now, consider \( f(z) := h(z) - \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy|, z \in G \). Then

\[
\int_{\gamma} f(z) |dz| = \int_{\gamma} \left( h(z) - \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right) |dz| = 0,
\]

\( f'(z) = h'(z) \) and by (3.1) we get

\[
\int_{\gamma} \left| h(z) - \frac{1}{\ell(\gamma)} \int_{\gamma} h(y) |dy| \right|^2 |dz| \leq \frac{1}{\pi^2} \ell(\gamma) \int_{\gamma} |h'(z)|^2 |dz|,
\]

and by (3.5) we get the desired result (3.3).

The second part follows by (3.2). \( \square \)
4. Complex Čebyšev Functional

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t), t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If $f$ and $g$ are continuous on $\gamma$, we consider the complex Čebyšev functional defined by

$$D_\gamma(f, g) := \frac{1}{w - u} \int_\gamma f(z) g(z) \, dz - \frac{1}{w - u} \int_\gamma f(z) \, dz \frac{1}{w - u} \int_\gamma g(z) \, dz.$$ 

We start with the following identity of interest:

**Lemma 4.1.** Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t), t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If $f$ and $g$ are continuous on $\gamma$, then

$$D_\gamma(f, g) = \frac{1}{2(w - u)^2} \int_\gamma \left( \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw \right) \, dz$$

$$= \frac{1}{2(w - u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dz \, dw$$

$$= \frac{1}{2(w - u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dz \, dw. \quad (4.1)$$

**Proof.** For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw$ exists and

$$I(z) := \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw$$

$$= \int_\gamma (f(z) g(z) + f(w) g(w) - g(z) f(w) - f(z) g(w)) \, dw$$

$$= f(z) g(z) \int_\gamma dw + \int_\gamma f(w) g(w) \, dw - g(z) \int_\gamma f(w) \, dw - f(z) \int_\gamma g(w) \, dw$$

$$= (w - u) f(z) g(z) + \int_\gamma f(w) g(w) \, dw - g(z) \int_\gamma f(w) \, dw - f(z) \int_\gamma g(w) \, dw.$$ 

The function $I(z)$ is also continuous on $\gamma$, then the integral $\int_\gamma I(z) \, dz$ exists and

$$\int_\gamma I(z) \, dz = \int_\gamma \left[ (w - u) f(z) g(z) + \int_\gamma f(w) g(w) \, dw \right.$$ 

$$- g(z) \int_\gamma f(w) \, dw - f(z) \int_\gamma g(w) \, dw \right] \, dz$$

$$= (w - u) \int_\gamma f(z) g(z) \, dz + (w - u) \int_\gamma f(w) g(w) \, dw$$

$$- \int_\gamma f(w) \, dw \int_\gamma g(z) \, dz - \int_\gamma g(w) \, dw \int_\gamma f(z) \, dz$$

$$= 2(w - u) \int_\gamma f(z) g(z) \, dz - 2 \int_\gamma f(z) \, dz \int_\gamma g(z) \, dz = 2(w - u)^2 P_\gamma(f, g),$$

where $P_\gamma(f, g)$ is a certain integral expression.
which proves the first equality in (4.1).

The rest follows in a similar manner and we omit the details.

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $f : \gamma \to \mathbb{C}$ a continuous function on $\gamma$. Note that, in particular:

$$P_{\gamma}(f, f) = \frac{1}{\ell(\gamma)} \int_{\gamma} |f(z)|^2 |dz| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|^2$$

(4.2)

We have:

**Theorem 4.2.** Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t), t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If $f$ and $g$ are continuous on $\gamma$, then

$$|D_{\gamma}(f, g)| \leq \frac{\ell^2(\gamma)}{|w - u|^2} \left[ P_{\gamma}(f, f) \right]^{1/2} \left[ P_{\gamma}(g, g) \right]^{1/2}. \tag{4.3}$$

**Proof.** Taking the modulus in the first equality in (4.1), we get

$$|D_{\gamma}(f, g)| = \frac{1}{2|w - u|^2} \left| \int_{\gamma} \left( \int_{\gamma} (f(z) - f(w))(g(z) - g(w)) \, dw \right) \, dz \right|$$

$$\leq \frac{1}{2|w - u|^2} \int_{\gamma} \left( \int_{\gamma} |f(z) - f(w)| \, dw \right)^2 |dz| =: A.$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\left| \int_{\gamma} (f(z) - f(w))(g(z) - g(w)) \, dw \right| \leq \left( \int_{\gamma} |f(z) - f(w)|^2 \, dw \right)^{1/2} \left( \int_{\gamma} |g(z) - g(w)|^2 \, dw \right)^{1/2},$$

which implies that

$$A \leq \frac{1}{2|w - u|^2} \int_{\gamma} \left( \int_{\gamma} |f(z) - f(w)|^2 \, dw \right)^{1/2} \left( \int_{\gamma} |g(z) - g(w)|^2 \, dw \right)^{1/2} |dz| =: B.$$
By the Cauchy-Bunyakovsky-Schwarz integral inequality, we also have

\[
\left( \int_{\gamma} \left( \int_{\gamma} |f(z) - f(w)|^2 |dw| \right)^{1/2} \left( \int_{\gamma} |g(z) - g(w)|^2 |dw| \right)^{1/2} |dz| \right) \leq \left( \int_{\gamma} \left( \left( \int_{\gamma} |f(z) - f(w)|^2 |dw| \right)^{1/2} \right)^2 |dz| \right)^{1/2} \times \left( \int_{\gamma} \left( \left( \int_{\gamma} |g(z) - g(w)|^2 |dw| \right)^{1/2} \right)^2 |dz| \right)^{1/2}
\]

\[
= \left( \int_{\gamma} \left( \int_{\gamma} |f(z) - f(w)|^2 |dw| \right) |dz| \right)^{1/2} \times \left( \int_{\gamma} \left( \int_{\gamma} |g(z) - g(w)|^2 |dw| \right) |dz| \right)^{1/2},
\]

which implies that

\[
B \leq \frac{1}{2 |w - u|^2} \left( \int_{\gamma} \left( \int_{\gamma} |f(z) - f(w)|^2 |dw| \right) |dz| \right)^{1/2} \times \left( \int_{\gamma} \left( \int_{\gamma} |g(z) - g(w)|^2 |dw| \right) |dz| \right)^{1/2}. \quad (4.4)
\]

Now, observe that

\[
\int_{\gamma} \left( \int_{\gamma} |f(z) - f(w)|^2 |dw| \right) |dz| = \int_{\gamma} \left( \int_{\gamma} |f(z)|^2 - 2 \Re \left( f(z) \overline{f(w)} \right) + |f(w)|^2 \right) |dw| \right) |dz| = \int_{\gamma} \left( \ell(\gamma) |f(z)|^2 - 2 \Re \left( f(z) \int_{\gamma} \overline{f(w)} |dw| + \int_{\gamma} |f(w)|^2 |dw| \right) |dz| = \ell(\gamma) \int_{\gamma} |f(z)|^2 |dz| - 2 \Re \left( \int_{\gamma} f(z) |dz| \right) \int_{\gamma} \overline{f(w)} |dw| + \ell(\gamma) \int_{\gamma} |f(w)|^2 |dw| = 2 \ell(\gamma) \int_{\gamma} |f(z)|^2 |dz| - 2 \Re \left( \int_{\gamma} f(z) |dz| \right) \int_{\gamma} \overline{f(w)} |dw| \right) = 2 \left( \ell(\gamma) \int_{\gamma} |f(z)|^2 |dz| - \int_{\gamma} f(z) |dz| \right)^2 = 2 \ell^2(\gamma) \mathcal{P}_\gamma(f, \bar{f}) \quad (4.5)
\]

and, similarly

\[
\int_{\gamma} \left( \int_{\gamma} |g(z) - g(w)|^2 |dw| \right) |dz| = 2 \ell^2(\gamma) \mathcal{P}_\gamma(g, \bar{g}). \quad (4.6)
\]

Making use of (4.5) and (4.6), we get

\[
B \leq \frac{1}{2 |w - u|^2} \left( 2 \ell^2(\gamma) \mathcal{P}_\gamma(f, \bar{f}) \right)^{1/2} \left( 2 \ell^2(\gamma) \mathcal{P}_\gamma(g, \bar{g}) \right)^{1/2} \leq \frac{\ell^2(\gamma)}{|w - u|^2} \left[ \mathcal{P}_\gamma(f, \bar{f}) \right]^{1/2} \left[ \mathcal{P}_\gamma(g, \bar{g}) \right]^{1/2},
\]

which proves the desired result (4.3). \( \square \)
Remark. For \( g = f \) we have
\[
D_\gamma (f, f) = \frac{1}{w-u} \int_\gamma f^2 (z) \, dz - \left( \frac{1}{w-u} \int_\gamma f (z) \, dz \right)^2
\]  
(4.7)
and by (4.3) we get
\[
\| D_\gamma (f, f) \| \leq \frac{\ell^2 (\gamma)}{|w-u|^2} P_\gamma (f, \overline{f}).
\]  
(4.8)
For \( g = \overline{f} \) we have
\[
D_\gamma (f, \overline{f}) = \frac{1}{w-u} \int_\gamma |f (z)|^2 \, dz - \frac{1}{w-u} \int_\gamma f (z) \, dz \overline{f (z)} \, dz
\]  
(4.9)
and by (4.3) we get
\[
\| D_\gamma (f, \overline{f}) \| \leq \frac{\ell^2 (\gamma)}{|w-u|^2} P_\gamma (f, f).
\]  
(4.10)

We have

**Theorem 4.3.** Let \( f \) and \( g \) be analytic in \( G \), a domain of complex numbers and suppose \( \gamma \subset G \) is a smooth path parametrized by \( z (t), t \in [a, b] \) from \( z (a) = u \) to \( z (b) = w \) and \( z' (t) \neq 0 \) for \( t \in (a, b) \). Then we have
\[
| D_\gamma (f, g) | \leq \frac{1}{\pi^2} \epsilon^2 (\gamma) \ell (\gamma) \left( \left( \int_\gamma |f' (z)|^2 \, |dz| \right)^{1/2} \left( \int_\gamma |g' (z)|^2 \, |dz| \right)^{1/2} \right.
\]
\[
+ \frac{1}{4} \left( \int_\gamma |f' (z)|^2 \, |dz| \right)^{1/2} \left( \int_\gamma |g' (z)|^2 \, |dz| \right)^{1/2}
\]
\[
\left. \right| \frac{1}{4} \left( \int_\gamma |f' (z)|^2 \, |dz| \right)^{1/2} \left( \int_\gamma |g' (z)|^2 \, |dz| \right)^{1/2}
\]
if \( f (u) = f (w) \),
\[
\frac{1}{\pi^2} \epsilon^2 (\gamma) \ell (\gamma) \left( \left( \int_\gamma |f' (z)|^2 \, |dz| \right)^{1/2} \left( \int_\gamma |g' (z)|^2 \, |dz| \right)^{1/2} \right.
\]
\[
+ \frac{1}{4} \left( \int_\gamma |f' (z)|^2 \, |dz| \right)^{1/2} \left( \int_\gamma |g' (z)|^2 \, |dz| \right)^{1/2}
\]
\[
\left. \right| \frac{1}{4} \left( \int_\gamma |f' (z)|^2 \, |dz| \right)^{1/2} \left( \int_\gamma |g' (z)|^2 \, |dz| \right)^{1/2}
\]
if \( f (u) = f (w) \) and \( g (u) = g (w) \),
\]  
(4.11)
where \( \epsilon (\gamma) := \frac{\ell (\gamma)}{|w-u|} \geq 1 \).

**Proof.** From (3.3) we have
\[
0 \leq P_\gamma (f, \overline{f}) \leq \frac{1}{\pi^2} \ell (\gamma) \int_\gamma |f' (z)|^2 \, |dz|
\]
and
\[
0 \leq P_\gamma (g, \overline{g}) \leq \frac{1}{\pi^2} \ell (\gamma) \int_\gamma |g' (z)|^2 \, |dz|,
\]
which together with the inequality (4.3), produce the first inequality in (4.11).

The rest follows in a similar way and we omit the details. □
References


