AMAZING ARCHIMEDES NUMBER 22/7

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Abstract: *The first mathematician in Ancient Greece who did not understand mathematics in an idealized way, but was occupied with counting and applications (logistics) was Archimedes. The aim of our work is to point out to some elements of Archimedes' work in Numerical Mathematics. A special attention should be paid to the work "On Measuring a Circle", where Archimedes got the optimal approximation of the number* π *and square roots. It can be confirmed that these approximations are optimal in the theory on continued fractions, which was created many centuries after Archimedes. Didn't Archimedes know about continued fractions?*

Key-words: *Archimedes, continued fractions, number 22/7, number* π*, the best approximation*

1. Introduction

Mathematics has had exclusively practical character until the appearance of Greek science (VII century B.C.), and calculations were approximate. The first approximation a man has made at the moment when he knew about 1,2, and all the rest of the numbers he replaced with the word "plenty". In early mathematics there were no proofs and generalization.

Greek mathematicians, starting with Thales (about 625-548.B.C.) began to prove the statements and in that way raised mathematics to the level of a science.

In the beginning, preserving the purity of mathematics, the Greeks detested any of its applications for practical purpose. For them, the application was unworthy of mathematics, whose aim was to bring a man closer to deity. Approximate numerical values computing was pejoratively called "logistics" by the Greeks. According to them, the task of science was not to count the values, but to deal with the relations of volumens, surfaces and segments.

Archimedes (about 287-212, B.C.) is considered to be one of the first scientists who broke off the conception of mathematics as an idealised science which should be separated from practice.

Archimedes does not have greater contributions in the field of numerical mathematics in most of his work. But, in one of his later works "On Measuring a Circle", Archimedes not only gave a contribution to numerical mathematics, but entered the history of numerical mathematics as the first scientist who evaluated an error and the way of determining the level of correctness of obtained result. For the first time in history Archimedes stated a task on measuring a circumference of a circle and determining the approhimate value of relation of a circumference and diameter. No doubt Archimedes, by intuition of a genial mathematician, felt and comprehended the special nature of a relation of a circumference and diameter of a circle. As approximate value for π Archimedes used 22/7. The aim of this study was to examine the secret of this number.

2. Archimede's Number 22/7

A basic theorem in the work "On Measuring a Circle" is as follows [9]:

The relation of any circle to its diameter is smaller than $3\frac{1}{7}$ *and bigger than* $3\frac{10}{71}$. *In other words, it applies*:

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

Archimedes came to this result by gradual inscription and circumscription of regular polygons in a circle, drawing out the circumference of a circle in that way. He starts with a triangle and stops at a 96-angle.Archimedes does not inform us where he takes the quoted limits for the number π from. He in a series of calculations where he uses square roots of many numbers, about which we do not know how he counts them, gets that the relation of a circumfered 96-angle to the diameter is:

$$< 14688 : 4673\frac{1}{2} < 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} < 3\frac{1}{7}$$

Similarly, the relation of a 96-angle inscribed in a circle, to the diameter is:

$$> (66 \cdot 96) : 2017\frac{1}{4} > 6336 : 2017\frac{1}{4} > 3\frac{10}{71}.$$

What we would like to present as a curiousity of the obtained result and genial Archimedes' intuition is the fact that the approximate value of $\frac{22}{7}$ present a better approhimation of the number π that all the fractions with a denominator smaller or equal to 7. Similarly, for $\sqrt{3}$ Archimedes in his work takes the number $\frac{265}{153}$, and that is a better approximation than all the fractions with a denominator smaller or equal to 153.

Although Archimedes does not inform us about the ways of getting the quoted approximations, with the help of Computational Mathematics we could state some hypotheses.

What in general does it mean to approximate some real number α with a fraction whose denominator is q? That means that from all the fractions with a denominator q it should be chosen the one which is the closest to the number α i.e., it should be found that kind of a fraction $\frac{p}{q}$ so the value $\left|\alpha - \frac{p}{q}\right|$ was the smallest. In the case the relation: $\frac{p-1}{q} < \alpha < \frac{p}{q}$ should be valid. The case when α coincides with one of these numbers is not interesting. We take for α that $\frac{p-1}{q}, \frac{p}{q}$, which is closer to α. Let it be $\frac{p}{q}$. Than the absolute error is $|\Delta| = \left|\alpha - \frac{p}{q}\right|$. It can be clearly seen in the picture that $|\Delta| \le \frac{1}{2q}$ is valid.

$\frac{p-1}{q}$ α $\frac{p}{q}$

Let

$$\lambda = \frac{\text{the upper limit of absolute error}}{\text{absolute error}}, \text{ i.e. } \lambda = \frac{\frac{1}{2q}}{|\Delta|}.$$

That number shows how many times the real absolute error is smaller than absolutely possible one. It could be seen that the value λ increases with the increase of correctness, i.e. with the decrease of absolute error.

Let us observe the following Table 1. Let number π be approximated by the fractions whose denominators are one after the other, numbers 1,2,3,4,5,6,7,89,10.

Table 1. Approximate value for π

q	Approximate value for π	The upper limit of absolute error	$\lvert\Delta\rvert$	λ
1	$\frac{3}{1}$	$\frac{1}{2} = 0.5000$	0.1416	3.5
2	$\frac{6}{2}$	$\frac{1}{4} = 0.2500$	0.1416	1.8
3	$\frac{9}{3}$	$\frac{1}{6} = 0.1667$	0.1416	1.2
4	$\frac{13}{4}$	$\frac{1}{8} = 0.1250$	0.1084	1.2
5	$\frac{16}{5}$	$\frac{1}{10} = 0.1000$	0.0584	1.7
6	$\frac{19}{6}$	$\frac{1}{12} = 0.0833$	0.0251	3.3
7	$\frac{22}{7}$	$\frac{1}{14} = 0.0714$	0.0013	54.9
8	$\frac{25}{8}$	$\frac{1}{16} = 0.0625$	0.0166	3.8
9	$\frac{28}{9}$	$\frac{1}{18} = 0.0556$	0.0305	1.8
10	$\frac{31}{10}$	$\frac{1}{20} = 0.0500$	0.0416	1.2

We can see from the Table 1 that the fraction with the denominator 7 gives the smallest absolute error, 0.0013. If we would search for a denominator in advance, so that the absolute error does not exceed 0.0013, we should get

$$\frac{1}{2q} \leq 0.0013, \ \ q \geq 385.$$

Archimedes has achieved, as it can be seen from the table, the corectness of 0.0013 for a great deal smaller denominator than 385, what is more appropriate for usage.

From the above we can conclude that Archimedes could not reach his result by chance in any case. How he felt a premonition of a denominator 7 in the fraction $\frac{22}{7}$ present a secret for us. A possible way to the quoted number is through continued fractions.

3. Continued fractions give the solution of the secret of number $\frac{22}{7}$

The expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + a_{s-1} + \cfrac{1}{a_s}}}}$$

where $a_1, a_2, ..., a_s$ are natural numbers, and a_0 is an integer, is called a continued or continuous fraction. L. Euler (Leonard Euler, 1707-1783) proved that an arbitrary rational number can be rationalized into a final, and an irrational into an endless continued fraction [4].

For further operation we can define the fraction $\frac{p_n}{q_n}$ in the following way:

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + a_{n-1} + a_n}}}.$$

Then the following theorem [1]:

If $\frac{p_n}{q_n}$ (n ≥1) is a fraction obtained from the continual development of a certain real number α in the above defined way, and $\frac{p}{q}$ is an arbitrary fraction, where $q \le q_n$, then it is

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p}{q}\right|.$$

This means that the fraction $\frac{p_n}{q_n}$ is a better approximation of the number α than all the fractions whose denominator is smaller or equal to q_n,and it can be improved only by increasing a denominator. It should be noted that a reverse theorem does not apply, i.e. it exists a fraction which is not obtained from a continued development of number α and that fraction could be a better approximation of number α than all the fractions with a denominator smaller than its. We should also have in mind that the theorem does not assert that the fraction $\frac{p_n}{q_n}$ is the best approximation in general.

Let us observe the continued development of the number π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}}$$

It follows that: $\frac{p_0}{q_0} = \frac{3}{1}, \frac{p_1}{q_1} = 3 + \frac{1}{7} = \frac{22}{7}, \frac{p_2}{q_2} = 3 + \cfrac{1}{7 + \frac{1}{15}} = \frac{333}{106}, \frac{p_3}{q_3} = \frac{355}{113}$, ...

We can see that $\frac{p_1}{q_1} = \frac{22}{7}$. So, according to the quoted theorem, Archimedes' number $\frac{22}{7}$ is the best approximation of the number π of all the fractions with a denominator smaller or equal to 7.

In a way, the secret of the number $\frac{22}{7}$ is solved. We can reach it with the help of continued fractions. Did Archimedes work in this way, too? This still remains the secret. However, there are some clues that he might have also used continued fractions [4]:

1. Archimedes did not know about decimal numbers, and the only way of reaching $\frac{22}{7}$ was through continued fractions;

2. The Egyptians used fractions a lot earlier than Archimedes, presenting them by means of the sum of fractions and numerator 1;

3. In the seventh book of The Elements, Euclid presents his algorhythm for finding a mutual divisior of numbers a and b, that directly leads to disassembling of fraction $\frac{a}{b}$ into a continued fraction.

The following approximation of the number π by means of continued fractions is $\frac{p_2}{q_2} = \frac{333}{106}$. This may clarify the fact why Archimedes was stopped by a 96-angle. The number $\frac{333}{106}$ is far more complex for usage than number $\frac{22}{7}$.

Let us mention one more fact which convinces us that Archimedes did not reach his results by chance. At determining the relation between a circumference of a circle and its diameter, Archimedes counted square roots of numbers, as well, and did not quote how he had obtained the values. So, he determinates the limits for $\sqrt{3}$ for example:

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}.$$

In the continued fraction of number $\sqrt{3}$ it is obtained: $\frac{p_{12}}{q_{12}} = \frac{1351}{780}$. Therefore, Archimedes chose of all the fractions with denominator 780 just the one that approximated $\sqrt{3}$ best.

4. Conclusion

We have seen how we get so easily to the approximation of real numbers by fractions which are optimal in relation to all other fractions with smaller or equal denominator with the help of continued fractions. Althought continued fractions have important part in numerical mathematics, they are rerely used due to their bulk size. Huygens (Christian Huygens, 1629-1695) made the theory of them. The very idea of disassembly of real numbers through continued fractions was used in the XI century. Euler was the first to introduce the usage of the term continued fractions.

As we have seen, Archimedes, a genius in everything he was occupied with, through one “little thing”-counting number π and square roots, has given genial results whose value we realize through a modern apparatus of Numerical Mathematics. The honorary place in the historical development of Numerical Mathematics belongs to him, because he has found the methods of obtaining numerical results and in that way made dealing with this kind of mathematics (logistics) official and public, and was the first to determine the accuracy of obtained results.

Finally, let's look at the importance of the number 22/7 in school education. Due to the lack of knowledge of irrational numbers, primary school students usually use the number 22/7 instead of pi. We think that it would be pedagogically very useful for students to use computers to gain knowledge about number 22/7. Then the students would know what is the meaning of pi, error, and so the best approximation. Using computer animation students will be able to understand the method of exhaustion circle by poligons, just like how Archimedes did [10], [11], [12].

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