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Inequalities Involving Noor Integral Operator

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Abstract

The object of the present paper is to give an application of Noor integral operator $I_{n+p-1}(n > -p; p \in N)$ to Miller and Mocanu's theorems.

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1. Introduction

Let $A(p)$ be the class of functions of the form :

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. For functions $f_j(z)(j = 1, 2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the Hadamard product (or convolution) $(f_1 * f_2(z))$ of functions $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2(z)) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k . \tag{1.3}$$

The integral operator $I_{n+p-1} : A(p) \to A(p)$ is defined as follows (see (Liu & Noor, 2000)):

For any integer n biger than $-p$, let $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$ and let $f_{n+p-1}(z)$ be defined such that

$$f_{n+p-1}(z) * f_{n+p-1}^{(-1)}(z) = \frac{z^p}{(1-z)^{1+p}} . \tag{1.4}$$

Then

$$I_{n+p-1} f(z) = f_{n+p-1}^{(-1)} * f(z) = \left[\frac{z^p}{(1-z)^{n+p}} \right]^{(-1)} * f(z). \tag{1.5}$$

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From (1.4) and (1.5) and a well-known identity for the Ruscheweyh derivative (see (Goel & Sohi, 1980) and (Ruscheweyh, 1975)), it follows that

$$z(I_{n+p}f(z))' = (n+p)I_{n+p-1}f(z) - nI_{n+p}f(z). \tag{1.6}$$

For $p = 1$, the identity (1.6) is given by Noor and Noor (Noor & Noor, 2005). If $f(z)$ is given by (1.1), then from (1.4) and (1.5), we deduce that

$$I_{n+p-1}f(z) = \left[z^p{}_2F_1(1, 1+p; n+p; z)\right] * f(z)(n > -p), \tag{1.7}$$

where ${}_2F_1$ is the hypergeometric function. We also note that $I_{p-1}f(z) = \frac{zf'(z)}{p}$ and $I_pf(z) = f(z)$. Moreover, the operator $I_{n+p-1}f(z)$ defined by (1.5) is called as Noor integral operator of $(n+p-1)$-th order of $f(z)$ (see (Liu & Noor, 2000)). For $p = 1$, the operator I_nf was introduced by Noor (Noor, 1999) and Noor and Noor (Noor & Noor, 2005). Several classes of analytic functions, defined by using the operator $I_{n+p-1}f(z)$, have been studied by many authors (see (Noor, 2004), (Noor, 2005) and (Patel & Cho, 2005)).

By using the operator I_{n+p-1} we define :

Definition 1.1. Let G_1 be the set of complex- valued functions $g(r, s, t)$;

$$g(r, s, t) : C^3 \to C \text{ (C is the complex plane)}$$

such that

(i) $g(r, s, t)$ is continuous in a domain $D \subset C^3$;
(ii) $(0, 0, 0) \in D$ and $|g(0, 0, 0)| < 1$;
(iii) $\left|g(e^{i\theta}, \frac{n+\zeta}{n+p}e^{i\theta}, \frac{(n-1)(n+2\zeta)e^{i\theta}+M}{(n+p)(n+p-1)})\right| > 1$
whenever $(e^{i\theta}, \frac{n+\zeta}{n+p}e^{i\theta}, \frac{(n-1)(n+2\zeta)e^{i\theta}+M}{(n+p)(n+p-1)}) \in D$ with $Re\{e^{-i\theta}M\} \geq \zeta(\zeta-1)$ for all $\theta \in R$, and for all $\zeta \geq p \geq 1$.

Definition 1.2. Let G_2 be the set of complex- valued functions $h(r, s, t)$;

$$h(r, s, t) : C^3 \to C$$

such that

(i) $h(r, s, t)$ is continuous in a domain $D \subset C^3$;
(ii) $(1, 1, 1) \in D$ and $|h(1, 1, 1)| < J \quad (J > 1)$;
(iii) $\left|h(Je^{i\theta}, \frac{\zeta-1+(n+p)Je^{i\theta}}{n+p-1}, \frac{1}{n+p-2}\{\zeta - 1 + \ + (n+p)Je^{i\theta} + \frac{\zeta-\zeta^2+(n+p)\zeta Je^{i\theta}+L}{\zeta-1+(n+p)Je^{i\theta}}\})\right| \geq J$,
whenever $(Je^{i\theta}, \frac{\zeta-1+(n+p)Je^{i\theta}}{n+p-1}, \frac{1}{n+p-2}\{\zeta - 1 + (n+p)Je^{i\theta} + \frac{\zeta-\zeta^2+(n+p)\zeta Je^{i\theta}+L}{\zeta-1+(n+p)Je^{i\theta}}\}) \in D$
with $Re\{L\} \geq \zeta(\zeta-1)$ for all $\theta \in R$ and for all $\zeta \geq \frac{J-1}{J+1}$.

2. Main Results

We begin with the statement of the following lemmas due to Miller and Mocanu (Miller & Mocanu, 1978).

Lemma 2.1. *(**Miller & Mocanu, 1978**). Let* $w(z) = b_pz^p + b_{p+1}z^{p+1} + ...(p \in N)$ *be regular in the unit disc U with* $w(z) \neq 0$ $(z \in U)$*. If* $z_0 = r_0e^{i\theta}(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z|\leq|z_0|}|w(z)|$ *then*

$$(i) \quad z_0w'(z_0) = \zeta w(z_0) \tag{2.1}$$

and

$$(ii) \quad Re\left\{1 + \frac{z_0w''(z_0)}{w'(z_0)}\right\} \geq \zeta \tag{2.2}$$

where ζ is real and $\zeta \geq p \geq 1$.

Lemma 2.2. *(Miller & Mocanu, 1978). Let* $w(z) = a + w_k z^k + ...$*be regular in* U *with* $w(z) \neq a$ *and* $k \geq 1$*. If* $z_0 = r_0 e^{i\theta}(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)|$*then*

$$(i) \quad z_0 w'(z_0) = \zeta w(z_0)$$

$$(ii) \quad Re\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \zeta$$

where ζ *is a real number and*

$$\zeta \geq k\frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k\frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}. \tag{2.3}$$

Making use of the above lemmas, we prove

Theorem 2.3. *Let* $g(r, s, t)$ *be in* G_1*, and let* $f(z)$ *belonging to the class* $A(p)$*, satisfy*

(i) $(I_{n+p}f(z), I_{n+p-1}f(z), I_{n+p-2}f(z)) \in D \subset C^3$
and
(ii) $\left|g(I_{n+p}f(z), I_{n+p-1}f(z), I_{n+p-2}f(z))\right| < 1$
where $n > -p, p \in N$ and $z \in U$. Then we have

$$\left|I_{n+p}f(z)\right| < 1 \quad (z \in U). \tag{2.4}$$

We define the function $w(z)$ by

$$I_{n+p}f(z) = w(z) \quad (n > -p; p \in N) \tag{2.5}$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z) \in A(p)$ and $w(z) \neq 0\,(z \in U)$. With the aid of the identily (1.6), we have

$$I_{n+p-1}f(z) = \frac{1}{n+p}\left\{nw(z) + zw'(z)\right\} \tag{2.6}$$

and

$$I_{n+p-2}f(z) = \frac{1}{(n+p)(n+p-1)}\left\{n(n-1)w(z) + 2(n-1)zw'(z) + z^2 w''(z)\right\}. \tag{2.7}$$

Suppose that $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1; \theta \in R)$ and

$$|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = 1 \tag{2.8}$$

Letting $w(z_0) = e^{i\theta}$ and using (2.1) of Lemma 2.1, we see that

$$L_{n+p}f(z_0) = w(z_0) = e^{i\theta}, \tag{2.9}$$

$$L_{n+p-1}f(z_0) = \frac{n+\zeta}{n+p}w(z_0) = \frac{n+\zeta}{n+p}e^{i\theta}, \tag{2.10}$$

and

$$\begin{aligned} L_{n+p-2}f(z_0) &= \frac{1}{(n+p)(n+p-1)}\left\{(n-1)(n+2\zeta)w(z_0) + z_0^2 w''(z_0)\right\} \\ &= \frac{(n-1)(n+2\zeta)e^{i\theta} + M}{(n+p)(n+p-1)}, \end{aligned} \tag{2.11}$$

where $M = z_0^2 w''(z_0)$ and $\zeta \geq p \geq 1$.
Further, an application of (2.2) in Lemma 2.1 gives

$$Re\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = Re\left\{\frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}}\right\} \geq (\zeta - 1), \tag{2.12}$$

or

$$Re\left\{e^{-i\theta}M\right\} \geq \zeta(\zeta - 1)(\theta \in R; \zeta \geq 1). \tag{2.13}$$

Since $g(r, s, t) \in G_1$, we have

$$\left|g(I_{n+p}f(z_0), I_{n+p-1}f(z_0), I_{n+p-2}f(z_0))\right|$$

$$= \left|g(e^{i\theta}, \frac{n+\zeta}{n+p}e^{i\theta}, \frac{(n-1)(n+2\zeta)e^{i\theta}+M}{(n+p)(n+p-1)})\right| > 1 \tag{2.14}$$

which contradicts the condition (ii) of Theorem 2.3. Therefore, we conclude that

$$|w(z)| = \left|I_{n+p}f(z)\right| < 1, \tag{2.15}$$

which $n > -p; p \in N$ and for all $z \in U$. This completes the proof of Theorem 2.3.

Corollary 2.4. *Let $g_0(r, s, t) = s$ and let $f(z)$ belonging to the class $A(p)$ satisfy the conditions in Theorem 2.3. Then*

$$\left|I_{n+p+i}f(z)\right| < 1 \quad (i = 0, 1, 2, ...; n > -p; p \in N; z \in U). \tag{2.16}$$

Proof. Note that $g_0(r, s, t) = s$ is in G_1, with the aid of Theorem 2.3, we have

$$\left|I_{n+p-1}f(z)\right| < 1 \Longrightarrow \left|I_{n+p}f(z)\right| < 1 \quad (n > -p; p \in N)$$

$$\Longrightarrow \left|I_{n+p+i}f(z)\right| < 1 \quad (i = 0, 1, 2, ...; n > -p; p \in N; z \in U).$$

□

Theorem 2.5. *.Let $h(r, s, t) \in G_2$, let $f(z)$ belonging to $A(p)$ satisfying*

$$(i)\ (\frac{I_{n+p-1}f(z)}{I_{n+p}f(z)}, \frac{I_{n+p-2}f(z)}{I_{n+p-1}f(z)}, \frac{I_{n+p-3}f(z)}{I_{n+p-2}f(z)}) \in D \subset C^3$$

and

$$(ii)\ \left|h(\frac{I_{n+p-1}f(z)}{I_{n+p}f(z)}, \frac{I_{n+p-2}f(z)}{I_{n+p-1}f(z)}, \frac{I_{n+p-3}f(z)}{I_{n+p-2}f(z)})\right| < J$$

for some n, J $(n > -p; p \in N; J > 1)$ and for all $z \in U$. Then we have

$$\left|\frac{I_{n+p-1}f(z)}{I_{n+p}f(z)}\right| < J \quad (z \in U). \tag{2.17}$$

Proof. We define the function $w(z)$ by

$$\frac{I_{n+p-1}f(z)}{I_{n+p}f(z)} = w(z) \qquad (n > -p; p \in N; z \in U) \tag{2.18}$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z)$ is either analytic or meromorphic in U, $w(0) = 1$, and $w(z) \neq 1$. With the aid of the identity (1.6), we have

$$\frac{I_{n+p-2}f(z)}{I_{n+p-1}f(z)} = \frac{1}{n+p-1}[(n+p)w(z) - 1 + \frac{zw'(z)}{w(z)}] \tag{2.19}$$

and

$$\frac{I_{n+p-3}f(z)}{I_{n+p-2}f(z)} = \frac{1}{n+p-2}\left\{(n+p)w(z) - 1 + \frac{zw'(z)}{w(z)} + \frac{(n+p)zw'(z) + \frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - (\frac{zw'(z)}{w(z)})^2}{(n+p)w(z) - 1 + \frac{zw'(z)}{w(z)}}\right\}. \tag{2.20}$$

Suppose that $z_0 = r_0 e^{i\theta} (0 < r_0 < 1; \theta \in R)$ and $|w(z_0)| = \max_{|z| \le |z_0|} |w(z)| = J$. Letting $w(z_0) = Je^{i\theta}$ and using Lemma 2.2 with $a = k = 1$, we see that

$$\frac{I_{n+p-2} f(z_0)}{I_{n+p-1} f(z_0)} = \frac{1}{n+p-1} [\zeta - 1 + (n+p) J e^{i\theta}] \tag{2.21}$$

and

$$\frac{I_{n+p-3} f(z_0)}{I_{n+p-2} f(z_0)} = \frac{1}{n+p-2} \Big[\zeta - 1 + (n+p) J e^{i\theta} + \frac{\zeta - \zeta^2 + (n+p)\zeta J e^{i\theta} + L}{\zeta - 1 + (n+p) J e^{i\theta}}\Big], \tag{2.22}$$

where $L = \frac{z_0^2 w''(z_0)}{w(z_0)}$ and $\zeta \ge \frac{J-1}{J+1}$.
Further, an application of (ii) in Lemma 2.2 gives

$$Re\{L\} \ge \zeta(\zeta - 1).$$

Since $h(r, s, t) \in G_2$, we also have

$$\left| h\Big(\frac{I_{n+p-1} f(z_0)}{I_{n+p} f(z_0)}, \frac{I_{n+p-2} f(z_0)}{I_{n+p-1} f(z_0)}, \frac{I_{n+p-3} f(z_0)}{I_{n+p-2} f(z_0)}\Big) \right|$$
$$= \left| h\Big(Je^{i\theta}, \frac{\zeta - 1 + (n+p) J e^{i\theta}}{n+p-1}, \frac{1}{n+p-2} \Big\{\zeta - 1 + (n+p) J e^{i\theta} + \frac{\zeta - \zeta^2 + (n+p)\zeta J e^{i\theta} + L}{\zeta - 1 + (n+p) J e^{i\theta}}\Big\}\Big) \right| \ge J, \tag{2.23}$$

which contradicts condition (ii) of Theorem 2.3. Therefore, we conclude that

$$|w(z)| = \left| \frac{I_{n+p-1} f(z)}{I_{n+p} f(z)} \right| < J \tag{2.24}$$

for $n > -p, p \in N$ and $z \in U$. This completes the proof of Theorem 2.3. □

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