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A Short Note

Certain New Classes of Analytic and Univalent Functions in the Unit Disk

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Abstract

We introduce the classes $H(\omega,\alpha)$ and $K(\omega,\alpha)$ of analytic functions with negative coefficients. In this work we give some properties of functions in these classes and we obtain coefficient estimates, neighborhood and integral means inequalities for function $f(z)$ belonging to these classes.

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1. Introduction

Let A be the class of function $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let S denote the subclass of A consisting of univalent functions $f(z)$ in E.

It is necessary here to recall the definitions of the well-known classes of starlike and convex functions

$$S^* = \left\{ f \in A : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$
$$S^c = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in E \right\}.$$

Let $A(\omega) \subset A$ denote the class of functions of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k \tag{1.2}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalized with $f(\omega) = 0$ and $f'(\omega) - 1 = 0$, and ω is a fixed point in E.

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In (Kanas & Ronning, 1999), S. Kanas and F. Ronning introduced and studied the following classes of functions.

$$S(\omega) = \{f \in A(\omega) : f \text{ is univalent in } E\}$$
$$ST(\omega) = S^*(\omega) = \left\{f \in S(\omega) : Re\left(\frac{(z-\omega)f'(z)}{f(z)}\right) > 0, z \in E\right\}$$
$$CV(\omega) = S^c = \left\{f \in S(\omega) : Re\left(1 + \frac{(z-\omega)f''(z)}{f'(z)}\right) > 0, z \in E\right\}$$

which are respectively the classes of univalent, starlike and convex functions and ω is a fixed point in E. These classes were further studied in (Acu & Owa, 2005).

Let $T(\omega)$ denote subclass of $S(\omega)$ whose elements can be expressed in the form

$$f(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k (z-\omega)^k. \tag{1.3}$$

Here we denote by $H(\omega, \alpha)$ and $K(\omega, \alpha)$ repectively the subfamilies of $S^*(\omega, \alpha)$ and $S^c(\omega, \alpha)$ obtained by taking intersection of $S^*(\omega, \alpha)$ and $S^c(\omega, \alpha)$ with $T(\omega)$ that is,

$$H(\omega, \alpha) = S^*(\omega, \alpha) \cap T(\omega)$$

and

$$K(\omega, \alpha) = S^c(\omega, \alpha) \cap T(\omega)$$

where $S^*(\omega, \alpha)$ and $S^c(\omega, \alpha)$ are respectively classes of starlike of order α and convex of order α (Oladipo, 2009).

Consequently, we have

$$H(\omega, 0) = S^*(\omega, 0) \cap T(\omega) \Rightarrow H(\omega) = S^*(\omega) \cap T(\omega)$$

and

$$K(\omega, 0) = S^c(\omega, 0) \cap T(\omega) \Rightarrow K(\omega) = S^c(\omega) \cap T(\omega).$$

Also let $P(\omega) \subset P$(class of Caratheodory functions) denote the class of functions of the form

$$p_\omega(z) = 1 + \sum_{k=2}^{\infty} B_k (z-\omega)^k \tag{1.4}$$

that are regular in E and satisfy $p_\omega(\omega) = 1$, $Re p_\omega(z) > 0$ for $z \in E$ and ω is a fixed point in E and

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k},\ k \geq 1,\ \text{and } d = |\omega|$$

(Kanas & Ronning, 1999), (Acu & Owa, 2005), (Wald, 1978).

2. Coefficient estimates

For our main results we first derive the following:

Lemma 2.1. *A function $f(z) \in T(\omega)$ is in the class $H(\omega, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} (k-\alpha)(1-d)^{k-1} a_k \leq 1 - \alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z-\omega| = 1-d < 1$. Then we have

$$\left|\frac{(z-\omega)f'(z)}{f(z)} - 1\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)a_k(z-\omega)^{k-1}}{1-\sum_{k=2}^{\infty}a_k(z-\omega)^{k-1}}\right|$$

$$\leq \frac{\sum_{k=2}^{\infty}(k-1)(1-d)^{k-1}a_k}{1-\sum_{k=2}^{\infty}(1-d)^{k-1}a_k} \leq 1-\alpha.$$

This shows that the values of $\frac{(z-\omega)f'(z)}{f(z)}$ lie in the circle centred at $\gamma = 1$ whose radius is $1-\alpha$. Hence $f(z)$ is in the class $H(\omega, \alpha)$. Then

$$Re\frac{(z-\omega)f'(z)}{f(z)} = Re\left\{\frac{1-\sum_{k=2}^{\infty}ka_k(z-\omega)^{k-1}}{1-\sum_{k=2}^{\infty}a_k(z-\omega)^{k-1}}\right\} > \alpha \tag{2.2}$$

for $z \in E$ and ω is a fixed point in E.
Choose values of z on the real axis so that $\frac{(z-\omega)f'(z)}{f(z)}$ is real. Upon clearing the denominator in (6) and letting $z \to 1^-$ through real values, we have

$$\alpha\left(1-\sum_{k=2}^{\infty}(1-d)^{k-1}a_k\right) \leq 1-\sum_{k=2}^{\infty}k(1-d)^{k-1}a_k \tag{2.3}$$

which obviousely is the required result.
Finally, we note that the assertion (2.1) of Lemma 2.1 is sharp, with the extremal function being

$$f(z) = (z-\omega) - \frac{1-\alpha}{(k-\beta)(1-d)^{k-1}}(z-\omega)^k. \tag{2.4}$$

□

Corollary 2.2. *Let $f(z) \in T(\omega)$ be in the class $H(\omega, \alpha)$. Then we have*

$$a_k \leq \frac{1-\alpha}{(k-\beta)(1-d)^{k-1}} \tag{2.5}$$

Equality in (2.5) holds true for the function $f(z)$ given by (2.4).

Lemma 2.3. *A function $f(z) \in T(\omega)$ is in the class $K(\omega, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty}k(k-\beta)(1-d)^{k-1}a_k \leq 1-\alpha. \tag{2.6}$$

The result is sharp.

Proof. The proof follows the same method as in Lemma 2.1 . The assertion of Lemma 2.2 is sharp with extremal function

$$f(z) = (z-\omega) - \frac{1-\alpha}{k(k-\beta)(1-d)^{k-1}}(z-\omega)^k. \tag{2.7}$$

□

Corollary 2.4. *Let $f(z) \in T(\omega)$ be in the class $K(\omega, \alpha)$. Then we have*

$$a_k \leq \frac{1-\alpha}{k(k-\beta)(1-d)^{k-1}} \tag{2.8}$$

Equality in (2.8) holds true for the function $f(z)$ given by (2.7).

Theorem 2.5. *Let $f(z) \in H(\omega, \alpha)$ and $f(z) = (z-\omega) - a_2(z-\omega)^2 - ...$ for $0 \leq \alpha < 1$, and ω is a fixed point in E. Then*

$$|a_2| \leq \frac{-2(1-\alpha)}{1-d^2} \tag{2.9}$$

$$|a_3| \leq -\left[\frac{(1-\alpha)}{(1-d^2)(1-d)} + \frac{2(1-\alpha)^2}{(1-d^2)^2}\right]$$

$$|a_4| \leq -\left[\frac{2(1-\alpha)}{3(1+d)(1-d)^3} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{3(1-d^2)^3}\right].$$

Proof. Let us define

$$p_\omega(z) = \frac{\frac{(z-\omega)f'(z)}{f(z)} - \alpha}{1-\alpha} \tag{2.10}$$

That is,

$$(z-\omega)f'(z) = f(z)\left[1 + (1-\alpha)\sum_{k=2}^{\infty} B_k(z-\omega)^k\right] \tag{2.11}$$

On comparing the coefficient in (2.11) the results follow.

Following the earlier investigations of Goodman (Goodman, 1957) and Ruschweyh (Ruscheweyh, 1981), we define the δ-neighborhood of function $f(z) \in T(\omega)$ by

$$N_\delta = \left\{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \ \sum_{k=2}^{\infty} k(1-d)^{k-1}|b_k| \leq \delta\right\} \tag{2.12}$$

and in particular, for the identity function

$$e(z) = (1 - \frac{\omega}{z})z \tag{2.13}$$

we immediately have

$$N_\delta(e) = \left\{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \ \sum_{k=2}^{\infty} k(1-d)^{k-1}|b_k| \leq \delta\right\}. \tag{2.14}$$

□

Theorem 2.6. *$H(\omega, \alpha) \subset N_\delta(e)$, where $\delta = \frac{2(1-\alpha)}{(2-\alpha)(1-d)}$.*

Proof. Let $f(z) \in H(\omega, \alpha)$. Then, in view of Lemma 2.1, since $(k-\alpha)(1-d)^{k-1}$ is an increasing function of $k(k \geq 2)$, we have

$$(2-\alpha)(1-d)\sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty}(k-\alpha)(1-d)^{k-1}a_k \leq 1-\alpha \tag{2.15}$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{(2-\alpha)(1-d)}. \tag{2.16}$$

On the other hand, we also find from (2.3) that

$$(1-d)\sum_{k=2}^{\infty} ka_k - \alpha(1-d)\sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty}(k-\alpha)(1-d)^{k-1} \le 1-\alpha \tag{2.17}$$

from (2.16) and (2.17), we have

$$\begin{aligned}(1-d)\sum_{k=2}^{\infty} ka_k &\le (1-\alpha) + \alpha(1-d)\sum_{k=2}^{\infty} a_k \\ &\le (1-\alpha) + \frac{\alpha(1-\alpha)}{(2-\alpha)} \\ &\le \frac{2(1-\alpha)}{2-\alpha}\end{aligned} \tag{2.18}$$

$$\sum_{k=2}^{\infty} ka_k \le \frac{2(1-\alpha)}{(2-\alpha)(1-d)} \tag{2.19}$$

which proved the theorem. □

3. Integral mean inequality

Lemma 3.1. *if f and g are analytic in E with $f \prec g$, then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\delta} d\theta \le \int_0^{2\pi} \left|f(re^{i\theta}\right|^{\delta} d\theta \tag{3.1}$$

where $\delta > 0, z = re^{i\theta}, \omega = de^{i\theta}$ and $0 < r + d < 1$.

Applying Lemma 3.1 and (1.2) we prove the following

Theorem 3.2. *Let $\delta > 0$. if $f(z) \in H(\omega, \alpha)$, then $z = re^{i\theta}, \omega = de^{i\theta}$ and $0 \le d < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\delta} d\theta \tag{3.2}$$

where

$$f_2(z) = (z-\omega) - \frac{1-\alpha}{(2-\alpha)(1-d)}(z-\omega)^2. \tag{3.3}$$

Proof. Let $f(z)$ defined by (1.3) and $f_2(z)$ be given by (3.3). We must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k(z-\omega)^{k-1}\right|^{\delta} d\theta \le \int_0^{2\pi} \left|1 - \frac{1-\alpha}{(2-\alpha)(1-d)}(z-\omega)\right|^{\delta} d\theta. \tag{3.4}$$

By Lemm A, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k(z-\omega)^{k-1} \prec 1 - \frac{1-\alpha}{(2-\alpha)(1-d)}(z-\omega). \tag{3.5}$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k(z-\omega)^{k-1} = 1 - \frac{1-\alpha}{(2-\alpha)(1-d)}h(z) \tag{3.6}$$

From (3.6) and (2.1), we obtain

$$|h(z)| = \left| \sum_{k=2}^{\infty} \frac{(2-\alpha)(1-d)}{1-\alpha} a_k (z-\omega)^{k-1} \right| \tag{3.7}$$
$$\leq |z-\omega| \sum_{k=2}^{\infty} \frac{(k-\alpha)(1-d)^{k-1}}{1-\alpha} a_k$$
$$\leq |z-\omega| \,.$$

This complete the proof. □

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