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An Application of Pescar's Univalence Criterion

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Abstract

For the operator $F_\alpha(z) = \left(\alpha \int\limits_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}}$, Pescar has obtained a generalization of Ahlfor's and Becker's criterion of univalence. In this paper we generalize the Pescar's univalence criterion for other two operators $G_{\alpha_1,\ldots,\alpha_n,n}(z)$ and $J_{\gamma_1,\ldots,\gamma_n}(z)$ and we obtain new univalence conditions of analytic functions in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

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2000 MSC: 30C45.

1. Introduction and preliminaries

Let $\mathcal{U} = \{z : |z| < 1\}$ the unit disk and $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition

$$f(0) = f'(0) - 1 = 0.$$

Theorem 1.1. *(Mocanu et al., 2009) (Maximum Modulus Principle) Let f be a nonconstant analytic function on a connected open set U. Then $|f|$ cannot attains maximum in U, i.e. there exists $\alpha \in U$ such that $|f(a)| \geq |f(z)|$ for all $z \in U$.*

The next lemma is a result given by J. Becker (Becker, 1972):

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Lemma 1.2. *(Becker, 1972) If $f(z) = z + a_2z^2 + \dots$ is analytic in $\mathcal{U}$ and*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then the function $f(z)$ is univalent in $\mathcal{U}$.

L. V. Ahlfors (Ahlfors, 1973) and J. Becker (Becker, 1973) has obtain the next univalence criterion:

Theorem 1.3. *((Ahlfors, 1973) and (Becker, 1973)) Let c be a complex number, $|c| \leq 1, c \neq -1$. If $f(z) = z + a_2z^2 + \dots$ is a regular function in $\mathcal{U}$ and*

$$\left|c|z|^2 + (1 - |z|^2)\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then the function f is regular and univalent in $\mathcal{U}$.

V. Pescar in (Pescar, 1996) obtain an univalence criterion which is a generalization of Ahlfor's and Becker's criterion of univalence and is given in next theorem.

Theorem 1.4. *(Pescar, 1996) Let α and c be complex numbers, $\operatorname{Re}\alpha > 0, |c| \leq 1, c \neq -1$. If $f(z) = z + a_2z^2 + \dots$ is a regular function in $\mathcal{U}$ and*

$$\left|c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}} = z + \dots$$

is regular and univalent in $\mathcal{U}$.

In (Pascu & Radomir, 1989) N.N. Pascu and I. Radomir has obtain:

Theorem 1.5. *(Pascu & Radomir, 1989) Let β and c complex numbers, $\operatorname{Re}\beta > 0, |c| \leq 1, c \neq -1$ and $f(z) = z + a_2z^2 + \dots$ be a regular function in $\mathcal{U}$. If*

$$\left|ce^{-2t\beta} + (1 - e^{-2t\beta})\frac{e^{-t}zf''(e^{-t}z)}{\beta f'(e^{-t}z)}\right| \leq 1$$

holds for every $z \in \mathcal{U}$ and $t \geq 0$, then the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}} = z + \dots$$

is regular and univalent in $\mathcal{U}$.

We define the operators

$$G_{\alpha_1,\alpha_2,\ldots,\alpha_n,n}(z) = \left(\left(\sum_{i=1}^{n}\alpha_i - n + 1\right)\int_0^z \prod_{i=1}^{n}(g_i(t))^{\alpha_i-1}dt\right)^{\frac{1}{\sum_{i=1}^{n}\alpha_i-n+1}} \tag{1.2}$$

for $g_i \in \mathcal{A}, i = \overline{1,n}$ and

$$J_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \left(\left(\sum_{j=1}^{n}\frac{1}{\gamma_j}\right)\int_0^z t^{-1}\prod_{j=1}^{n}(f_j(t))^{\frac{1}{\gamma_j}}dt\right)^{\frac{1}{\sum_{j=1}^{n}\frac{1}{\gamma_j}}}. \tag{1.3}$$

for $f_j \in \mathcal{A}, j = \overline{1,n}$.
The operator $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,n}(z)$ is a generalization of an operator defined by Breaz et all in (Breaz *et al.*, 2009).

2. Main results

Theorem 2.1. *Let α_i and c complex numbers, $n \in \mathbb{N}, n \geq 1, i = \overline{1,n}$, $\operatorname{Re}\left(\sum_{i=1}^{n}\alpha_i - n + 1\right) > 0, |c| < 1, c \neq -1$. We suppose that the function f defined by (1.1) is analytic in $\mathcal{U}$. If*

$$\left| c|z|^{2(\sum_{i=1}^{n}\alpha_i-n+1)} + \left(1 - |z|^{2(\sum_{i=1}^{n}\alpha_i-n+1)}\right)\frac{zf''(z)}{(\sum_{i=1}^{n}\alpha_i - n + 1)f'(z)}\right| \leq 1 \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,n}(z)$ defined by (1.2) is analytic and univalent in $\mathcal{U}$.

Proof. For $z \in \mathcal{U}$ from (2.1) we have that $f'(z) \neq 0$ and from here the function

$$w(z,t) = c \cdot e^{-2t(\sum_{i=1}^{n}\alpha_i-n+1)} + (1 - e^{-2t(\sum_{i=1}^{n}\alpha_i-n+1)})\frac{e^{-t}zf''(e^{-t}z)}{(\sum_{i=1}^{n}\alpha_i - n + 1)f'(e^{-t}z)} \tag{2.2}$$

is analytic in $\overline{\mathcal{U}} = \{z : |z| \leq 1\}$, for $t > 0$. To the function $w(z,t)$ we apply the maximum modulus principle and we have that

$$\begin{aligned} |w(z,t)| &< \max_{|z|=1}|w(z,t)| = \left|w(e^{i\theta},t)\right| \\ &= \left|ce^{-2t(\sum_{i=1}^{n}\alpha_i-n+1)} + (1 - e^{-2t(\sum_{i=1}^{n}\alpha_i-n+1)})\frac{e^{-t+i\theta}f''(e^{-t+i\theta})}{(\sum_{i=1}^{n}\alpha_i - n + 1)f'(e^{-t+i\theta})}\right| \end{aligned} \tag{2.3}$$

where $\theta = \theta(t) \in \mathbb{R}$.
We note $\phi = e^{-t+i\theta}$. From here we have that $|\phi| = e^{-t}$.
If in relation (2.3) we replace $e^{-t+i\theta}$ with ϕ we obtain:

$$\left|w(e^{i\theta},t)\right| = \left|c \cdot |\phi|^{2(\sum_{i=1}^{n}\alpha_i-n+1)} + (1 - |\phi|^{2(\sum_{i=1}^{n}\alpha_i-n+1)})\frac{\phi f''(\phi)}{(\sum_{i=1}^{n}\alpha_i - n + 1)f'(\phi)}\right| \tag{2.4}$$

Because $|\phi| = e^{-t}$, for all $t > 0$ it results that $\phi \in \mathcal{U}$.
For $z = \phi$ using the relations (2.1) and (2.4) we obtain that

$$\left|w(e^{i\theta}, t)\right| \leq 1 \tag{2.5}$$

From (2.3) and (2.5) we have $|w(z,t)| < 1$, for $z \in \mathcal{U}, t > 0$.
For $t = 0$ we have that $w(z,0) = c$. Using the hypothesis we obtain $|w(z,0)| < 1, z \in \mathcal{U}$. So, $|w(z,t)| < 1, z \in \mathcal{U}, t \geq 0$ and from here and using Theorem 1.5 for $\beta = \sum\limits_{i=1}^{n} \alpha_i - n + 1$ results that $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,n}(z)$ is a analytic and univalent function in $\mathcal{U}$. □

Corollary 2.2. *Let $\alpha \in \mathbb{C}, n \in \mathbb{N}, n \geq 1, \mathrm{Re}[\alpha - n + 1] > 0$ and $c \in \mathbb{C}, |c| < 1, c \neq -1$. We suppose that the function f given by (1.1) is analytic in $\mathcal{U}$. If*

$$\left|c|z|^{2(\alpha-n+1)} + (1 - |z|^{2(\alpha-n+1)}\frac{zf''(z)}{(\alpha - n + 1)f'(z)})\right| \leq 1$$

for all $z \in \mathcal{U}$, then the function

$$G_{n,\alpha}(z) = \left((\alpha - n + 1)\int\limits_0^z (g_1(t))^{\alpha-1} \ldots (g_n(t))^{\alpha-1} dt\right)^{\frac{1}{\alpha-n+1}}$$

is analytic and univalent in $\mathcal{U}$.

Proof. Similar with the proof of previous theorem for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ □

Remark. For $n = 1$ in Theorem 2.1 we obtain the Pescar's criterion of univalence.

Remark. For $n = 1$ and $\alpha = 1$ in Theorem 2.1 we obtain Ahlfor's and Becker's univalence criterion.

Theorem 2.3. *Let $\gamma_j \in \mathbb{C}$, $j = \overline{1,n}$, $\mathrm{Re}\left(\sum\limits_{j=1}^{n} \frac{1}{\gamma_i}\right) > 0$ and $c \in \mathbb{C}, |c| \leq 1$, $c \neq -1$. We suppose that the function f defined by (1.1) is analytic in $\mathcal{U}$. If*

$$\left| c|z|^{2\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}} + (1 - |z|^{2\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}})\frac{zf''(z)}{\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}f'(z)} \right| \leq 1 \tag{2.6}$$

for all $z \in \mathcal{U}$, then the function $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)$ defined by (1.3) is analytic and univalent in $\mathcal{U}$.

Proof. From (2.6) we have that $f'(z) \neq 0$, for $z \in \mathcal{U}$. We have that the function

$$v(z,t) = c \cdot e^{-2t\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}} + \left(1 - e^{-2t\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}}\right)\frac{e^{-t}zf''(e^{-t}z)}{\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}f'(e^{-t}z)} \tag{2.7}$$

is analytic in $\overline{\mathcal{U}}$, for $t > 0$. □

For the function $v(z,t)$ we apply the maximum modulus principle and we obtain

$$\begin{aligned} |v(z,t)| < \max_{|z|=1} |v(z,t)| &= \left|v(e^{j\theta},t)\right| \\ &= \left| ce^{-2t\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}} + \left(1 - e^{-2t\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}}\right) \frac{e^{-t+j\theta} f''(e^{-t+j\theta})}{\sum\limits_{j=1}^{n} \frac{1}{\gamma_j} f'(e^{-t+j\theta})} \right| \end{aligned} \tag{2.8}$$

where $\theta = \theta(t) \in \mathbb{R}$.
We note with $\psi = e^{-t+j\theta}$ and we have that $|\psi| = e^{-t}, \forall t > 0$.
If in (2.8)we replace $e^{-t}e^{j\theta}$ with ψ we obtain

$$\left|v(e^{j\theta},t)\right| = \left| c|\psi|^{2\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}} + \left(1 - |\psi|^{2\sum\limits_{j=1}^{n}\frac{1}{\gamma_j}}\right) \frac{\psi \cdot f''(\psi)}{\sum\limits_{j=1}^{n} \frac{1}{\gamma_j} f'(\psi)} \right| \tag{2.9}$$

But $|\psi| = e^{-t} < 1$ for $t > 0$ implies that $\psi \in \mathcal{U}$.
Using (2.6) and (2.9) for $z = \psi$ we obtain:

$$\left|v(e^{j\theta},t)\right| \le 1 \tag{2.10}$$

From (2.8) and (2.10) we have that $|v(z,t)| < 1$, for all $z \in \mathcal{U}, t > 0$. For $t = 0$ we obtain $v(z,0) = c$. Using the hypothesis we obtain that $|v(z,0)| < 1$ for all $z \in \mathcal{U}$. So, $|v(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \ge 0$. Hence and from Theorem 1.5 for $\beta = \sum\limits_{j=1}^{n} \frac{1}{\gamma_j}$ we obtain that $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)$ is univalent and analytic in $\mathcal{U}$.

Corollary 2.4. *Let $\gamma \in \mathbb{C}, \mathrm{Re}\left(\frac{1}{\gamma}\right) > 0$ and $c \in \mathbb{C}, |c| \le 1$, $c \ne -1$. We suppose that the function f defined by (1.1) is analytic in $\mathcal{U}$. If*

$$\left| c|z|^{\frac{2}{\gamma}} + (1 - |z|^{\frac{2}{\gamma}}) \frac{zf''(z)}{\frac{1}{\gamma} f'(z)} \right| \le 1$$

for all $z \in \mathcal{U}$, then the function

$$J_\gamma(z) = \left(\frac{1}{\gamma} \int_0^z t^{-1} (f(t))^{\frac{1}{\gamma}} dt \right)^{\frac{1}{\gamma}}$$

is analytic and univalent in $\mathcal{U}$.

Remark. For $\gamma_1 = \gamma_2 = \cdots = \gamma_n = \gamma = 1$ in Theorem 2.3 we obtain Ahlfor's and Becker's criterion of univalence.

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