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An Extension of Kuttner's Theorem

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Abstract

If $0 < p < 1$ and X is a locally convex FK - space, then $X \supset l_\infty$ whenever $X \supset w_0(p)$ (Kuttner's theorem see (B.Thorpe, 1981)).The aim of this paper is to give some extensions of this theorem by replacing $w_0(p)$ by $[c_A, M]_0$.

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1. Introduction

A real function g on a linear space X is called an F - norm if

[i] $g(x) = 0$ if and only if $x = 0$,
[ii] $|\alpha| \leq 1 (\alpha \in K) \Rightarrow g(\alpha x) \leq g(x)$ for all $x \in X$,
[iii] $g(x + y) \leq g(x) + g(y)$ for all $x, y \in X$,
[iv] $\lim_n \alpha_n = 0 \quad (\alpha_n \in K), x \in X \quad \Rightarrow \lim_n g(\alpha_n x) = 0$.

An F -norm g in a sequence space X is called absolutely monotone if $|x_k| \leq |y_k|, k \in \mathbb{N} \Rightarrow g(x) \leq g(y)$, for all $x = (x_k), y = (y_k) \in X$.

An F -space is defined as a complete F - normed space. If a sequence space X is an F - space on which the coordinate functionals $\pi_k(x) = x_k$ are continuous, then X is called an FK- space. An FK - space with normable topology is called a BK - space . Some authors include local convexity in the definition of a Fréchet Space and of an FK - space . We do not and we follow the definition used by Maddox and by Wilansky (Wilansky, 1964).

Let ϕ be the space of all finite sequences. An F- space X containing ϕ is called an AK- space if $x = \lim_n \sum_{k=1}^{n} x_k e_k$, for all $x = (x_k) \in X$.

For a sequence space X we denote by X^α and X^β the Köthe - Toeplitz duals of X , i.e.

$$X^\alpha = \left\{ \alpha = (\alpha_k) : \quad \sum_{k=1}^{n} |\alpha_k x_k| < \infty \text{ for all } (x_k) \in X \right\} \tag{1.1}$$

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and

$$X^{\beta} = \left\{ \alpha = (\alpha_k) : \sum_{k=1}^{n} \alpha_k x_k \text{ converges for all } (x_k) \in X \right\}. \tag{1.2}$$

For an F - normed sequence space X we denote by X' the topological dual of X and in the case $\phi \subset X$, we use the notation

$$X^{\phi} = \{ f(e_k)) : f \in X' \}. \tag{1.3}$$

A sequence space X is called solid (or normal), if $(\alpha_k x_k) \in X$, whenever $(x_k) \in X$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1, k \in \mathbb{N}$.

A sequence space X is called monotone, if X contains the canonical pre-images of all its step spaces.

Lemma 1.1. *If a sequence space X is solid then X is monotone.*

Let X and Y be any two sequence spaces and $A = (a_{nk})_{n,k=1}^{\infty}$ an infinite matrix. We say that the matrix A maps X into Y if for each $x \in X$, the sequence $Ax = (A_n(x)) \in Y$, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k, \quad n = 1, 2, \cdots \tag{1.4}$$

provided that the series on the right converges for each n. We denote by (X, Y) the class of all matrices A which map X into Y.

Let S be a subset of a real linear space.

[a] The set S is called convex if for all $x, y \in S$

$$\lambda x + (1 - \lambda) y \in S \quad \forall \lambda \in [0, 1], \tag{1.5}$$

[b] If S is a nonempty and convex set , we say that a functional $f : S \to \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y), \quad \forall \lambda \in [0, 1] \text{ and } \forall \; x, y \in S. \tag{1.6}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Let $0 < p \leq 1$. A function $M : [0, \infty) \to [0, \infty)$ is called $p - convex$ if

$$M(\alpha x + \beta y) \leq \alpha^p M(x) + \beta^p M(y) \tag{1.7}$$

for all $x, y \geq 0$ and $\alpha^p + \beta^p = 1$.

In this paper we consider $p - convex$ $(0 < p < 1)$ Orlicz functions. Note that the notion of $1 - convex$ functions coincides with the notion of convex functions.

Example. The function $M(t) = t^p$, $0 < p < 1$ is $p - convex$ and it is not $r - convex$ if $r > p$.

If convexity of an Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y) \tag{1.8}$$

then this function is called a modulus function, defined and discussed by Nakano (Nakano, 1953), Ruckle (Ruckle, 1973), Maddox (Maddox, 1986) and others. Lindenstrauss and Tzafriri (Lindenstrauss & Tzafriri, 1971) used the idea of an Orlicz function to construct the sequence space

$$\ell_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}. \tag{1.9}$$

The space ℓ_M with the norm

$$\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(x) = x^p$, $1 \leq p < \infty$, the space ℓ_M coincide with the classical sequence space l_p.

An Orlicz function M is said to satisfy the $\triangle_2$ - condition for all values x, if there exists a constant $K > 0$, such that

$$M(2x) \leq KM(x) \text{ for all } x \geq 0. \tag{1.10}$$

The $\triangle_2$ - condition is equivalent to

$$M(Lx) \leq KLM(x), \text{ for all values of } x \geq 0, \text{ and for } L > 1. \tag{1.11}$$

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt, \tag{1.12}$$

where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$. Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1. \tag{1.13}$$

Let $A = (a_{nk})$ be an infinite matrix with $a_{nk} \geq 0$ and let c_A be the summability field of matrix method A (see Virge Soomer (Soomer, 2003))i.e.

$$c_A = \left\{ x = (x_k) : A(x) = \lim_n \sum_k a_{nk} x_k \ exists \right\}. \tag{1.14}$$

Then, passing to strong summability,

$$[c_A] = \left\{ x = (x_k) : \text{ there exists } L, \ \lim_n \sum_k a_{nk}|x_k - L| = 0 \right\} \tag{1.15}$$

and

$$[c_A]_0 = \left\{ x = (x_k) : \ \lim_n \sum_k a_{nk}|x_k| = 0 \right\} \tag{1.16}$$

are the spaces of strongly A - summable and strongly A - summable to zero sequences , respectively.

Thorpe (B.Thorpe, 1981) gave the following generalization of Kuttner's theorem.

Theorem 1.2. *If* $0 < p < 1$ *and* X *is a locally convex* FK *- space, then* $X \supset l_\infty$ *whenever* $X \supset w_0(p)$*).*

Kuttner (Kuttner, 1946) proved this result in the case $X = c_A$, where A is a regular matrix method (Kuttner's theorem).

If the matrix $A = (a_{nk})$ satisfies the condition

$$\sup_n a_{nk} > 0 \quad \text{for each} \quad k \in \mathbb{N}, \tag{1.17}$$

then $[c_A]_0$ is a solid $AK - BK$ - space with the norm

$$\|x\| = \sup_n \sum_{k=1}^{\infty} a_{nk}|x_k|. \tag{1.18}$$

Since for every solid $AK - BK$ - space X we have

$$X^\alpha = X^\beta = X^\phi, \tag{1.19}$$

this is also true for $X = [c_A, M]_0$.

For a positive matrix method $A = (a_{nk})$ Virge Soomer (Soomer, 2003) defined

$$D(A,p) = \left\{ x = (x_k) : \quad \lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p}|x_k| = 0 \right\}. \tag{1.20}$$

A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. The space of lacunary strongly convergent sequences L_θ was defined by Freedman et al (Freedman *et al.*, 1978) as :

$$L_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0 \ \text{ for some } \ 1 \right\}. \tag{1.21}$$

The space L_θ is a solid AK-BK - space with the norm

$$\|x\|_\theta = \sup_r \frac{1}{h_r} \sum_{k \in I_r} |x_k|. \tag{1.22}$$

L_θ^0 denotes the subset of L_θ those sequences for which $l = 0$ in the definition of L_θ. Then L_θ^0 is the strong null summability field of the matrix method $A_\theta = (a_{rk}^\theta)$ where

$$a_{rk}^\theta = \begin{cases} \frac{1}{h_r} & (k_r \le k \le k_{r+1} - 1, \quad r,k \in \mathbb{N}) \\ 0 & \text{otherwise} \end{cases} \tag{1.23}$$

For $\theta = (2^r)$ we have $L_\theta^0 = w_0(1)$ and the norm $\|x\|_\theta$ is equivalent to the usual norm

$$\|x\| = \sup_n \frac{1}{n+1} \sum_{k=0}^{n} |x_k| \ \text{ in } \ w_0(1) \text{ (see (Maddox, 1970))}. \tag{1.24}$$

Note that $L_\theta^0(M)$ is a solid $AK - FK$ - space with the F - norm

$$g_M(x) = \sup_r \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{|x_k|}{\rho}\right), \text{for some } \rho > 0. \tag{1.25}$$

T. Bilgin (Bilgin, 2003) defined the following sequence spaces :

$$L_\theta^0(M,p)_\triangle = \left\{ x=(x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} M\left(\frac{|\triangle x_k|}{\rho}\right)^{p_k} = 0,\ \text{for some } \rho > 0 \right\}. \tag{1.26}$$

$$L_\theta(M,p)_\triangle = \left\{ x=(x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} M\left(\frac{|\triangle x_k - l|}{\rho}\right)^{p_k} = 0,\ \text{for some l and } \rho > 0 \right\}. \tag{1.27}$$

$$L_\theta^\infty(M,p)_\triangle = \left\{ x=(x_k) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} M\left(\frac{|\triangle x_k|}{\rho}\right)^{p_k} < \infty,\ \text{for some } \rho > 0 \right\}. \tag{1.28}$$

If we take x_k instead of $\triangle x_k$ and $p_k = 1$ for all k, then we have the following sequence spaces :

$$L_\theta^0(M) = \left\{ x=(x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} M\left(\frac{|x_k|}{\rho}\right) = 0,\ \text{for some } \rho > 0 \right\}. \tag{1.29}$$

$$L_\theta(M) = \left\{ x=(x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} M\left(\frac{|x_k - l|}{\rho}\right) = 0,\ \text{for some l and } \rho > 0 \right\}. \tag{1.30}$$

$$L_\theta^\infty(M) = \left\{ x=(x_k) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ \text{for some } \rho > 0 \right\}. \tag{1.31}$$

Virge Soomer (Soomer, 2003) defined the sequence space

$$H_\theta(p) = \left\{ \alpha = (\alpha_k) : \sum_{r=0}^{\infty} h_r^{1/p} \max_{k\in I_r} |\alpha_k| < \infty \right\}. \tag{1.32}$$

The aim of this paper is to give some extensions of Theorem 1.2 by replacing $w_0(p)$ by $[c_A, M]_0$.

2. Main results

In this paper we define the sequence space :

$$[c_A, M]_0 = \left\{ x=(x_k) : \lim_n \sum_k a_{nk} M\left(\frac{|x_k|}{\rho}\right) = 0,\ \text{for some } \rho > 0 \right\}. \tag{2.1}$$

If $M(x) = x^p$, $p \geq 1$, then we have $[c_A, M]_0 = [c_A]_0^p$, the space of the sequences that are strongly A -summable to zero with index p. By taking $A = (C, 1)$, the Cesàro matrix , and for $0 < p < \infty$ the space $[c_A]_0^p$ is usually denoted by $w_0(p)$, i.e.

$$w_0(p) = \left\{ x=(x_k) : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p = 0 \right\}. \tag{2.2}$$

Theorem 2.1. *Let M be an Orlicz function and let $A = (a_{nk})$ be positive regular matrix method with finite rows satisfying the conditions*

$$\sup_n a_{nk} > 0 \quad \text{for each } k \in \mathbb{N}, \tag{2.3}$$

and

$$\sum_{k=1}^{\infty} a_{nk} = 1 \quad \text{for each } n \in \mathbb{N}. \tag{2.4}$$

Then the following statements hold :

[i] $D(A, p)$ is a solid $AK - BK$ - space with the norm

$$q(x) = \sup_n \sum_{k=1}^{\infty} a_{nk}^{1/p} |x_k|. \tag{2.5}$$

[ii] If M is p - convex , then $[c_A, M]_0 \subset D(A, p)$.

[iii] $l_\infty \subset D(A, p)$ if and only if $\lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p} = 0$.

Proof.

[i] The proof is straightforward.

[ii] Since M is p - convex and $\alpha_k \geq 0$, $\sum_{k=1}^{n} \alpha_k^p = 1$, $t_k \geq 0$, then

$$M\left(\sum_{k=1}^{n} \alpha_k t_k\right) \leq \sum_{k=1}^{n} \alpha_k^p M(t_k). \tag{2.6}$$

Putting $\alpha_k = a_{nk}^{1/p}$ and $t_k = \frac{|x_k|}{\rho}$ we get (note that the matrix A has finite rows and satisfies $\sum_{k=1}^{\infty} a_{nk} = 1$ for each $n \in \mathbb{N}$)

$$M\left(\sum_{k=1}^{\infty} a_{nk}^{1/p} \frac{|x_k|}{\rho}\right) \leq \sum_{k=1}^{\infty} a_{nk} M\left(\frac{|x_k|}{\rho}\right). \tag{2.7}$$

Then [ii] follows by the properties of Orlicz functions.

[iii] It is clear that (see (Boos, 2000), Theorem 2.4.1(of Schur)) that the matrix method $A_p = (a_{nk}^{1/p})$ sums all bounded sequences if and only if

$$\lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p} = 0. \tag{2.8}$$

Theorem 2.2. *Let X be a locally convex FK - space. If the matrix method A and the Orlicz function M satisfy conditions of Theorem 2.1 and*

$$([c_A, M]_0)^{\phi} \subset (D(A, p))^{\phi}, \quad \text{then the condition} \tag{2.9}$$

$$\lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p} = 0 \quad \text{is sufficient for} \tag{2.10}$$

$$X \supset [c_A, M]_0 \Longrightarrow X \supset l_\infty. \tag{2.11}$$

Proof. Let

$$X \supset [c_A, M]_0, \text{ then } X^{\phi} \subset ([c_A, M]_0)^{\phi} \tag{2.12}$$

and

$$X^{\phi} \subset (D(A,p))^{\phi}, \text{ (since } ([c_A, M]_0)^{\phi} \subset (D(A,p))^{\phi}). \tag{2.13}$$

Since the BK - space $D(A, p)$ is an AK - space and hence also an AD - space. (i.e. ϕ is dence in $D(A, p)$), $X \supset D(A, p)$ follows from Theorem 4 of (Snyder & Wilansky, 1972).Thus , by Theorem 1[ii], we get $X \supset l_\infty$.

Theorem 2.3. *Let M be an unbounded p - convex Orlicz function satisfying the condition*

$$M(t^{\frac{1}{p}}) = O(t), t \to \infty. \tag{2.14}$$

Then

$$(L^0_\theta(M))^\alpha = H_\theta(p). \tag{2.15}$$

Proof.

Suppose that $x = (x_k) \in L^0_\theta(M)$,$\alpha = (\alpha_k) \in H_\theta(p)$ and let M^{-1} be the inverse function of M. Let $A_{rk} = |\alpha_k| h_r^{1/p}$ $(r, k \in \mathbb{N})$. Then

$$\sum_{k\in I_r} |\alpha_k x_k| \leq \max_{k\in I_r} A_{rk} (h_r^{1/p})^{-1} \sum_{k\in I_r} |x_k| = \rho \max_{k\in I_r} M^{-1}\left[M\left((h_r^{1/p})^{-1} \sum_{k\in I_r} \frac{|x_k|}{\rho} \right)\right]. \tag{2.16}$$

By applying p-convexity of M we have

$$\sum_{k\in I_r} |\alpha_k x_k| \leq \max_{k\in I_r} A_{rk} M^{-1}\left[(h_r)^{-1} \sum_{k\in I_r} M\left(\frac{|x_k|}{\rho}\right)\right] = \max_{k\in I_r} A_{rk} M^{-1}[g_M(x)]. \tag{2.17}$$

and

$$\sum_{r=0}^{\infty} |\alpha_r x_r| = \sum_{r=0}^{\infty} \sum_{k\in I_r} |\alpha_k x_k| \leq M^{-1}[g_M(x)] \sum_{r=0}^{\infty} (h_r^{1/p}) \max_{k\in I_r} |\alpha_k| < \infty. \tag{2.18}$$

Hence $\alpha = (\alpha_k) \in (L^0_\theta(M))^\alpha$ and thus $H_\theta(p) \subset (L^0_\theta(M))^\alpha$.

Now suppose that $\alpha = (\alpha_k) \notin H_\theta(p)$. Then the series in $H_\theta(p)$ is divergent , and therefore there exists a sequence (c_r), $0 < c_r \to 0$, $r \to \infty$ such that

$$\sum_{r=0}^{\infty} c_r (h_r^{1/p}) \max_{k\in I_r} |\alpha_k| = \infty. \tag{2.19}$$

Let $\max_{k\in I_r} |\alpha_k| = |\alpha_{k_r}|$ and let $\bar{x} = (\bar{x}_k)$ be defined by

$$\bar{x}_k = \begin{cases} \rho c_r h_r^{1/p} & \text{for } k = k_r \\ 0 & \text{for } k \neq k_r \quad r, k \in \mathbb{N} \end{cases} \tag{2.20}$$

Since $c_r \to 0$, $r \to \infty$, we have $c_r < 1$ for sufficiently large r. Now by convexity of M, by the definition $M(0) = 0$ and by the given condition (2.14) we have

$$(h_r)^{-1} \sum_{k\in I_r} M\left(\frac{|x_k|}{\rho}\right) = (h_r)^{-1} M(c_r h_r^{1/p}) \leq \frac{c_r^p M(h_r^{1/p})}{h_r} = O(1), \text{ as } r \to \infty. \tag{2.21}$$

Hence $x \in L_\theta^0(M)$.

But

$$\sum_{k \in I_r} |\alpha_k \bar{x}_k| = |\alpha_{k_r}| c_r h_r^{1/p} \tag{2.22}$$

so that by (3.2) the series $\sum_{k=0}^{\infty} |\alpha_k \bar{x}_k|$ diverges and therefore $(\alpha_k) \notin (L_\theta^0(M))^\alpha$.This completes the proof.

Theorem 2.4. *Let X be a locally convex FK - space and let M be an unbounded p - convex Orlicz function satisfying the condition*

$$H_\theta(p) = \left\{ \alpha = (\alpha_k) : \lim_r \frac{1}{h_r^{1/p}} \max_{k \in I_r} |\alpha_k| < \infty \right\}. \tag{2.23}$$

Then the following statements holds :

[i] $(L_\theta^0(M))^\alpha \subset (D(A_\theta, p))^\alpha$,

[ii] $X \supset L_\theta^0(M) \Longrightarrow X \supset l_\infty$.

Proof.

[i] Since $L_\theta^0(M)$ and $D(A_\theta, p)$ are solid $AK - FK$ spaces. This implies that their α - duals and ϕ - duals are equal and so it is sufficient to prove $(L_\theta^0(M))^\phi \subset (D(A_\theta, p))^\phi$,. By Theorem 2.3 it is sufficient to show $H_\theta(p) \subset (D(A_\theta, p))^\alpha$.

Suppose that $\alpha = (\alpha_k) \in H_\theta(p)$, then for each $x = (x_k) \in D(A_\theta, p)$ we have

$$\sum_{r=0}^{\infty} |\alpha_r x_r| = \sum_{r=0}^{\infty} \sum_{k \in I_r} |\alpha_k x_k| \tag{2.24}$$

$$\leq \sum_{r=0}^{\infty} h_r^{1/p} \max_{k \in I_r} |\alpha_k| \frac{1}{h_r^{1/p}} \sum_{k \in I_r} |x_k| \leq q(x) \sum_{r=0}^{\infty} h_r^{1/p} \max_{k \in I_r} |\alpha_k| < \infty. \tag{2.25}$$

This implies that $(\alpha_k) \in (D(A_\theta, p))^\phi$.

[ii] The matrix $A_\theta = (a_{nk}^\theta)$ satisfies conditions of Theorem 2.1, $(L_\theta^0(M))^\phi \subset (D(A_\theta, p))^\phi$ by [i] and

$$\lim_r (a_{nk}^\theta)^{1/p} = \lim_r h_r^{1-1/p} = 0. \tag{2.26}$$

Consequently proof of [ii] follows immediately by Theorem 2.2.

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