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Univalence Conditions for a New Integral Operator

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Abstract

In this paper, we study the univalence conditions for a new integral operator defined by Al-Oboudi differential operator. Many known univalence conditions are written to prove our main results.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \tag{1.2}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0. \tag{1.3}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.
For $f \in \mathcal{A}$, Al-Oboudi (Al-Oboudi, 2004) introduced the following operator

$$D^0 f(z) = f(z), \tag{1.4}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z), \qquad \delta \geq 0 \tag{1.5}$$

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$$D^n f(z) = D_\delta\left(D^{n-1} f(z)\right), \qquad (n \in \mathbb{N} = \{1, 2, ...\}), \tag{1.6}$$

If f is given by (1.1), then from (1.5) and (1.6) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\,\delta]^n a_k z^k, \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup 0), \tag{1.7}$$

with $D^n f(0) = 0$.

Remark. When $\delta = 1$, we get Sălăgean differential operator (Sălăgean, 1983).

Here, in our present investigation, we introduce a new general integral operator by means of the Al-Oboudi differential operator as follows

$$F_{n,m,\beta}(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{D^m f_i(t)}{t}\right)^{\alpha_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt\right)^{\frac{1}{\beta}} \tag{1.8}$$

$\alpha_i, \gamma_i \in \mathbb{C}$, $\beta \in \mathbb{C} - \{0\}$, $f_i, g_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$, D^m is the Al-Oboudi differential operator, $m \in \mathbb{N}_0$ and $\frac{D^m f_i(z)}{z} \neq 0$.

In this paper, we study the univalence conditions involving the general integral operator defined by (1.8).

In the proof of our main results (Theorem 2.1) we need the following univalence criterion. The univalence criterion, asserted by Theorem 1.1, is a generalization of Ahlfors's and Becker's univalence criterion; it was proven by Pescar (Pescar, 1996).

Theorem 1.1. (Pescar, 1996) *Let β be a complex number, $Re\beta > 0$, and c a complex number, $|c| \leq 1$, $c \neq -1$ and $f(z) = z + ...$ a regular function in U. If*

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{z f''(z)}{\beta f'(z)} \right| \leq 1, \tag{1.9}$$

for all $z \in U$, then the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt\right)^{\frac{1}{\beta}} = z + ... \tag{1.10}$$

is regular and univalent in U.

In (Yang & Liu, 1999) is defined the class $\mathcal{S}(p)$. For $0 < p \leq 2$, let $\mathcal{S}(p)$ denote the class of functions $f \in \mathcal{A}$ which satisfies the conditions $f(z) \neq 0, (0 < |z| < 1)$ and $\left|\left(\frac{z}{f(z)}\right)''\right| \leq p, (z \in \mathbb{U})$. Also, if $f \in \mathcal{S}(p)$ then the following property is true

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq p\,|z|^2, (z \in \mathbb{U}) \tag{1.11}$$

relation proved in (Singh, 2000).

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, (Nehari, 1952)).

Lemma 1.2. (Nehari, 1952) *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger or equal to m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m , (z \in \mathbb{U}_R) . \tag{1.12}$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m, \tag{1.13}$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let the functions $f_i \in \mathcal{A}$ satisfy the conditions*

$$\left| \frac{z^2 \left(D^m f_i(z)\right)'}{[D^m f_i(z)]^2} - 1 \right| \leq p_i |z|^2 , (z \in \mathbb{U}; 0 < p_i \leq 2) , \tag{2.1}$$

$$\frac{D^m f_i(z)}{z} \neq 0, (z \in \mathbb{U}; m \in \mathbb{N}_0) \tag{2.2}$$

and $g_i \in \mathcal{A}$ with

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| \leq 1, (z \in \mathbb{U}) \tag{2.3}$$

for all $i \in \{1, 2, ..., n\}$. Also, let $\alpha_i, \gamma_i, \beta$ be complex numbers with the property

$$\mathrm{Re}\beta \geq \sum_{i=1}^{n} \left[|\alpha_i| \left((1 + p_i) M_i + 1\right) + 2 |\gamma_i| N_i \right] > 0, (i \in \{1, 2, ..., n\}) . \tag{2.4}$$

If for all $i \in \{1, 2, ..., n\}$

$$\left|D^m f_i(z)\right| \leq M_i, (z \in \mathbb{U}; M_i \geq 1; m \in \mathbb{N}_0) , \tag{2.5}$$

$$|g_i(z)| \leq N_i \, (z \in \mathbb{U}; N_i \geq 1) \tag{2.6}$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[|\alpha_i| \left((p_i + 1) M_i + 1\right) + 2 |\gamma_i| N_i \right] \tag{2.7}$$

then the integral operator $F_{n,m,\beta}(z)$ defined by (1.8) is in the class $\mathcal{S}$.

Proof. By (1.7), we have

$$\frac{D^m f_i(z)}{z} = 1 + \sum_{k=2}^{\infty} [1 + (k-1)\,\delta]^m a_{k,i}\, z^{k-1}, (m \in \mathrm{N}_0) \tag{2.8}$$

for all $i \in \{1, 2, ..., n\}$. Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{D^m f_i(t)}{t} \right)^{\alpha_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt, \tag{2.9}$$

then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$h'(z) = \prod_{i=1}^{n} \left(\frac{D^m f_i(z)}{z}\right)^{\alpha_i} \left(e^{g_i(z)}\right)^{\gamma_i} \tag{2.10}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \left[\alpha_i \left(\frac{z\,(D^m f_i(z))'}{D^m f_i(z)} - 1\right) + \gamma_i z g_i'(z)\right]. \tag{2.11}$$

From the equation (2.11), we have

$$\begin{aligned}
\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)}\right| &= \left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \sum_{i=1}^{n} \left(\alpha_i \left(\frac{z\,(D^m f_i(z))'}{D^m f_i(z)} - 1\right) + \gamma_i z g_i'(z)\right)\right| \\
&\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} \left(|\alpha_i| \left(\left|\frac{z\,(D^m f_i(z))'}{D^m f_i(z)}\right| + 1\right) + |\gamma_i| \left|z g_i'(z)\right|\right) \\
&\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} \left[|\alpha_i| \left(\left|\frac{z^2\,(D^m f_i(z))'}{[D^m f_i(z)]^2}\right| \left|\frac{D^m f_i(z)}{z}\right| + 1\right)\right. \\
&\qquad \left. + |\gamma_i| \left|\frac{z g_i'(z)}{g_i(z)}\right| |g_i(z)|\,\right].
\end{aligned} \tag{2.12}$$

From the hypothesis, we have

$$\left|D^m f_i(z)\right| \leq M_i, (z \in \mathbb{U}), |g_i(z)| \leq N_i, (z \in \mathbb{U}),$$

then by the General Schwarz Lemma for the functions f_i $(i \in \{1, 2, ..., n\})$, we obtain

$$\left|D^m f_i(z)\right| \leq M_i\,|z|, (z \in \mathbb{U}; i \in \{1, 2, ..., n\}).$$

We apply this result in the inequality (2.12) and from (2.1), (2.3) we obtain

$$\begin{aligned}
\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)}\right| &\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} \left[|\alpha_i| \left(\left(\left|\frac{z^2\,(D^m f_i(z))'}{[D^m f_i(z)]^2} - 1\right| + 1\right) M_i + 1\right)\right. \\
+ |\gamma_i| \left(\left|\frac{z g_i'(z)}{g_i(z)} - 1\right| + 1\right) N_i\Big] &\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} \left[\,|\alpha_i| \left(\left(p_i\,|z|^2 + 1\right) M_i + 1\right) + 2\,|\gamma_i|\,N_i\right] \\
&\leq |c| + \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[\,|\alpha_i|\,((p_i + 1)\,M_i + 1) + 2\,|\gamma_i|\,N_i\right].
\end{aligned} \tag{2.13}$$

So, from (2.7) we have

$$\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)}\right| \leq 1. \tag{2.14}$$

Applying Theorem 1.1, we obtain that $F_{n,m,\beta}(z)$ is in the class $\mathcal{S}$. □

If we set $m = 0$ in Theorem 2.1, we can obtain the following interesting consequence of this theorem.

Corollary 2.2. *Let $f_i \in \mathcal{A}$ satisfy the condition*

$$\left|\frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1\right| \le p_i |z|^2, (z \in \mathbb{U}; 0 < p_i \le 2) \tag{2.15}$$

and $g_i \in \mathcal{A}$ with

$$\left|\frac{z g_i'(z)}{g_i(z)} - 1\right| \le 1, (z \in \mathbb{U}) \tag{2.16}$$

for all $i \in \{1, 2, ..., n\}$. Also, let $\alpha_i, \gamma_i, \beta$ be complex numbers with the property

$$\mathrm{Re}\beta \ge \sum_{i=1}^{n} [\,|\alpha_i| ((1 + p_i) M_i + 1) + 2 |\gamma_i| N_i] > 0, (i \in \{1, 2, ..., n\}). \tag{2.17}$$

If for all $i \in \{1, 2, ..., n\}$

$$|f_i(z)| \le M_i \,(z \in \mathbb{U}; M_i \ge 1), \tag{2.18}$$

$$|g_i(z)| \le N_i \,(z \in \mathbb{U}; N_i \ge 1) \tag{2.19}$$

and

$$|c| \le 1 - \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} [\,|\alpha_i| ((p_i + 1) M_i + 1) + 2 |\gamma_i| N_i] \tag{2.20}$$

then the integral operator

$$F_{n,\beta}(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt\right)^{\frac{1}{\beta}} \tag{2.21}$$

is in the class S.

Setting $n = 1$ in Theorem 2.1 we have:

Corollary 2.3. *Let $f \in \mathcal{A}$ satisfies the conditions*

$$\left|\frac{z^2 (D^m f(z))'}{[D^m f(z)]^2} - 1\right| \le p |z|^2, (z \in \mathbb{U}; 0 < p \le 2), \tag{2.22}$$

$$\frac{D^m f(z)}{z} \ne 0, (z \in \mathbb{U}; m \in \mathbb{N}_0) \tag{2.23}$$

and $g \in \mathcal{A}$ with

$$\left|\frac{z g'(z)}{g(z)} - 1\right| \le 1, (z \in \mathbb{U}). \tag{2.24}$$

Also, let α, γ, β be complex numbers with the property

$$\mathrm{Re}\beta \ge [\,|\alpha| ((1 + p) M + 1) + 2 |\gamma| N] > 0. \tag{2.25}$$

If

$$\left|D^m f(z)\right| \le M, (z \in \mathbb{U}; M \ge 1; m \in \mathbb{N}_0), \tag{2.26}$$

$$|g(z)| \leq N, (z \in \mathbb{U}; N \geq 1)\,, \tag{2.27}$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta}[\,|\alpha|\,((p+1)\,M+1)+2\,|\gamma|\,N] \tag{2.28}$$

then the integral operator

$$F_{m,\beta}(z) = \left(\beta \int_0^z t^{\beta-1}\left(\frac{D^m f(t)}{t}\right)^{\alpha}\left(e^{g(t)}\right)^{\gamma} dt\right)^{\frac{1}{\beta}} \tag{2.29}$$

is in the class S.

If we set $m = 0$ in Corollary 2.3 we have another interesting consequence:

Corollary 2.4. *Let* $f \in \mathcal{A}$ *satisfies the condition (1.11) and* $g \in \mathcal{A}$ *with*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq 1 \qquad (z \in \mathbb{U})\,. \tag{2.30}$$

Also, let α, γ, β *be complex numbers with the property*

$$\mathrm{Re}\beta \geq [\,|\alpha|\,((1+p)\,M+1)+2\,|\gamma|\,N] > 0. \tag{2.31}$$

If

$$|f(z)| \leq M, (z \in \mathbb{U}; M \geq 1)\,, \tag{2.32}$$

$$|g(z)| \leq N, (z \in \mathbb{U}; N \geq 1) \tag{2.33}$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta}[\,|\alpha|\,((p+1)\,M+1)+2\,|\gamma|\,N] \tag{2.34}$$

then the integral operator

$$F_{\beta}(z) = \left(\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\alpha}\left(e^{g(t)}\right)^{\gamma} dt\right)^{\frac{1}{\beta}} \tag{2.35}$$

is in the class $\mathcal{S}$.

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