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On Some I-Convergent Sequence Spaces Defined by a Modulus Function

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Abstract

In this article we introduce the sequence spaces $c_0^I(f)$, $c^I(f)$ and $l_\infty^I(f)$ for a modulus function f and study some of the properties of these spaces.

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1. Introduction

Throughout the article $\mathbb{N}, \mathbb{R}, \mathbb{C}$ and ω denotes the set of natural,real,complex numbers and the class of all sequences respectively. The notion of the statistical convergence was introduced by (Fast, 1951). Later on it was studied by (Fridy, 1985) and (Fridy, 1993) from the sequence space point of view and linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Salat and Wilezynski in (Kostyrko *et al.*, 2000). Later on it was studied by Salat, Tripathy and Ziman in (Šalát *et al.*, 2004) and Demirci in (Demirci, 2001).

Here we give some preliminaries about the notion of I-convergence. Let X be a non empty set. A set $I \subseteq 2^X$ (2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \pounds(I)$,for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

An Ideal I$\subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\pounds(I)$ corresponding to I. i.e $\pounds(I) = \{K \subseteq N : K^c \in I\}$,where K$^c = N - K$.

The idea of modulus was structured in 1953 by Nakano. (See (Nakano, 1953)).

A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if:

(1) f(t) $= 0$ if and only if $t = 0$,

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(2) $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is nondecreasing, and,
(4) f is continuous from the right at zero.

Ruckle in (Ruckle, 1968) used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\} \tag{1.1}$$

This space is an FK space, and (Ruckle, 1967) proved that the intersecton of all such $X(f)$ spaces is ϕ, the space of all finite sequences.

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus (Ruckle, 1973) proved that, for any modulus f.

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = l_\infty \tag{1.2}$$

Where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\} \tag{1.3}$$

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See (Ruckle, 1973)).} \tag{1.4}$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch in (Gramsch, n.d.). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by D. J. H Garling in (Garling, 1966) and (Garling, 1968), G. Kothe in (Köthe, 1970)and W. H. Ruckle in (Ruckle, 1968) and (Ruckle, 1967).

Definition 1.1. A sequence space E is said to be solid or normal if $(x_k) \in$ E implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

Definition 1.2. A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Definition 1.3. A sequence space E is said to be covergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.4. A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k) \in E$, $(y_k) \in E$.

Definition 1.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$ where $\pi(k)$ is a permutation on $\mathbb{N}$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I - \lim x_k = L$.

The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \ : |x_k - L| \geq \epsilon\} \in I, \text{for some } L \in \mathbb{C}\}. \tag{1.5}$$

Definition 1.7. A sequence $(\mathrm{x}_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.9. A sequence$(\mathrm{x}_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in N : |x_k| > M\} \in I$

Definition 1.10. A modulus function f is said to satisfy $\triangle_2$ condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

Definition 1.11. Take for I the class I_f of all finite subsets of $\mathbb{N}$. Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X.(see (Kostyrko *et al.*, 2000)).

Definition 1.12. For $I = I_\delta$ and $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ-convergence is said to be logarithmic statistical covergence.(see (Kostyrko *et al.*, 2000)).

Definition 1.13. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$ (see (Šalát *et al.*, 2004)).

Definition 1.14. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}. \tag{1.6}$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (See (Šalát *et al.*, 2004)).

Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function (see (Šalát *et al.*, 2004)).
(c.f (Connor & Kline, 1996), (Dems, 2005), (Gurdal, 2004), (Jones & Retherford, 1967), (Kamthan & Gupta, 1980), (Maddox, 1970), (Maddox, 1986), (Maddox, 1969), (Šalát, 1980), (Singer, 1970), (Wilansky, 1964))

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded , I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces:

$$c^I(f) = \{(x_k) \in \omega : I - \lim f(|x_k|) = L \text{ for some L}\} \in I \tag{1.7}$$

$$c_0^I(f) = \{(x_k) \in \omega : I - \lim f(|x_k|) = 0\} \in I \tag{1.8}$$

$$l_\infty^I(f) = \{(x_k) \in \omega : \sup_k f(|x_k|) < \infty\} \in I \tag{1.9}$$

We also denote by

$$m^I(f) = c^I(f) \cap l_\infty(f) \tag{1.10}$$

and

$$m_0^I(f) = c_0^I(f) \cap l_\infty(f) \tag{1.11}$$

The following Lemmas will be used for establishing some results of this article:

Lemma 1.1. *Let E be a sequence space. If E is solid then E is monotone.*

Lemma 1.2. *Let $K \in £(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.*

Lemma 1.3. *If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.*

2. Main results

Theorem 2.1. *For any modulus function f, the classes of sequences $c^I(f), c_0^I(f), m^I(f)$ and $m_0^I(f)$ are linear spaces.*

Proof. We shall prove the result for the space $c^I(f)$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in c^I(f)$ and let α, β be scalars. Then

$$I - \lim f(|x_k - L_1|) = 0, \text{ for some } L_1 \in c; \tag{2.1}$$

$$I - \lim f(|y_k - L_2|) = 0, \text{ for some } L_2 \in c; \tag{2.2}$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in N : f(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{2.3}$$

$$A_2 = \{k \in N : f(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{2.4}$$

Since f is a modulus function, we have

$$f(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) \leq f(|\alpha||x_k - L_1|) + f(|\beta||y_k - L_2|) \leq f(|x_k - L_1|) + f(|y_k - L_2|) \tag{2.5}$$

Now, by (2.3) and (2.4),

$$\{k \in N : f(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2. \tag{2.6}$$

Therefore

$$(\alpha x_k + \beta y_k) \in c^I(f). \tag{2.7}$$

Hence $c^I(f)$ is a linear space. □

Theorem 2.2. *A sequence $x = (x_k) \in m^I(f)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(f) \tag{2.8}$$

Proof. Suppose that$L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I(f). \text{ For all} \epsilon > 0. \tag{2.9}$$

Fix an $N_\epsilon \in B_\epsilon$.Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{2.10}$$

which holds for all $k \in B_\epsilon$. Hence

$$\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(f). \tag{2.11}$$

Conversely, suppose that $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(f)$.
That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(f)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(f) \text{for all} \epsilon > 0. \tag{2.12}$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(f)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(f)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(f)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi \tag{2.13}$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I(f) \tag{2.14}$$

that is

$$diam J \leq diam J_\epsilon \tag{2.15}$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq .... \tag{2.16}$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(f)$ for $(k = 1, 2, 3, 4, ...)$.
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f(\xi) = I - \lim f(x)$, that is $L = I - \lim f(x)$.

□

Theorem 2.3. *Let f and g be modulus functions that satisfy the $\triangle_2$-condition.If X is any of the spaces c^I, c_0^I, m^I and m_0^I etc, then the following assertions hold.*

(1) $X(g) \subseteq X(f.g)$,

(4) $X(f) \cap X(g) \subseteq X(f + g)$.

Proof. (1) Let $(x_k) \in c_0^I(g)$.Then

$$I - \lim_k g(|x_k|) = 0 \tag{2.17}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t < \delta$. Write $y_k = g(|x_k|)$ and consider $\lim_k f(y_k) = \lim_k f(y_k)_{y_k<\delta} + \lim_k f(y_k)_{y_k>\delta}$. We have

$$\lim_k f(y_k) \leq f(2) \lim_k (y_k). \tag{2.18}$$

For $y_k > \delta$, we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since f is non-decreasing, it follows that

$$f(y_k) < f(1 + \frac{y_k}{\delta}) < \frac{1}{2} f(2) + \frac{1}{2} f(\frac{2y_k}{\delta}). \tag{2.19}$$

Since f satisfies the $\triangle_2$-condition, we have

$$f(y_k) < \frac{1}{2} K \frac{y_k}{\delta} f(2) + \frac{1}{2} K \frac{y_k}{\delta} f(2) = K \frac{y_k}{\delta} f(2). \tag{2.20}$$

Hence

$$\lim_k f(y_k) \leq max(1, K)\delta^{-1} f(2) \lim_k(y_k). \tag{2.21}$$

From (2.17), (2.18) and (2.21), we have $(x_k) \in c_0^I(f.g)$.

Thus $c_0^I(g) \subseteq c_0^I(f.g)$. The other cases can be proved similarly.

(2) Let $(x_k) \in c_0^I(f) \cap c_0^I(\mathrm{g})$. Then

$$I - \lim_k f(|x_k|) = 0 \tag{2.22}$$

and

$$I - \lim_k g(|x_k|) = 0 \tag{2.23}$$

The rest of the proof follows from the following equality

$$\lim_k (f + g)(|x_k|) = \lim_k f(|x_k|) + \lim_k g(|x_k|). \tag{2.24}$$

□

Corollary 2.4. *$X \subseteq X(f)$ for $X = c^I, c_0^I, m^I$ and m_0^I.*

Theorem 2.5. *The spaces $c_0^I(f)$ and $m_0^I(f)$ are solid and monotone .*

Proof. We shall prove the result for $c_0^I(f)$. Let $x_k \in c_0^I(f)$.Then

$$I - \lim_k f(|x_k|) = 0 \tag{2.25}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from (2.25) and the following inequality

$$f(|\alpha_k x_k|) \leq |\alpha_k| f(|x_k|) \leq f(|x_k|) \text{ for all } k \in \mathbb{N}. \tag{2.26}$$

That the space c_0^I(f) is monotone follows from the Lemma 1.1.
For $m_0^I(f)$ the result can be proved similarly. □

Theorem 2.6. *The spaces $c^I(f)$ and $m^I(f)$ are neither solid nor monotone in general.*

Proof. Here we give a counter example.
Let $I = I_\delta$ and $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows.
Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y_k) = \begin{cases} (x_k), \text{if } k \text{ is even}, \\ 0, otherwise. \end{cases} \tag{2.27}$$

Consider the sequence (x_k) defined by $(x_k) = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in c^I(f)$ but its K-stepspace preimage does not belong to $c^I(f)$. Thus $c^I(f)$ is not monotone. Hence $c^I(f)$ is not solid. □

Theorem 2.7. *The spaces $c^I(f)$ and $c_0^I(f)$ are sequence algebras.*

Proof. We prove that $c_0^I(f)$ is a sequence algebra. Let $(x_k), (y_k) \in c_0^I(f)$. Then

$$I - \lim f(|x_k|) = 0$$

and

$$I - \lim f(|y_k|) = 0. \tag{2.28}$$

Then we have

$$I - \lim f(|(x_k.y_k)|) = 0. \tag{2.29}$$

Thus $(x_k.y_k) \in c_0^I(f)$ is a sequence algebra. For the space $c^I(f)$, the result can be proved similarly. □

Theorem 2.8. *The spaces $c^I(f)$ and $c_0^I(f)$ are not convergence free in general.*

Proof. Here we give a counter example.
Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \quad and \quad y_k = k \text{ for all } k \in \mathrm{N}$$

Then $(x_k) \in c^I(f)$ and $c_0^I(f)$, but $(y_k) \notin c^I(f)$ and $c_0^I(f)$.
Hence the spaces $c^I(f)$ and $c_0^I(f)$ are not convergence free.

□

Theorem 2.9. *If I is not maximal and $I \neq I_f$, then the spaces $c^I(f)$ and $c_0^I(f)$ are not symmetric.*

Proof. Let $A \in I$ be infinite and $f(x) = x$ for all x$\in [0, \infty)$.
If

$$x_k = \begin{cases} 1, \text{for } k \in A, \\ 0, otherwise. \end{cases}$$

Then by lemma 1.3 $x_k \in c_0^I(f) \subset c^I(f)$
let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), \text{for } k \in K, \\ \psi(k), otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin c^I(f)$ and $x_{\pi(k)} \notin c_0^I(f)$.
Hence $c_0^I(f)$ and $c^I(f)$are not symmetric. □

Theorem 2.10. *Let f be a modulus function. Then $c_0^I(f) \subset c^I(f) \subset l_\infty^I(f)$ and the inclusions are proper.*

Proof. Let $x_k \in c^I(f)$. Then there exists $L \in C$ such that

$$I - \lim f(|x_k - L|) = 0. \tag{2.30}$$

We have $f(|x_k|) \leq \frac{1}{2} f(|x_k - L|) + f\frac{1}{2}(|L|)$.
Taking the supremum over k on both sides we get $x_k \in l_\infty(f)$.
The inclusion $c_0^I(f) \subset c^I(f)$ is obvious. □

Theorem 2.11. *The function $\hbar : m^I(f) \to \mathbb{R}$ is the Lipschitz function,where $m^I(f) = c^I(f) \cap l_\infty(f)$, and hence uniformly cotinuous.*

Proof. Let $x, y \in m^I(f)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq \|x - y\|\} \in I, \tag{2.31}$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq \|x - y\|\} \in I. \tag{2.32}$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \|x - y\|\} \in m^I(f), \tag{2.33}$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \|x - y\|\} \in m^I(f). \tag{2.34}$$

Hence also $B = B_x \cap B_y \in m^I(f)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3\|x - y\|. \tag{2.35}$$

Thus $\hbar$ is a Lipschitz function. For $m_0^I(f)$ the result can be proved similarly. □

Theorem 2.12. *If $x, y \in m^I(f)$, then $(x.y) \in m^I(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I(f), \tag{2.36}$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I(f). \tag{2.37}$$

Now,

$$|x_k y_k - \hbar(x)\hbar(y)| = |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)|$$

$$\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)| \tag{2.38}$$

As $m^I(f) \subseteq l_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$. Using (2.38) we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon.$$

For all $k \in B_x \cap B_y \in m^I(f)$. Hence $(x.y) \in m^I(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For $m_0^I(f)$ the result can be proved similarly. □

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