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On Vector Valued Periodic Distributions

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Abstract

In this paper we consider vector valued (X-valued with X a Banach space) distributions on the euclidean space $\mathbb{R}^d$, extending the T-periodicity, and the T-periodic transform with $T = (T_1, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$ from the scalar case to the Banach space valued case.

Besides immediate basic properties of these concepts, a realization of the space of X-valued T-periodic distributions, up to a toplinear isomorphism, as the space of all bounded linear operators from the space of T-periodic test functions to the Banach space X is given.

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1. Introduction

It is well known the part played by the concept of "periodicity" in the mathematical description of the state of a "phenomenon" with some rhythmic evolutions, appearing in different particular sciences.

But in spite of fact that mathematical models are often well described in terms of vector valued periodic functions, there are many situations in which the ordinary concept of function is not satisfactory. Such situations are mainly determined by the absence of derivability of such functions, especially when the evolutions of the phenomena to be modeled must satisfy a law expressed by a differential equation. Such difficulties are well overcome in the more general setting of distributions, or, if we wish to describe a class of larger and more complex situations, of vector valued distributions.

It is the aim of this paper to enlarge the domains (the possibilities) of application of vector valued periodic functions, extending some important results on scalar periodic distributions to the vector valued case.

Let us mention that there is a very rich literature regarding distributions and even their periodicity in the scalar case (see (Schwartz, 1950), (Zemanian, 1965), (Kecs, 1978)), as well as the new developments connected especially to the theory of topological linear spaces, including some general aspects from the

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vector valued case (see (Schwartz, 1953*a*), (Schwartz, 1953*b*), (Gaşpar & Gaşpar, 2009), (Schwartz, 1957)), which we shall use elsewhere.

The content of the paper runs as follows.

In Section 2 we recall and complete some necessary basic results on the spaces of test functions and of locally r-summable functions on the euclidean space $\mathbb{R}^d$ with respect to the Lebesgue measure $m_d(\cdot)$, the T-periodicity with respect to a general period $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, the T-periodic transform taking a special place.

The Section 3 is devoted to the main results of the note.

Considering the class of X-valued T-periodic distributions as a subspace of the space of all X-valued distributions (X a Banach space), which is invariant to the multiplication operator with T-periodic test functions (Proposition 3.2) and to derivation (Proposition 3.3), the T-periodic transform is extended from the space of compactly supported test functions to the space of compactly supported X-valued distributions (Theorem 3.1).

It is also proved that the space of X-valued T-periodic distributions is isomorphic as linear topological space to the space of all bounded linear operators from space of T-periodic test functions on X (Theorem 3.2 and Theorem 3.3).

2. Periodic functions

In this section we define the T-periodicity for test and locally summable scalar functions, as well as the T-periodic transform on the space of scalar test functions.

Definition 2.1. (see (Zemanian, 1965), chap. 11, § 2, p. 314) An ordinary function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be periodic if there exists $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, such that $(L_T f)(t) = f(t)$, $t \in \mathbb{R}^d$, where L_τ, $\tau \in \mathbb{R}^d$ means the translation operator on $\mathbb{R}^d$. T is called a period of f. The set of all periods of f is kT ($kT = (k_1 T_1, ..., k_d T_d)$, $k \in \mathbb{Z}^d$). The "smallest" period is called the fundamental period of f.

We will denote by $[0, T]$ the d-dimensional "parallelepiped" $[0, T_1] \times [0, T_2] \times ... \times [0, T_d]$, $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, $i \in \mathbb{N}$.

Definition 2.2. (see (Zemanian, 1965), chap. 11, § 2, p. 314) A function $\theta : \mathbb{R}^d \to \mathbb{C}$ will be called T-periodic test function, if it is periodic of period T and infinitely smooth. The space of all such T-periodic test functions will be denoted by $\mathcal{D}_T(\mathbb{R}^d)$ or $\mathcal{D}_{d,T}$.

Let us recall the basic well known spaces of test functions used in distributions theory (see (Gaşpar & Gaşpar, 2009), (Schwartz, 1950)): $\mathcal{D}(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{E}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$,$\dot{\mathcal{B}}(\mathbb{R}^d)$ and $O_M(\mathbb{R}^d)$ which we shall briefly denote $\mathcal{D}_d$, $\mathcal{S}_d$, $\mathcal{E}_d$, $\mathcal{B}_d$, $\dot{\mathcal{B}}_d$ and $O_{d,M}$. We also denote the Lebesgue spaces L^r_d of r-summable complex functions on $\mathbb{R}^d$ with respect to the Lebesgue measure m_d on $\mathbb{R}^d$ and $L^r_{d,loc}$ of locally r-summable complex valued functions on $\mathbb{R}^d$, while $L^r_{d,T}$ means the set of all elements from $L^r_{d,loc}$, which are T-periodic, where $1 \le r \le \infty$. For the space of all complex functions from $\mathcal{E}_d$ which together with all derivatives are in L^r_d we use the notation $\mathcal{D}_{d,L^r}$ $1 \le r \le \infty$ and $\mathcal{B}_d = \mathcal{D}_{d,L^\infty}$ (see (Schwartz, 1950), p. 55). For $r = 1$ we obtain the space of summable test functions $\mathcal{D}_{d,L^1}$.

These spaces satisfy the inclusions (with continuous embeddings):

$$\mathcal{D}_d \subset \mathcal{S}_d \subset \mathcal{D}_{d,L^1} \subset \mathcal{D}_{d,L^r} \subset \dot{\mathcal{B}}_d \subset \mathcal{B}_d \subset O_{d,M} \subset \mathcal{E}_d \tag{2.1}$$

(see (Schwartz, 1950))

Remark. $\mathcal{D}_{d,T}$ is a linear space and following inclusions hold

$$\mathcal{D}_{d,T} \subset \mathcal{B}_d \subset O_{d,M} \subset \mathcal{E}_d. \tag{2.2}$$

The space $\mathcal{D}_{d,T}$ will be endowed with the topology induced from $\mathcal{B}_d$, i.e. a sequence $\{\theta_k\}_{k\in\mathbb{N}}$ from $\mathcal{D}_{d,T}$ converges to zero, if the sequences of all derivatives $\{D^\alpha\theta_k\}_{k\in\mathbb{N}}$ $(\alpha \in \mathbb{N}^d)$ converge uniformly to zero.

Definition 2.3. The T-periodic transform on $\mathcal{D}_d$ denoted by ϖ_T (for T=(1,...,1) see (Schwartz, 1950), p. 85) is the $\mathcal{D}_{d,T}$-valued operator on $\mathcal{D}_d$ defined by

$$(\varpi_T\varphi)(t) = \sum_{n\in\mathbb{Z}^d} \varphi(t - nT) = \sum_{n\in\mathbb{Z}^d} (L_{nT}\varphi)(t),\ t \in \mathbb{R}^d,\ \varphi \in \mathcal{D}_d. \tag{2.3}$$

A function ξ from $\mathcal{D}_d$ is called a T-unitary function, or a T-partition of unity (see (Kecs, 1978), chap. 3, § 2, p. 133 and (Zemanian, 1965), chap. 11, § 2, p. 315), if $\varpi_T\xi = 1$. The space of all such functions ξ will be denoted by $\mathcal{U}_T(\mathbb{R}^d)$, or $\mathcal{U}_{d,T}$.

Remark. ϖ_T is a continuous linear operator from $\mathcal{D}_d$ onto $\mathcal{D}_{d,T}$.

Indeed, it is easy to see that ϖ_T is linear in φ and, if φ_j converges to zero $(j \to \infty)$ in $\mathcal{D}_d$, then $\varpi_T\varphi_j$ converges to zero in $\mathcal{D}_{d,T}$. Moreover ϖ_T is an onto mapping, since for any $\theta \in \mathcal{D}_{d,T}$ and a fixed $\xi \in \mathcal{U}_{d,T}$, we have $\xi\theta \in \mathcal{D}_d$ and $\varpi_T(\xi\theta) = \theta$. In this context it is obvious that the mapping

$$\mathcal{D}_{d,T} \ni \theta \mapsto \xi\theta \in \mathcal{D}_d, \tag{2.4}$$

is a linear continuous "inverse" of ϖ_T.

Remark. For each $\varphi \in \mathcal{D}_d$ the sum $\sum_{n\in\mathbb{Z}^d} (L_{nT}\varphi)(t)$ is finite and because $L_T(\sum_{n\in\mathbb{Z}^d} (L_{nT}\varphi)) = \sum_{n\in\mathbb{Z}^d} (L_{nT}\varphi)$, it defines a function from $\mathcal{D}_{d,T}$.

Remark. ϖ_T can be extended in a natural way to the space $\mathcal{D}_{d,L^1}$ (compare with (Schwartz, 1950), p. 86).

Remark. It is immediately seen that

$$\mathcal{D}_d = \mathcal{U}_{d,T}\mathcal{D}_{d,T}, \tag{2.5}$$

holds.

Let us mention that this T-periodic transform on the space of test functions is used in the study of scalar periodic distributions by extending this transform from test functions to distributions. Namely such a T-periodic transform is extended to the space of compactly supported distributions, $\mathcal{E}'_d$ (see (Kecs, 1978), p. 138) and to the space $\mathcal{D}'_{d,L^1}$ of summable distributions (see (Schwartz, 1950), p. 86). We try to do that for the case of vector valued distributions in the next Section.

3. *T*-periodic transform of *X*-valued distributions

At the beginning let us recall some general facts.

Definition 3.1. (see (Schwartz, 1957), chap. II, § 2) Let X be a Banach space. Any linear and continuous operator $U : \mathcal{D}_d \to X$ is an X-valued distribution on $\mathbb{R}^d$. The set of all X-valued distributions on $\mathbb{R}^d$ will be denoted by $\mathcal{D}'_d(X)$.

Analogously, we can introduce the spaces $\mathcal{S}'_d(X)$ of X-valued tempered distributions, $\mathcal{E}'_d(X)$ of X-valued "compactly" supported distributions and $\mathcal{B}'_d(X)$ of X-valued bounded distributions.

Remark. $\mathcal{D}'_d(X) = \mathcal{D}'_d\varepsilon X$, $\mathcal{S}'_d(X) = \mathcal{S}'_d\varepsilon X$, $\mathcal{E}'_d(X) = \mathcal{E}'_d\varepsilon X$, $\mathcal{B}'_d(X) = \mathcal{B}'_d\varepsilon X$, where by ε we have denoted the ε - product (see (Schwartz, 1957), chap. I, § 2).

Considering also X-valued test functions and the corresponding spaces the following inclusions hold with continuous embeddings:

$$\begin{array}{cccccccc} \mathcal{D}_d(X)\subset & \mathcal{S}_d(X)\subset & \mathcal{D}_{d,L^1}(X)\subset & \mathcal{D}_{d,L^r}(X)\subset & \dot{\mathcal{B}}_d(X)\subset & \mathcal{B}_d(X)\subset & \mathcal{O}_{d,M}(X)\subset & \mathcal{E}_d(X) \\ \cap & \cap & \cap & \cap & \cap & \cap & \cap & \cap \\ \mathcal{E}'_d(X)\subset & \mathcal{O}'_{d,c}(X)\subset & \mathcal{D}'_{d,L^1}(X)\subset & \mathcal{D}'_{d,L^r}(X)\subset & \dot{\mathcal{B}}'_d(X)\subset & \mathcal{B}'_d(X)\subset & \mathcal{S}'_d(X)\subset & \mathcal{D}'_d(X), \end{array} \tag{3.1}$$

(see (Schwartz, 1950), (Popa, 2007)).

Analogously to the Lebesque type spaces L^r_d, $L^r_{d,loc}$, $L^r_{d,T}$ of complex valued functions, we associate in an obvious way the corresponding spaces $L^r_d(X)$, $L^r_{d,loc}(X)$, $L^r_{d,T}(X)$ $(1 \le r \le \infty)$ of X-valued functions.

Let us consider now $F \in L^1_{d,loc}(X)$. The operator U_F defined by

$$U_F(\varphi) := \int_{\mathbb{R}^d} \varphi(t)F(t)dt,\ \varphi \in \mathcal{D}_d \tag{3.2}$$

is clearly linear and continuous on $\mathcal{D}_d$, hence $U_F \in \mathcal{D}'_d(X)$.

Identifying F with U_F, the following continuous embeddings holds

$$L^r_d(X) \subseteq L^r_{d,loc}(X) \subseteq L^1_{d,loc}(X) \subseteq \mathcal{D}'_d(X). \tag{3.3}$$

For any $\varphi \in \mathcal{D}_d$ we recall the definition of the following operators on the spaces of X-valued distributions defined with the help of corresponding operators on the spaces of test functions:

- The translations $(\boldsymbol{L}_\tau U)(\varphi) := U(L_{-\tau}\varphi)$, $\tau \in \mathbb{R}^d$;
- Multiplications with functions $(\boldsymbol{M}_\psi U)(\varphi) := U(M_\psi\varphi)$, $\psi \in \mathcal{E}_d$;
- The derivation $(\boldsymbol{D}^\alpha U)(\varphi) := (-1)^{|\alpha|}U(D^\alpha\varphi)$, $\alpha \in \mathbb{Z}^d$, $|\alpha| = \alpha_1 + ... + \alpha_d$.

Now we try to extend to the vector valued case and $d > 1$ some results regarding the scalar periodic distributions treated in (Schwartz, 1950), (Zemanian, 1965), (Kecs, 1978).

Definition 3.2. A vector valued distribution $U \in \mathcal{D}'_d(X)$ is said to be T-periodic, where $T = (T_1, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, when $\boldsymbol{L}_T U = U$. T is called a period of U. The set of all periods of the distribution U is kT, $k \in \mathbb{Z}^d$. The "smallest" period is called the fundamental period of U (see (Zemanian, 1965) for $d = 1$)

By $\mathcal{D}'_T(\mathbb{R}^d, X)$, or $\mathcal{D}'_{d,T}(X)$, we shall denote the space of all such X-valued T-periodic distributions having the same period $T \in \mathbb{R}^d$, $T_i > 0$ (T - fixed).

In the next Theorem we extend the T-periodic transform from the space of compactly supported test functions, to the space of compactly supported X-valued distributions.

Theorem 3.1. *If $V \in \mathcal{E}'_d(X)$, then $\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}V$ defines an X-valued T-periodic distribution U. Conversely, any X-valued T-periodic distribution $U \in \mathcal{D}'_{d,T}(X)$ can be written as follows*

$$U = \sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}V, \tag{3.4}$$

where $V \in \mathcal{E}'_d(X)$.

Proof. Since X is a Banach space, $\mathcal{E}'_d(X)$ consists just of the compactly supported X-valued distributions (see (Schwartz, 1957), p. 62), hence the sum $\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}V$ contains a finite nonzero terms. Denoting by U this X-valued distribution, we successively have

$$\mathbf{L}_T U = L_T \sum_{n\in\mathbb{Z}^d} \mathbf{L}_{nT}V = \sum_{n\in\mathbb{Z}^d} \mathbf{L}_T(\mathbf{L}_{nT}V) = \sum_{k\in\mathbb{Z}^d} \mathbf{L}_{kT}V = U,$$

i.e. $U \in \mathcal{D}'_{d,T}(X)$.

Conversely, let us consider $U \in \mathcal{D}'_{d,T}(X)$ an X-valued T-periodic distribution and $\xi \in \mathcal{U}_{d,T}$. Now the X-valued distribution $V = \boldsymbol{M}_\xi U$ (which is obvious from $\mathcal{E}'_d(X)$) satisfies

$$\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}V = \sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}(\xi U) = U.$$

□

From Theorem 3.1 it results that the operator ϖ_T defined by $U = \varpi_T V$ given by (3.4) is an onto mapping from $\mathcal{E}'_d(X)$ onto $\mathcal{D}'_{d,T}(X)$. It will be called the *T-periodic transform* on X-valued distributions.

Remark. It is a simple matter to observe that an analog of (2.5) also holds:

$$\mathcal{E}'_d(X) = \mathcal{U}_{d,T}\mathcal{D}'_{d,T}(X). \tag{3.5}$$

Remark. When $V \in \mathcal{D}'_{d,L^r}(X)$, then is not difficult to see that $\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}V$ also makes sense, meaning that ϖ_T can be naturally extended to $\mathcal{D}'_{d,L^1}(X)$.

Remark. Regarding the mapping ϖ_T, from the successive equalities

$$(\varpi_T V)(\varphi) = \sum_{n\in\mathbb{Z}^d} (\boldsymbol{L}_{nT}V)(\varphi) = V(\sum_{n\in\mathbb{Z}^d} L_{-nT}\varphi) = V(\varpi_T\varphi),\ \varphi \in \mathcal{D}_d,$$

we see that ϖ_T is the restrictions to $\mathcal{E}'_d(X)$ of the "adjoint" operator $\varpi'_T \in \mathcal{B}(\mathcal{B}'_d(X), \mathcal{D}'_d(X))$.

This transformation enables us to identify, up to an isomorphism, the space of X-valued T-periodic distributions on $\mathbb{R}^d$ with the "dual" of $\mathcal{D}_{d,T}$, i.e. with the space $\mathcal{B}(\mathcal{D}_{d,T}, X)$ of all bounded linear operators from $\mathcal{D}_{d,T}$ to X. Indeed now we can prove

Theorem 3.2. *(a) For each $U \in \mathcal{D}'_{d,T}(X)$, the operator U_T defined by*

$$U_T(\theta) := (U, \xi\theta),\ \theta \in \mathcal{D}_{d,T}, \tag{3.6}$$

is correctly defined, being independent of the choice of $\xi \in \mathcal{U}_{d,T}$.
(b) $U_T \in \mathcal{B}(\mathcal{D}_{d,T}, X)$.

Proof. (a) For any $U \in \mathcal{D}'_d(X)$ we have that $\eta U \in \mathcal{E}'_d(X)$, $\eta \in \mathcal{U}_{d,T}$, where $\eta U(\varphi) = U(\eta\varphi)$, $\varphi \in \mathcal{D}_d$. Also, because

$$\sum_{n\in\mathbb{Z}^d} (\boldsymbol{L}_{nT}\eta U)(\varphi) = U(\sum_{n\in\mathbb{Z}^d} L_{nT}\eta)(\varphi) = U(\varphi),\ \varphi \in \mathcal{D}_d,$$

we have

$$U = \sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}\eta U.$$

For any ξ and $\eta \in \mathcal{U}_{d,T}$, assuming that U is T-periodic, we can write $U = \boldsymbol{L}_{nT}U$ and for any $\theta \in \mathcal{D}_{d,T}$ we have

$$(U, \xi\theta) = (\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}\eta U, \xi\theta) = \sum_{n\in\mathbb{Z}^d} (UL_{nT}\eta\xi, \theta) = \sum_{n\in\mathbb{Z}^d} (\boldsymbol{L}_{nT}\eta U, \xi\theta) = \sum_{n\in\mathbb{Z}^d} (\eta U, L_{-nT}\xi\theta)$$

$$= (U, \eta(\sum_{n\in\mathbb{Z}^d} L_{-nT}\xi)\theta) = (U, \eta\theta),$$

for any ξ, $\eta \in \mathcal{U}_{d,T}$ and $\theta \in \mathcal{D}_{d,T}$.

(b) We show that U_T from (3.6) is a linear and continuous operator between $\mathcal{D}_{d,T}$ and X, i.e. $U_T \in \mathcal{B}(\mathcal{D}_{d,T}, X)$.

For linearity, let us consider the functions θ_1, $\theta_2 \in \mathcal{D}_{d,T}$ and α, $\beta \in \mathbb{C}$. Then

$$U_T(\alpha\theta_1 + \beta\theta_2) = (U, \xi(\alpha\theta_1 + \beta\theta_2)) =$$

$$= \alpha(U, \xi\theta_1) + \beta(U, \xi\theta_2) = \alpha U_T(\theta_1) + \beta U_T(\theta_2).$$

For continuity of U_T we consider the sequence $\{\theta_k\}_{k=1}^{\infty}$ converging to 0 in $\mathcal{D}_{d,T}$. Because, in this case, $\xi\theta_k \to 0$, $(k \to \infty)$ in $\mathcal{D}_d$ it results

$$U_T(\theta_k) = (U, \xi\theta_k) \to 0, \ (k \to \infty).$$

□

In this way $\mathcal{D}'_{d,T}(X)$ is linearly continuously embedded in $\mathcal{B}(\mathcal{D}_{d,T}, X)$.

Before proving that $U \mapsto U_T$ is a toplinear isomorphism let us put in evidence the embedding of vector valued T-periodic summable functions in $\mathcal{D}'_{d,T}(X)$.

Proposition 3.1. *If U_F is a distribution corresponding to the locally integrable X-valued T-periodic function F, then $(U_F)_T$ from (3.2) will be expressed by the integral on a parallelepiped of the form*

$$[a, a+T] = [a_1, a_1 + T_1] \times [a_2, a_2 + T_2] \times ... \times [a_d, a_d + T_d], \ a \in \mathbb{R}^d.$$

Proof. Let $F \in L^1_T(\mathbb{R}^d, X) \subset L^1_{loc}(\mathbb{R}^d, X)$. Then for the distribution $U_F \in \mathcal{D}'(\mathbb{R}^d, X)$ from (3.2) and $\theta \in \mathcal{D}_{d,T}$, $\xi \in \mathcal{U}_{d,T}$, a, $T \in \mathbb{R}^d$, $T_i > 0$, we have

$$(U_F)_T(\theta) = (U_F, \xi\theta) = \int_{\mathbb{R}^d} F(t)\xi(t)\theta(t)dt = \sum_{n\in\mathbb{Z}^d} \int_{[a+nT, a+nT+T]} F(t)\xi(t)\theta(t)dt =$$

$$= \sum_{n\in\mathbb{Z}^d} \int_{[a,a+T]} F(t+nT)\xi(t+nT)\theta(t+nT)dt = \int_{[a,a+T]} F(t)\theta(t) \sum_{n\in\mathbb{Z}^d} \xi(t+nT)dt = \int_{[a,a+T]} F(t)\theta(t)dt,$$

because F and θ are T-periodic, and $\sum_{n\in\mathbb{R}^d} \xi(t+nT) = 1$. □

Remark. The map $L^1_T(X)(\subset L^1_{loc}(X)) \ni F \mapsto U_F \in \mathcal{D}'_{d,T}(X)$ being linear and injective, the space $L^1_T(X)$ is linear continuous embedded in $\mathcal{D}'_{d,T}(X)$ through $F \equiv U_F$, where (compare with (3.2) and (3.3))

$$(U_F)_T(\theta) = \int_{[0,T)} F(t)\theta(t)dt, \quad \theta \in \mathcal{D}_{d,T}. \tag{3.7}$$

Proposition 3.2. *The multiplication of a vector valued T-periodic distribution $U \in \mathcal{D}'_{d,T}(X)$ with a T-periodic test function $\psi \in \mathcal{D}_{d,T}$ is also a vector valued T-periodic distribution, i.e.*

$$\boldsymbol{M}_\psi U \in \mathcal{D}'_{d,T}(X).$$

Proof. We consider the vector valued periodic distribution $U \in \mathcal{D}'_{d,T}(X)$ and the periodic test function $\psi \in \mathcal{D}_{d,T}$.

We show that $\boldsymbol{M}_\psi U \in \mathcal{D}'_{d,T}(X)$. Because $(\boldsymbol{M}_\psi U)(\varphi) = U(\varphi\psi)$, $\varphi \in \mathcal{D}_d$ is easy to see that $\boldsymbol{M}_\psi U$ is linear and continuous as operator from $\mathcal{D}_d$ to X. It remains to show that $\boldsymbol{M}_\psi U$ is an X-valued periodic distribution of period T. Applying $\boldsymbol{L}_T U = U$ and $L_T\psi = \psi$, we have:

$$(\boldsymbol{L}_T\psi U)(\varphi) = (\psi U)(L_{-T}\varphi) = U(\psi L_{-T}\varphi) = U(L_{-T}(L_T\psi)\varphi) =$$

$$= \boldsymbol{L}_T U((L_T\psi)\varphi) = U(\varphi\psi) = (\psi U)(\varphi),\ \varphi \in \mathcal{D}_d.$$

□

Remark. $\mathcal{D}'_{d,T}(X) = \mathcal{D}'_{d,T}\varepsilon X.$

Indeed, $\mathcal{D}_{d,T}$ have the topology γ, i.e. $((\mathcal{D}_{d,T})'_c)'_c = \mathcal{D}_{d,T}$, where $(\mathcal{D}_{d,T})'_c$ is the dual of $\mathcal{D}_{d,T}$ endowed with the uniform convergence topology on the absolutely convex and compact sets from $\mathcal{D}_{d,T}$, and

$$(\mathcal{D}_{d,T}(X))'_c \approx \mathcal{L}_c(\mathcal{D}_{d,T}, X) \approx \mathcal{L}_\varepsilon(X'_c, (\mathcal{D}_{d,T})'_c) \approx (\mathcal{D}_{d,T})'_c\widehat{\otimes}_\varepsilon X$$

(compare with (Schwartz, 1953*a*), (Schwartz, 1953*b*) and (Schwartz, 1957))

Proposition 3.3. *The subspace $\mathcal{D}'_{d,T}(X)$ of $\mathcal{D}'_d(X)$ is invariant to the derivation operators $\boldsymbol{D}^\alpha$, $\alpha \in \mathbb{N}^d$.*

Proof. We successively have

$$U \in \mathcal{D}'_{d,T}(X) \Rightarrow \boldsymbol{L}_T U = U \Rightarrow$$

$$\Rightarrow (\boldsymbol{D}^\alpha U)(\varphi) = (-1)^{|\alpha|}U(D^\alpha\varphi) = (-1)^{|\alpha|}(\boldsymbol{L}_T U)(D^\alpha\varphi) = (-1)^{|\alpha|}U(L_{-T}D^\alpha\varphi)$$

and

$$(\boldsymbol{L}_T\boldsymbol{D}^\alpha U)(\varphi) = (\boldsymbol{D}^\alpha U)(L_{-T}\varphi) = (-1)^{|\alpha|}U(D^\alpha L_{-T}\varphi),$$

respectively.
Because $L_{-T}D^\alpha\varphi = D^\alpha L_{-T}\varphi$ it follows that $\boldsymbol{L}_T(\boldsymbol{D}^\alpha U) = \boldsymbol{D}^\alpha U$. □

Finally we shall prove that the map constructed in Theorem 3.2,

$$\mathcal{D}'_{d,T}(X) \ni U \mapsto U_T \in \mathcal{B}(\mathcal{D}_{d,T}, X) \tag{3.8}$$

is a toplinear isomorphism.

By applying the properties of the T-periodic transform ϖ_T on $\mathcal{D}_d$, because of (3.6), for any $\varphi \in \mathcal{D}_d$, we have $(U, \varphi) = U_T(\varpi_T \varphi) = (\boldsymbol{\varpi}'_T U_T)(\varphi)$, i.e.

$$U = \boldsymbol{\varpi}'_T U_T. \tag{3.9}$$

Therefore, for each $\lambda_1, \lambda_2 \in \mathbb{C}$ and every $\theta \in \mathcal{D}_{d,T}$, we have

$$(\lambda_1 U_1 + \lambda_2 U_2)_T(\theta) = (\lambda_1 U_1 + \lambda_2 U_2)(\xi\theta) =$$

$$= \lambda_1 U_1(\xi\theta) + \lambda_2 U_2(\xi\theta) = \left(\lambda_1 (U_1)_T + \lambda_2 (U_2)_T\right)(\theta).$$

The injectivity results from the successive implications

$$U_T = 0 \Rightarrow \boldsymbol{\varpi}'_T U_T = 0 \Rightarrow U = 0.$$

For continuity we have that

$$U_n \overset{\mathcal{D}'_{d,T}}{\longrightarrow} 0 \Rightarrow U_n(\varphi) \longrightarrow 0,\ \varphi \in \mathcal{D}_d \Rightarrow \boldsymbol{\varpi}'_T (U_n)_T(\theta_\varphi) \longrightarrow 0 \Rightarrow$$

$$(U_n)_T(\varpi_T \varphi) \longrightarrow 0,\ \varphi \in \mathcal{D}_d \Rightarrow (U_n)_T(\theta) \longrightarrow 0,\ \theta \in \mathcal{D}_{d,T} \Rightarrow (U_n)_T \overset{\mathcal{B}(\mathcal{D}_{d,T}, X)}{\longrightarrow} 0.$$

Let us consider an element V from $\mathcal{B}(\mathcal{D}_{d,T}, X)$ and define U by $U(\varphi) = V(\varpi_T \varphi)$, $\varphi \in \mathcal{D}_d$. So U is an X-valued T-periodic distribution from $\mathcal{D}'_d(X)$, i.e. $U \in \mathcal{D}'_{d,T}(X)$. Indeed U satisfies $\boldsymbol{L}_T U = U$, because, from

$$\varpi_T(L_{-T}\varphi) = \varpi_T(\varphi),\ \varphi \in \mathcal{D}_d,$$

we have:

$$(\boldsymbol{L}_T U)(\varphi) = U(L_{-T}\varphi) = V(\varpi_T \varphi) = U(\varphi),\ \varphi \in \mathcal{D}_d.$$

Hence we have constructed just the inverse of (3.8), which is easy to see that it is also continuous. Thus we obtain

Theorem 3.3. *The mapping (3.8) is a toplinear isomorphism from $\mathcal{D}'_{d,T}(X)$ onto $\mathcal{B}(\mathcal{D}_{d,T}, X)$.*

Proof. It only remains to prove that $\{(U_k)_T\}_{k=1}^{\infty}$ converges in $\mathcal{B}(\mathcal{D}_{d,T}, X)$ to zero then $\{(U_k)\}_{k=1}^{\infty}$ converges in $\mathcal{D}'_d(X)$ to zero. Indeed, for ξ in $\mathcal{U}_{d,T}$ and θ in $\mathcal{D}_{d,T}$, we have

$$(U_k)_T(\theta) = (U_k, \xi\theta) \to 0,\ \ (k \to \infty),$$

which means $U_k \to 0$ $(k \to \infty)$ in $\mathcal{D}'_d(X)$. □

References

Gaşpar, D. and P. Gaşpar (2009). *Analiză funcţională, Ed. a 2-a*. Ed. de Vest,Timişoara.

Kecs, W. (1978). *Produsul de convoluţie şi aplicaţii*. Editura Academiei RSR, Bucureşti.

Popa, L. (2007). On vector valued distributions. In: *Section Mathematics and Computer Science of Proceedings of the International Symposium "Research and Education in an Innovation era", Arad Romania, 16 - 18 Nov, 2006*. Ed. Mirton, Timişoara, Romania. pp. 112–117.

Schwartz, L. (1950). *Théorie des distributions vol. I, Actualits Scientifiques et Industrielles, no. 1091=Publ. Inst. Math. Univ. Strasbourg 9, Paris, Hermann, 1950, 148 pp.; vol. II, Actualits Scientifiques et Industrielles no. 1122=Publ. Inst. Math. Univ. Strasbourg 10, Paris, Hermann, 1951. 169 pp.*

Schwartz, L. (1953*a*). Distributions à valeurs vectorielles, tome 1 (1953-1954) no. 20. In: *Séminaire Schwartz* (Faculté des sciences de Paris, Ed.). Secrétariat mathématique, Paris.

Schwartz, L. (1953*b*). Sur certains espaces de functions diffèrentiables à valeurs vectorielles, tome 1 (1953-1954) no. 10. In: *Séminaire Schwartz* (Faculté des sciences de Paris, Ed.). Secrétariat mathématique, Paris.

Schwartz, L. (1957). Théorie des distributions à valeurs vectorielles I.. *Ann. Inst. Fourier* **7**, 1–141.

Zemanian, A. H. (1965). *Distributions theory and transform analysis*. New York - Saint Louis -San Francisco - Toronto - London - Sydney, Mc Graw-Hill Book Co., Inc.