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The Reduced Differential Transform Method for the Exact Solutions of Advection, Burgers and Coupled Burgers Equations

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Abstract

Reduced differential transform method (RDTM) is employed to obtain the solution of simple homogeneous advection, Burgers and coupled Burgers equations exactly. The RDTM produces a solution with few and easy computation. The method is simple, accurate and efficient.

Keywords: Reduced differential transform method, advection equation, Burgers and coupled Burgers equations.
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1. Introduction

The concept of differential transform method has been introduced to solve linear and non linear initial value problems in electric circuit analysis, It was first introduced by (Zhou, 1986). Burgers equation generally appears in fluid mechanics. This equation incorporates both convection and diffision in fluid dynamics, and is used to describe the structure of shock waves. Coupled Burgers equation is a simple model of sedimentation or evalution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity. Reseachers have used other methods such as tanh method, HAM, VIM in (Hassan, 2009), (Alomari *et al.*, 2008) and (Abdou & Soliman, 2005) respectively. In this letter, RDTM is used to obtain the exact solution of simple homogeneous advection equation, Burgers equation and coupled Burgers equation.

2. Analysis of the method

The basic definitions of reduced differential transform method are introduced as follows:

Definition 2.1. If the function $u(x,t)$ is analytic and differentiated continously with respect to time t and space X in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0} \tag{2.1}$$

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where the t-dimensional spectrum function $U_k(x)$ is the transformed function.

Definition 2.2. The differential inverse transform of $U_k(x)$ is defined as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{2.2}$$

The fundamental mathematical operations performed by RDTM as given by (Keskin & c, 2010*a*) and (Keskin & c, 2010*b*) are provided in Table1:

Table 1
The fundamental mathematical operations performed by RDTM.

Functional form	Transformed form
$u(x,t)$	$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k} u(x,t)]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k = \alpha U_k(x)\ \alpha$ is a constant
$w(x,t) = x^m t^n$	$W_k = x^m \delta(k-n), \delta(k) = \begin{cases} 1, & k = 0 \\ 0 & k \neq 0 \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r U_{k-r}(x) = \sum_{r=0}^{k} U_r V k - r(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}(x,t)$	$W_k(x) = (k+1)...(k+r)U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
$w(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$	$W_k(x) = \frac{\partial^2}{\partial x^2} U_k(x)$

3. Applications

Example1: Consider the homogeneous advection equation given by (Alomari *et al.*, 2008) as,

$$u_t + uu_x = 0, \quad u(x,0) = -x. \tag{3.1}$$

Here $u_t = -uu_x$. Now taking the reduced differential transform of 3.1 we have

$$(k+1)U_{k+1} = -\sum_{r=0}^{k} U_r \frac{\partial}{\partial x} U_{k-r}, \tag{3.2}$$

with $U_0(x) = -x$ we can then obtain $U_k(x)$ values successively as $U_1(x) = U_2(x) = U_3(x) = ... = U_k(x) = -x$.

Using the differential inverse transform 2.2 we have:

$$u(x,t) = -x \sum_{n=0}^{\infty} t^n \tag{3.3}$$

equation 3.3 is a taylor series that converges to

$$u(x,t) = \frac{x}{t-1} \tag{3.4}$$

under $|t| < 1$ which is the exact solution.

Example2: Consider the one dimensional Burgers equation given by (Alomari *et al.*, 2008), that has the form

$$u_t + uu_x - \nu u_{xx} = 0 \tag{3.5}$$

subject to the boundary condition

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)e^{\gamma}}{1 + e^{\gamma}}, \tag{3.6}$$

where $\gamma = \alpha(\frac{x}{\nu})$ and the parameters α, β, ν are arbitrary constants.
Taking the reduced differential transform of 3.5 we have

$$(k+1)U_{k+1}(x) = -\sum_{r=0}^{k} U_r(x)\frac{\partial}{\partial x}U_{k-r}(x) + \nu\frac{\partial^2}{\partial x^2}U_k(x) \tag{3.7}$$

$U_0 = \frac{\alpha+\beta+(\beta-\alpha)e^{\gamma}}{1+e^{\gamma}}$ we then obtain $U_k(x)$ values successively as
$U_1 = -U_0\frac{\partial}{\partial x}U_0 + \nu\frac{\partial^2}{\partial x^2}U_0(x)$

$$= \frac{1\alpha^2\beta e^{\gamma}}{\nu(1+e^{\gamma})^2}$$

$U_2 = -\frac{1}{2}(U_0(x)\frac{\partial}{\partial x}U_1(x) + U_1(x)\frac{\partial}{\partial x}U_0(x) + \nu\frac{\partial^2}{\partial x^2}U_1(x))$

$$= \frac{\alpha^3\beta^2(e^{\gamma}-1)e^{\gamma}}{\nu^2(1+e^{\gamma})^3}$$

$$U_3 = \frac{\alpha^4\beta^3 e^{\gamma}(1-4e^{\gamma}-e^{2\gamma})}{3\nu^3(1+e^{\gamma})^4}$$

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Using the differential inverse transform 2.2 we have:

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)e^{\gamma}}{1+e^{\gamma}} + \frac{1\alpha^2\beta e^{\gamma}}{\nu(1+e^{\gamma})^2}t + \frac{\alpha^3\beta^2(e^{\gamma}-1)e^{\gamma}}{\nu^2(1+e^{\gamma})^3}t^2 + \frac{\alpha^4\beta^3 e^{\gamma}(1-4e^{\gamma}-e^{2\gamma})}{3\nu^3(1+e^{\gamma})^4}t^3 + ... \tag{3.8}$$

which in its closed form gives

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)e^{\frac{\alpha}{\nu}(x-\beta t)}}{1+e^{\frac{\alpha}{\nu}(x-\beta t)}}. \tag{3.9}$$

Example3: Consider the following system of coupled Burgers equation given in (Alomari *et al.*, 2008) as

$$u_t - uu_{xx} - 2uu_x + (uv)_x = 0, \tag{3.10}$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \tag{3.11}$$

subject to the initial conditions

$$u(x,0) = sin(x), \quad v(x,0) = sin(x). \tag{3.12}$$

Taking the reduced differential differential transform of 3.10 and 3.11, we have

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + 2\sum_{r=0}^{k} U_r(x)\frac{\partial}{\partial x}U_{k-r} - \frac{\partial}{\partial x}\sum_{r=0}^{k} U_r V_{k-r}, \tag{3.13}$$

$$(k+1)V_{k+1}(x) = \frac{\partial^2}{\partial x^2}V_k(x) + 2\sum_{r=0}^{k} V_r(x)\frac{\partial}{\partial x}V_{k-r} - \frac{\partial}{\partial x}\sum_{r=0}^{k} U_r V_{k-r}. \tag{3.14}$$

Using equation 3.13 and 3.14 with
$U_0 = V_0 = sin(x)$
we recursively obtain
$U_1 = V_1 = -sin(x)$,
$U_2 = V_2 = \frac{1}{2}sin(x)$,
$U_3 = V_3 = -\frac{1}{6}sin(x)$,
.
.
.

Using the differential inverse transform 2.2 we have

$$u(x,t) = sin(x) - sin(x)t + \frac{1}{2!}sin(x)t^2 - \frac{1}{3!}sin(x)t^3 + ..., \tag{3.15}$$

$$v(x,t) = sin(x) - sin(x)t + \frac{1}{2!}sin(x)t^2 - \frac{1}{3!}sin(x)t^3 + ..., \tag{3.16}$$

which is

$$u(x,t) = sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...), \tag{3.17}$$

$$v(x,t) = sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...), \tag{3.18}$$

and finally in its closed form gives

$$u(x,t) = e^{-t}sin(x) \tag{3.19}$$

and

$$v(x,t) = e^{-t}sin(x), \tag{3.20}$$

which are the exact solution of the coupled Burgers equation.

4. Conclusion

Exact solutions of simple homogeneous advection equation, Burgers equation and Coupled Burgers equation were presented via the reduced differential transform method (RDTM). The method is applied in a direct way without any linearization or descretization. The computational size of this method is small compared with those of DTM, HAM, HPM and Adomian decomposition method. Hence, this method is a powerful and an efficient technique in finding the exact solutions for wide classes of problems, also the speed of the convergence is very fast.

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