Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 15–22

$\mathcal{I}$-limit Points in Random 2-normed Spaces

Mehmet Gürdal[a,*], Mualla Birgül Huban[a]

[a]*Suleyman Demirel University, Department of Mathematics, East Campus, 32260, Isparta, Turkey.*

Abstract

In this article we introduce the notion $\mathcal{I}$-cluster points, and investigate the relation between $\mathcal{I}$-cluster points and limit points of sequences in the topology induced by random 2-normed spaces and prove some important results.

Keywords: t-norm, random 2-normed space, ideal convergence, $\mathcal{I}$-cluster points, F-topology.
2000 MSC: 40A35, 46A70, 54E70.

1. Introduction

Probabilistic metric (PM) spaces were first introduced by Menger (Menger, 1942) as a generalization of ordinary metric spaces and further studied by Schweizer and Sklar (Schweizer & Sklar, 1960, 1983). The idea of Menger was to use distribution function instead of non-negative real numbers as values of the metric. In this theory, the notion of distance has a probablistic nature. Namely, the distance between two points x and y is represented by a distribution function F_{xy}; and for $t > 0$, the value $F_{xy}(t)$ is interpreted as the probability that the distance from x to y is less than t. After that it was developed by many authors. Using this concept, Śerstnev (Śerstnev, 1962) introduced the concept of probabilistic normed space. It provides an important method of generalizing the deterministic results of linear normed spaces. It has also very useful applications in various fields, e.g., continuity properties (Alsina *et al.*, 1997), topological spaces (Frank, 1971), linear operators (Golet, 2005), study of boundedness (Guillén *et al.*, 1999), convergence of random variables (Guillén & Sempi, 2003), statistical and ideal convergence of probabilistic normed space or 2-normed space (Karakus, 2007), (Mohiuddine & Savaş, 2012), (Mursaleen, 2010), (Mursaleen & Mohiuddine, 2010), (Mursaleen & Mohiuddine, 2012), (Mursaleen & Alotaibi, 2011), (Rahmat & Harikrishnan, 2009), (Tripathy *et al.*, 2012) etc.

The concept of 2-normed spaces was initially introduced by Gähler (Gähler, 1963), (Gähler, 1964) in the 1960's. Since then, many researchers have studied these subjects and obtained various results (Gunawan & Mashadi, 2001), (Gürdal & Pehlivan, 2004), (Gürdal, 2006), (Gürdal & Açik, 2008), (Gürdal *et al.*, 2009), (Savaş, 2011), (Siddiqi, 1980), (Şahiner *et al.*, 2007).

P. Kostyrko et al (cf. (Kostyrko *et al.*, 2000); a similar concept was invented in (Katětov, 1968)) introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such

*Corresponding author
Email addresses: gurdalmehmet@sdu.edu.tr (Mehmet Gürdal), btarhan03@yahoo.com (Mualla Birgül Huban)

convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. The notion of statistical convergence of sequences of real numbers was introduced by H. Fast in (Fast, 1951) and H. Steinhaus in (Steinhaus, 1951).

There are many pioneering works in the theory of $\mathcal{I}$-convergence. The aim of this work is to introduce and investigate the relation between $\mathcal{I}$-cluster points and ordinary limit points of sequence in random 2-normed spaces.

2. Definitions and Notations

First we recall some of the basic concepts, which will be used in this paper.

Definition 2.1. ((Freedman & Sember, 1981), (Fast, 1951)) A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \chi_E(k)$ exists. A number sequence $(x_n)_{n\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0$. If $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to L we write st-lim $x_n = L$, which is necessarily unique.

Definition 2.2. ((Kelley, 1955), (Kostyrko *et al.*, 2000)) A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if **(i)** $\emptyset \in \mathcal{I}$; **(ii)** $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; **(iii)** $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ in Y is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, i.e., $\{x\} \in \mathcal{I}$ for each $x \in Y$.

Let $\mathcal{I} \subset P(Y)$ be a non-trivial ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \backslash A\}$ is a filter on Y, called the filter associated with the ideal $\mathcal{I}$.

Definition 2.3. ((Kostyrko *et al.*, 2000), (Kostyrko *et al.*, 2005)) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - L\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Definition 2.4. ((Gähler, 1963) (Gähler, 1964)) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies **(i)** $\|x, y\| = 0$ if and only if x and y are linearly dependent; **(ii)** $\|x, y\| = \|y, x\|$; **(iii)** $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$; **(iv)** $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Also, if x, y and z are linearly dependent, then $\|x, y + z\| = \|x, y\| + \|x, z\|$ or $\|x, y - z\| = \|x, y\| + \|x, z\|$. Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for every $y \in X$.

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar (Schweizer & Sklar, 1983).

Definition 2.5. Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0, 1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2.6. A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutive, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 2.7. Let X be a linear space of dimension greater than one, τ is a triangle, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:

(2.2.1) $F(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $F(x, y; t)$ denotes the value of $F(x, y)$ at $t \in \mathbb{R}$,

(2.2.2) $F(x, y; t) \neq H_0(t)$ if x and y are linearly independent,

(2.2.3) $F(x, y; t) = F(y, x; t)$ for all $x, y \in X$,

(2.2.4) $F(\alpha x, y; t) = F(x, y; \frac{t}{|\alpha|})$ for every $t > 0, \alpha \neq 0$ and $x, y \in X$,

(2.2.5) $F(x + y, z; t) \geq \tau(F(x, z; t), F(y, z; t))$ whenever $x, y, z \in X$ and $t > 0$,

If (2.2.5) is replaced by

$(2.2.5)'$ $F(x + y, z; t_1 + t_2) \geq F(x, z; t_1) * F(y, z; t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_+$;

then $(X, F, *)$ is called a random 2-normed space (for short, RTN space).

Remark. Note that every 2-norm space $(X, \|., .\|)$ can be made a random 2-normed space in a natural way, by setting

(i) $F(x, y; t) = H_0(t - \|x, y\|)$, for every $x, y \in X, t > 0$ and $a * b = \min\{a, b\}, a, b \in S$;

(ii) $F(x, y; t) = \frac{t}{t + \|x, y\|}$ for every $x, y \in X, t > 0$ and $a * b = ab$ for $a, b \in S$.

Let $(X, F, *)$ be an RTN space. Since $*$ is a continuous t-norm, the system of (ε, λ)-neighborhoods of θ (the null vector in X)

$$\{\mathcal{N}_\theta(\varepsilon, \lambda) : \varepsilon > 0,\ \lambda \in (0, 1)\},$$

where

$$\mathcal{N}_\theta(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\}.$$

determines a first countable Hausdorff topology on X, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice versa.

A sequence $x = (x_n)$ in X is said to be F-convergence to $L \in X$ if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and for each nonzero $z \in X$ there exists a positive integer N such that

$$x_n, z - L \in \mathcal{N}_\theta(\varepsilon, \lambda) \text{ for each } n \geq N$$

or equivalently,

$$x_n, z \in \mathcal{N}_L(\varepsilon, \lambda) \text{ for each } n \geq N.$$

In this case we write $F\text{-}\lim x_n, z = L$.

3. The Relation Between $\mathcal{I}$-cluster Points and Ordinary Limit Points in Random 2-Normed Spaces

It is known (see (Fridy, 1993)) that statistical cluster Γ_x and statistical limit points set Λ_x of a given sequence (x_n) are not altered by changing the values of a subsequence the index set of which has density zero. Moreover, there is a strong connection between statistical cluster points and ordinary limit points of a given sequence. We will prove that these facts are satisfied for $\mathcal{I}$-cluster and $\mathcal{I}$-limit point sets of a given sequences in the topology induced by random 2-normed spaces

Definition 3.1. Let $(X, F, *)$ be an RTN space, $\mathcal{I}$ be an admissible ideal and $x = (x_n) \in X$.

(i) An element $L \in X$ is said to be an $\mathcal{I}$-limit point of the sequence x with respect to the random 2-norm F (or $\mathcal{I}_F^2(x)$-limit point) if there is a set $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $F\text{-}\lim_{k\to\infty} x_{n_k}, z = L$ for each nonzero z in X. The set of all $\mathcal{I}_F^2$-limit points of x is denoted by $\mathcal{I}\left(\Lambda_F^2(x)\right)$.

(ii) An element $L \in X$ is said to be an $\mathcal{I}$-cluster point of x with respect to the random 2-norm F (or $\mathcal{I}_F^2$-cluster point) if for each $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero z in X

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}.$$

The set of all $\mathcal{I}_F^2$-cluster points of x is denoted by $\mathcal{I}\left(\Gamma_F^2(x)\right)$.

Proposition 3.1. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal. Then for each sequence $x = (x_n)_{n\in\mathbb{N}}$ of X we have $\mathcal{I}\left(\Lambda_F^2(x)\right) \subset \mathcal{I}\left(\Gamma_F^2(x)\right)$ and the set $\mathcal{I}\left(\Gamma_F^2(x)\right)$ is a closed set.*

Proof. Let $L \in \mathcal{I}\left(\wedge_F^2(x)\right)$. Then there exists a set $M = \{n_1 < n_2 < ...\} \notin \mathcal{I}$ such that

$$F\text{-}\lim_{k\to\infty} x_{n_k}, z = L \tag{3.1}$$

for each nonzero z in X. According to 3.1, for each $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero z in X there exists a positive integer k_0 such that for $k > k_0$ we have $x_{n_k}, z \in \mathcal{N}_L(\varepsilon, \lambda)$. Hence

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \supset M \backslash \{n_1, ..., n_{k_0}\}$$

and so

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I},$$

which means that $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$.

Let $y \in \overline{\mathcal{I}\left(\Gamma_F^2\right)}$. Take $\varepsilon > 0$ and $\lambda \in (0, 1)$. There exists $L \in \mathcal{I}\left(\Gamma_F^2(x)\right) \cap \mathcal{N}_\theta(y, \varepsilon, \lambda)$. Choose $\eta > 0$ such that $\mathcal{N}_\theta(L, \eta, \lambda) \subset \mathcal{N}_\theta(y, \varepsilon, \lambda)$. We obviously have

$$\{n \in \mathbb{N} : y - x_n, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \supset \{n \in \mathbb{N} : L - x_n, z \in \mathcal{N}_\theta(\eta, \lambda)\}.$$

Hence $\{n \in \mathbb{N} : y - x_n, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \notin \mathcal{I}$ and $y \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. □

Definition 3.2. Let $(X, F, *)$ be an RTN space, $\mathcal{I}$ be an admissible ideal and $x = (x_n)_{n\in\mathbb{N}}$ be a sequence in X.

If $K = \{k_1 < k_2 < ...\} \in \mathcal{I}$, then the subsequence $x_K = (x_k)_{n\in\mathbb{N}}$ in X is called $\mathcal{I}_F^2$-thin subsequence of the sequence x in X.

If $M = \{m_1 < m_2 < ...\} \notin \mathcal{I}$, then the subsequence $x_M = (x_m)_{n\in\mathbb{N}}$ in X is called $\mathcal{I}_F^2$-nonthin subsequence of the sequence x in X.

It is clear that if L is a $\mathcal{I}_F^2$-limit point of $x \in X$, then there is a $\mathcal{I}_F^2$-nonthin subsequence x_M that convergent to L with respect to the random 2-norm F.

Definition 3.3. Let $(X, F, *)$ be an RTN space and $x = (x_n)_{n\in\mathbb{N}} \in X$. An element $L \in X$ is said to be limit point of the sequence $x = (x_n)$ with respect to the random 2-norm F if there is subsequence of the sequence x which converges to L with respect to the random 2-norm F. By $L_F^2(x)$, we denote the set of all limit points of the sequence $x = (x_n)$ with respect to the random 2-norm F.

It is obvious $\mathcal{I}\left(\Lambda_F^2(x)\right) \subseteq L_F^2(x)$, $\mathcal{I}\left(\Gamma_F^2(x)\right) \subseteq L_F^2(x)$: Take $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$, then $\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}$ for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X. If $L \notin L_F^2(x)$, then there is $\varepsilon' > 0$ such that $\mathcal{N}_L(\varepsilon', \lambda)$ contains only a finite number of elements of x in X. Then $\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon', \lambda)\} \in \mathcal{I}$, but it contradicts to $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. Hence $x \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. Thus $x \in L_F^2(x)$, and so $\mathcal{I}\left(\Gamma_F^2(x)\right) \subseteq L_F^2(x)$.

Lemma 3.1. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal. For a sequence $x = (x_n) \in X$, if x is $\mathcal{I}_F$-convergent with respect to the random 2-norm F, then $\mathcal{I}\left(\Lambda_F^2(x)\right)$ and $\mathcal{I}\left(\Gamma_F^2(x)\right)$ are both equal to the singleton set $\{\mathcal{I}_F\text{-}\lim x_n, z\}$ for each nonzero z in X.*

Proof. Let $\mathcal{I}_F\text{-}\lim_n x_n, z = L$. Show that $L \in \mathcal{I}\left(\Lambda_F^2(x)\right)$. By definition of $\mathcal{I}_F$-convergence we have

$$A(\varepsilon, \lambda) = \{n \in \mathbb{N} : x_n, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{I}$$

for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Since $\mathcal{I}$ is an admissible ideal we can choose the set $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ such that $n_k \notin A\left(\frac{1}{k}, \lambda\right)$ and $x_{n_k}, z \in \mathcal{N}_L\left(\frac{1}{k}, \lambda\right)$ for all $k \in \mathbb{N}$. That is $F\text{-}\lim_{k\to\infty} x_{n_k}, z = L$. Suppose $M \in \mathcal{I}$. Since $M \subset \{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(1, \lambda)\}$,

$$(\mathbb{N}\backslash M) \cap \{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(1, \lambda)\} = \emptyset,$$

but $\mathbb{N}\backslash M \in \mathcal{F}(\mathcal{I})$ and

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(1, \lambda)\} \in \mathcal{F}(\mathcal{I}).$$

This contradiction gives $M \notin \mathcal{I}$. Hence we get $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ and $M \notin \mathcal{I}$ such that $F\text{-}\lim_{k\to\infty} x_{n_k}, z = L$, i.e., $L \in \mathcal{I}\left(\Lambda_F^2(x)\right)$. Since $\mathcal{I}\left(\Lambda_F^2(x)\right) \subset \mathcal{I}\left(\Gamma_F^2(x)\right)$, $\xi \in \mathcal{I}\left(\Gamma_F^2(x)\right)$.

Now we suppose there is $\eta \in \mathcal{I}\left(\Gamma_F^2(x)\right)$ such that $\eta \neq L$. It is clear that

$$A = \left\{n \in \mathbb{N} : x_n, z \notin \mathcal{N}_L\left(\frac{|\eta - L|}{2}, \lambda\right)\right\} \in \mathcal{I}$$

and

$$B = \left\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L\left(\frac{|\eta - L|}{2}, \lambda\right)\right\} \notin \mathcal{I}$$

for $\lambda \in (0,1)$ and each nonzero $z \in X$. We have $B \subset A \in \mathcal{I}$. This contradiction shows $\mathcal{I}\left(\Gamma_F^2(x)\right) = \{L\}$. Hence from inclusion $\mathcal{I}\left(\Lambda_F^2(x)\right) \subset \mathcal{I}\left(\Gamma_F^2(x)\right) = \{L\}$, we have $\mathcal{I}\left(\Lambda_F^2(x)\right) = \mathcal{I}\left(\Gamma_F^2(x)\right) = L$. The lemma is proved. □

Theorem 3.2. *Let $(X, F, *)$ be an RTN space, $\mathcal{I}$ be an admissible ideal and $x = (x_n)$, $y = (y_n)$ are sequences in X such that*

$$M = \{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}.$$

Then $\mathcal{I}\left(\Lambda_F^2(x)\right) = \mathcal{I}\left(\Lambda_F^2(y)\right)$ and $\mathcal{I}\left(\Gamma_F^2(x)\right) = \mathcal{I}\left(\Gamma_F^2(y)\right)$.

Proof. Let $M = \{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$. If $L \in \mathcal{I}\left(\Lambda_F^2(x)\right)$, then there is a set $K = \{n_1 < n_2 < ...\} \notin \mathcal{I}$ such that $F\text{-}\lim_k x_{n_k}, z = L$. Given $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $N \in \mathbb{N}$ such that $x_{n_k}, z \notin \mathcal{N}_L(\varepsilon, \lambda)$ for $k > N$ and nonzero $z \in X$. Since $K_1 = \{n \in \mathbb{N} : n \in K \wedge x_n \neq y_n\} \subset M \in \mathcal{I}$,

$$K_2 = \{n \in \mathbb{N} : n \in K \wedge x_n = y_n\} \notin \mathcal{I}.$$

Indeed, if $K_2 \in \mathcal{I}$, then $K = K_1 \cup K_2 \in \mathcal{I}$, but $K \notin \mathcal{I}$. Hence the sequence $y_{K_2} = (y_n)_{n \in K_2}$ is an $\mathcal{I}_F^2$-nonthin subsequence of $y = (y_n)_{n\in\mathbb{N}}$ and y_{K_2} convergent to L with respect to the random 2-norm F. This implies that $L \in \mathcal{I}\left(\Lambda_F^2(y)\right)$. Similarly we can show that $\mathcal{I}\left(\Lambda_F^2(y)\right) \subset \mathcal{I}\left(\Lambda_F^2(x)\right)$. Hence $\mathcal{I}\left(\Lambda_F^2(y)\right) = \mathcal{I}\left(\Lambda_F^2(x)\right)$. Now let $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. Then

$$B_1 = \{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}$$

for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ and

$$B_2 = \{n \in \mathbb{N} : n \in B_1 \wedge x_n = y_n\} \notin \mathcal{I}.$$

Therefore, $B_2 \subset \{n \in \mathbb{N} : y_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\}$. It shows that $\{n \in \mathbb{N} : y_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}$, i.e., $L \in \mathcal{I}\left(\Gamma_F^2(y)\right)$. The theorem is proved. □

The next theorem proves a strong connection between $\mathcal{I}_F^2$-cluster and limit points of a given sequence with respect to the random 2-norm F.

Definition 3.4. (Kostyrko *et al.*, 2000) An admissible ideal $\mathcal{I} \subset P(\mathbb{N})$ is said to satisfy the property (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets of $\mathcal{I}$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{I}$.

Theorem 3.3. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal with property (AP) and $x = (x_n)$ be a sequence in X. Then there is a sequence $y = (y_n) \in X$ such that $L_F^2(y) = \mathcal{I}\left(\Gamma_F^2(x)\right)$ and $\{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$.*

Proof. If $\mathcal{I}\left(\Gamma_F^2(x)\right) = L_F^2(x)$, then $y = x$ and this case is trivial. Let $\mathcal{I}\left(\Gamma_F^2(x)\right)$ be a proper subset of $L_F^2(x)$. Then $L_F^2(x) \setminus \mathcal{I}\left(\Gamma_F^2(x)\right) \neq \emptyset$ for each $L \in L_F^2(x) \setminus \mathcal{I}\left(\Gamma_F^2(x)\right)$. There is an $\mathcal{I}_F^2$-thin subsequence $\left(x_{j_k}\right)_{k\in\mathbb{N}}$ of x such that $\lim_k x_{j_k}, z = L$, i.e., given $\varepsilon > 0$, $\lambda \in (0,1)$ there exists a positive integer N such that $x_{j_k}, z \notin \mathcal{N}_L(\varepsilon, \lambda)$ for $k > N$ and nonzero $z \in X$. Hence there exists an $\mathcal{N}_L(\varepsilon, \lambda)$ such that $\{k \in \mathbb{N} : x_k, z \in \mathcal{N}_L = \mathcal{N}_L(\delta, \lambda)\} \in \mathcal{I}$ for each $\delta > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$.

It is obvious that the collection of all $\mathcal{N}_L$'s is an open cover of $L_F^2(x) \setminus \mathcal{I}\left(\Gamma_F^2(x)\right)$. So by Covering Theorem there is a countable and mutually disjoint subcover $\left\{\mathcal{N}_j\right\}_{j=1}^{\infty}$ such that each $\mathcal{N}_j$ contains an $\mathcal{I}_F^2$-thin subsequence of $(x_n) \in X$.

Now let

$$A_j = \left\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_j = \mathcal{N}_j(\delta, \lambda),\ j \in \mathbb{N}\right\}$$

for each $\delta > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. It is clear that $A_j \in \mathcal{I}$ $(j = 1, 2, ...)$ and $A_i \cap A_j = \emptyset$. Then by (AP) property of $\mathcal{I}$ there is a countable collection $\left\{B_j\right\}_{j=1}^{\infty}$ of subsets of $\mathbb{N}$ such that $B = \cup_{j=1}^{\infty} B_j \in \mathcal{I}$ and $A_j \diagdown B$ is a finite set for each $j \in \mathbb{N}$. Let $M = \mathbb{N} \setminus B = \{m_1 < m_2 < ...\} \subset \mathbb{N}$. Now the sequence $y = (y_k) \in X$ is defined by $y_k = x_{m_k}$ if $k \in B$ and $y_k = x_k$ if $k \in M$. Obviously, $\{k \in \mathbb{N} : x_k \neq y_k\} \subset B \in \mathcal{I}$, and so by Theorem 3.2, $\mathcal{I}\left(\Gamma_F^2(y)\right) = \mathcal{I}\left(\Gamma_F^2(x)\right)$. Since $A_j \setminus B$ is a finite set, the sequence $y_B = (y_k)_{k\in B}$ has no limit point with respect to the random 2-norm F that is not also an $\mathcal{I}_F^2$-limit point of y, i.e., $L_F^2(y) = \mathcal{I}\left(\Gamma_F^2(y)\right)$. Therefore, we have proved $L_F^2(y) = \mathcal{I}\left(\Gamma_F^2(x)\right)$. □

Acknowledgements

We thank to the referees for their very carefully reading of the paper and making useful remarks that improved the presentation of the paper.

This work is supported by Süleyman Demirel University with Project 2947-YL-11.

References

Alsina, C., B. Schweizer and A. Sklar (1997). Continuity properties of probabilistic norms. *Journal of Mathematical Analysis and Applications* **208**(2), 446–452.

Fast, H. (1951). Sur la convergence statistique. *Colloquium Mathematicum* **2**, 241–244.

Frank, M.J. (1971). Probabilistic topological spaces. *Journal of Mathematical Analysis and Applications* **34**(1), 67–81.

Freedman, A.R. and J.J. Sember (1981). Densities and summability. *Pacific Journal of Mathematics* **95**(2), 293–305.

Fridy, J.A. (1993). Statistical limit points. *Proceedings of the American Mathematical Society* **118**(4), 1187–1192.

Gähler, S. (1963). 2-metrische räume und ihre topologische struktur. *Mathematische Nachrichten* **26**(1-4), 115–148.

Gähler, S. (1964). Lineare 2-normietre räume. *Mathematische Nachrichten* **28**(1-2), 1–43.

Golet, I. (2005). On probabilistic 2-normed spaces. *Novi Sad Journal of Mathematics* **35**(1), 95–102.

Guillén, B.L. and C. Sempi (2003). Probabilistic norms and convergence of random variables. *Journal of Mathematical Analysis and Applications* **280**(1), 9–16.

Guillén, B.L., J.A.R. Lallena and C. Sempi (1999). A study of boundedness in probabilistic normed spaces. *Journal of Mathematical Analysis and Applications* **232**(1), 183–196.

Gunawan, H. and Mashadi (2001). On finite dimensional 2-normed spaces. *Soochow Journal of Mathematics* **27**(3), 321–329.

Gürdal, M. (2006). On ideal convergent sequences in 2-normed spaces. *Thai Journal of Mathematics* **4**(1), 85–91.

Gürdal, M., A. Şahiner and I. Açik (2009). Approximation theory in 2-banach spaces. *Nonlinear Analysis: Theory, Methods & Applications* **71**(5-6), 1654–1661.

Gürdal, M. and I. Açik (2008). On $\mathcal{I}$-cauchy sequences in 2-normed spaces. *Mathematical Inequalities and Applications* **11**(2), 349–354.

Gürdal, M. and S. Pehlivan (2004). The statistical convergence in 2-banach spaces. *Thai Journal of Mathematics* **2**(1), 107–113.

Karakus, S. (2007). Statistical convergence on probabilistic normed spaces. *Mathematical Communications* **12**(1), 11–23.

Katětov, M. (1968). Products of filters. *Commentationes Mathematicae Universitatis Carolinae* **9**(1), 173–189.

Kelley, J.L. (1955). *General Topology*. Springer-Verlag. New York.

Kostyrko, P., M. Macaj and T. Salat (2000). $\mathcal{I}$-convergence. *Real Analysis Exchange* **26**(2), 669–686.

Kostyrko, P., M. Macaj, T. Salat and M. Sleziak (2005). $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. *Mathematica Slovaca* **55**(4), 443–464.

Menger, K. (1942). Statistical metrics. *Proceedings of the National Academy of Sciences of the United States of America* **28**(12), 535–537.

Mohiuddine, S.A. and E. Savaş (2012). Lacunary statistically convergent double sequences in probabilistic normed spaces. *Annali dell'Universita di Ferrara*. doi:10.1007/s11565-012-0157-5.

Mursaleen, M. (2010). On statistical convergence in random 2-normed spaces. *Acta Scientiarum Mathematicarum (Szeged)* **76**(1-2), 101–109.

Mursaleen, M. and A. Alotaibi (2011). On $\mathcal{I}$-convergence in random 2-normed spaces. *Mathematica Slovaca* **61**(6), 933–940.

Mursaleen, M. and S.A. Mohiuddine (2010). On ideal convergence of double sequences in probabilistic normed spaces. *Mathematical Reports* **12**(62), 359–371.

Mursaleen, M. and S.A. Mohiuddine (2012). On ideal convergence in probabilistic normed spaces. *Mathematica Slovaca* **62**(1), 49–62.

Rahmat, M.R.S. and K.K. Harikrishnan (2009). On $\mathcal{I}$-convergence in the topology induced by probabilistic norms. *European Journal of Pure and Applied Mathematics* **2**(2), 195–212.

Şahiner, A., M. Gürdal, S. Saltan and H. Gunawan (2007). Ideal convergence in 2-normed spaces. *Taiwanese Journal of Mathematics* **11**(4), 1477–1484.

Savaş, E. (2011). A-sequence spaces in 2-normed space defined by ideal convergence and an orlicz function. *Abstract and Applied Analysis*. doi:10.1155/2011/741382, 9 pages.

Schweizer, B. and A. Sklar (1960). Statistical metric spaces. *Pacific Journal of Mathematics* **10**, 313–334.

Schweizer, B. and A. Sklar (1983). *Probabilistic metric spaces*. North Holland, New York–Amsterdam–Oxford.

Śerstnev, A.N. (1962). Random normed space: Questions of completeness. *Kazan Gos. Univ. Uchen. Zap.* **122**(4), 3–20.

Siddiqi, A.H. (1980). 2-normed spaces. *The Aligarh Bulletin of Mathematics* pp. 53–70.

Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotique. *Colloquium Mathematicum* **2**, 73–74.

Tripathy, B. C., M. Sen and S. Nath (2012). $\mathcal{I}$-convergence in probabilistic n-normed space. *Soft Computing- A Fusion of Foundations, Methodologies and Applications*. doi: 10.1007/s00500-011-0799-8.