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On λ-Zweier Convergent Sequence Spaces

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Abstract

In this paper we introduce a new concept of λ-Zweier convergence and λ-statistical Zweier convergence and give some relations between these two kinds of convergence.

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1. Preliminaries

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ and l_∞, c and c_0 for the sets of all bounded, convergent sequences and null sequences, respectively.

A sequence space X with linear topology is called a K -space if each of the maps $P_i : X \to \mathbb{C}$ defined by $P_i(x) = x_i$ is continuous for $i = 1, 2, \cdots$.

A Fréchet space is a complete linear metric space, or equivalently , a complete totally paranormed space. In otherwords a locally convex space is called a Fréchet space if it is metrizable paranormed space and the Fréchet space is complete.

K -space X is called an FK-space if X is complete linear metric space. In otherwords we say that X is an FK-space if X is Fréchet space with continuous coordinate projection, we mean if $x^{(n)} \to x$ $(n \to \infty)$ in the metric of X then $x_k^{(n)} \to x_k$ $(n \to \infty)$ for each $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$, the linear functional $P_k(x) = x_k$ is such that P_k is continuous on X, i.e. X is K-space. Note that ω is a locally convex FK space with its usual metric. A BK-space is a normed FK-space (Choudhry & Nanda, 1989).

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers and $x \in \omega$. We write

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n = 0, 1, 2, \cdots)$$

and

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$$A(x) = (A_n(x))_{n=0}^{\infty}.$$

For any subset X of ω, the set

$$X_A = \{x = (x_k) \in \omega : A(x) \in X\}$$

is called the matrix domain of A in X.

Let $\lambda = (\lambda_n)$ be a non decereasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Vallee - Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V,λ)-summable to a number l if $t_n(x) \to l$ as $n \to \infty$ (Leindler, 1965). We write

$$[V, \lambda]^0 = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0\},$$

$$[V, \lambda] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - le| = 0, \text{for some l} \in \mathbb{C}\},$$

$$[V, \lambda]^{\infty} = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty\}.$$

For the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Valle- Poussin method.In the special case when $\lambda_n = n$ for $n = 1, 2, 3, \cdots$ the sets $[V, \lambda]^0$, $[V, \lambda]$ and $[V, \lambda]^{\infty}$ reduce the sets w_0 , w and w_{∞} introduced and studied by Maddox (Maddox, 1986).

In (Sengönül, 2007), Sengönül introduced Z and Z_0 spaces as the set of all sequences such that £ -transforms of them are in the spaces c and c_0 , respectively , i.e.

$$Z = \{x = (x_k) \in \omega : \text{£} \in c\},$$

$$Z_0 = \{x = (x_k) \in \omega : \text{£} \in c_0\},$$

where £ $= (z_{nk})$, $(n, k = 0, 1, 2, \cdots)$ denotes by the matrix

$$z_{nk} = \begin{cases} \frac{1}{2}, & k \leq n \leq k + 1, \quad (n, k \in \mathbb{N}) \\ 0, & otherwise \end{cases}$$

This matrix is called Zweier matrix.

The concept of statistical convergence was first introduced by Fast (Fast, 1951) and further studied by Salat in (Salat, 1980), Fridy in (Fridy, 1985), Connor in (Connor, 1988), Kolk in (Kolk, 1996), (Kolk, 1993), M. K. Khan and C. Orhan in (Khan & Orhan, 2007), Fridy and Orhan in (Fridy & Orhan, 1997), (Fridy & Orhan, 1993) and many others. Let $\mathbb{N}$ be the set of natural numbers and E$\subset \mathbb{N}$. Then the natural density of E is denoted by

$$\delta(E) = \lim_{n \to \infty} n^{-1} |\{k \leq n : k \in E\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

The sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every $\epsilon > 0$, the set $\{k : |x_k - l| \geq \epsilon\}$ has natural density 0, and we write $l = \text{st} - \lim x$. We shall also write S to denote the set of all statistically convergent sequences.

2. Main Results

We introduce the sequence spaces $[V,\lambda]^0[Z], [V,\lambda][Z]$ and $[V,\lambda]^\infty[Z]$ as the set of all sequences such that Z-transforms of them are in the $[V,\lambda]^0, [V,\lambda]$ and $[V,\lambda]^\infty$ respectively i.e

$$[V,\lambda]^0[Z] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(x_k + x_{k-1})| = 0\},$$

$$[V,\lambda][Z] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(x_k + x_{k-1}) - l| = 0, \text{ for some l} \in \mathbb{C}\},$$

$$[V,\lambda]^\infty[Z] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(x_k + x_{k-1})| < \infty\},$$

The $\pounds = (z_{nk})_{n,k\geq 0}$ matrix is well known as a regular matrix (Boos, 2000).Define the sequence y which will be frequently used, as £ - transform of the sequence x i.e.,

$$y_k = \frac{1}{2}(x_k + x_{k-1}), \quad (k \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *The sets $[V,\lambda]^0[Z], [V,\lambda][Z]$ and $[V,\lambda]^\infty[Z]$ are the linear spaces with the coordinatewise addition and scalar multiplication with the norm*

$$||x||_{[V,\lambda]^0[Z]} = ||x||_{[V,\lambda][Z]} = ||x||_{[V,\lambda]^\infty[Z]} = ||\pounds x||_\lambda.$$

Proof. Suppose that x,y $\in [V,\lambda]^0[Z]$ and α,β are complex numbers. Then

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}[\alpha(x_k + x_{k-1}) + \beta(y_k + y_{k-1})]| \leq \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} (|\frac{1}{2}\alpha(x_k + x_{k-1})| + |\frac{1}{2}\beta(y_k + y_{k-1})|)$$

$$= \lim_n \frac{\alpha}{\lambda_n} \sum_{k\in I_n} (|\frac{1}{2}(x_k + x_{k-1})| + \lim_n \frac{\beta}{\lambda_n} \sum_{k\in I_n} (|\frac{1}{2}(x_k + x_{k-1})| = 0, \quad \text{as } r \to \infty$$

Furthermore , since for any subset X of ω , the set

$$X_A = \{x = (x_k) \in \omega : A(x) \in X\} \qquad \text{(is called matrix domian of A in X)},$$

holds and $[V,\lambda]^0, [V,\lambda]$ are BK-spaces with respect to the norm defined by

$$||x||_{[V,\lambda]} = \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} |x_k|$$

and the matrix $\pounds = (z_{nk})$ is normal, that is $z_{nk} \neq 0$, for $0 \leq k \leq n$ and $z_{nk} = 0$ for $k > n$ for all $n,k \in \mathbb{N}$ and also by Theorem 4.3.2 of Wilansky (Wilansky, 1984) gives the fact that the spaces $[V,\lambda]^0[Z]$ and $[V,\lambda][Z]$ are BK spaces. □

Theorem 2.2. *The sequence spaces $[V,\lambda]^0[Z], [V,\lambda][Z]$ and $[V,\lambda]^\infty[Z]$ are linearly isomorphic to the spaces $[V,\lambda]^0, [V,\lambda]$ and $[V,\lambda]^\infty$ respectively.*

Proof. We want to show the existance of the linear bijection between the spaces $[V, \lambda]^0[Z]$ and $[V, \lambda]^0$.Consider the transformation £ defined by (1), from $[V, \lambda]^0[Z]$ to $[V, \lambda]^0$ by

$$£ : [V, \lambda]^0[Z] \to [V, \lambda]^0$$

$$x \to £x = y, \quad y = (y_k), \quad y_k = \frac{1}{2}(x_k + x_{k-1}), \quad (k \in \mathbb{N}).$$

The linearity of £ is clear. Further it is trivial that $x = 0$ when $£x = 0$ and hence £ is injective. Let $y \in [V, \lambda]^0$ and define the sequence $x = (x_k)$ by

$$x_k = 2\sum_{i=0}^{k}(-1)^{k-i}y_i \qquad (n \in \mathbb{N}).$$

Then

$$\|x\|_{[V,\lambda]^0[Z]} = \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(x_k + x_{k-1})|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(2\sum_{k=0}^{i}(-1)^{k-i}y_i + 2\sum_{k=0}^{i}(-1)^{(k-1)-i}y_i)|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |y_i|.$$

This implies that $x \in [V, \lambda]^0$[Z]. Also

$$\|x\|_{[V,\lambda]^0[Z]} = \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(x_k + x_{k-1})|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} |\frac{1}{2}(2\sum_{k=0}^{i}(-1)^{i-k}y_k + 2\sum_{k=0}^{i}(-1)^{i-k-1}y_k)|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} |y_k| = \|y\|^0_{[V,\lambda]}.$$

Thus we have that $x \in [V, \lambda]^0$[Z] and consequently £ is surjective. Hence £ is linear bijection which therefore says us that the spaces $[V, \lambda]^0[Z]$ and $[V, \lambda]^0$ are linearly isomorphic. It is clear here that if the spaces $[V, \lambda]^0[Z]$ and $[V, \lambda]^0$ replaced by the spaces $[V, \lambda][Z]$ and $[V, \lambda]$ or $[V, \lambda]^\infty[Z]$and $[V, \lambda]^\infty$, respectively. Then

$$[V, \lambda][Z] \cong [V, \lambda]^\infty[Z] \quad \text{or} \quad [V, \lambda]^\infty[Z] \cong [V, \lambda]^\infty.$$

This completes the proof. □

A sequence $x = (x_k)$ is said to be λ-statistical Zweir convergent to a number l if for $\epsilon > 0$.

$$R_\lambda[Z] = \{\frac{1}{\lambda_n} \sum_{k\in I_n} |£M_\lambda(\epsilon)| = 0\},$$

where

$$£M_\lambda(\epsilon) = \{[n - \lambda_n + 1, n] : |\frac{1}{2}(x_k + x_{k-1}) - l \geq \epsilon|\}.$$

Let

$$[n-\lambda_n+1,n]^* = \{[n-\lambda_n+1,n] : |\frac{1}{2}(x_k+x_{k-1}) - l \geq \epsilon|\} = CM_\lambda(\epsilon)$$

and

$$[n-\lambda_n+1,n]^{**} = \{[n-\lambda_n+1,n] : |\frac{1}{2}(x_k+x_{k-1}) - l| < \epsilon\}.$$

Theorem 2.3. *If* $x_k \to l[V,\lambda][Z] \Longrightarrow x_k \to l(R_\lambda[Z])$.

Proof. Let $\epsilon > 0$ and $x_k \to l[V,\lambda][Z]$, then

$$\frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]}|\frac{1}{2}(x_k+x_{k-1})-l|$$

$$\geq \frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]^*}|\frac{1}{2}(x_k+x_{k-1})-l|$$

$$\geq \frac{1}{\lambda_n}|£M_\lambda(\epsilon)|.$$

This implies that $x_k \to l(R_\lambda[Z])$. □

Theorem 2.4. *If* $x \in [V,\lambda]^\infty[Z]$ *and* $x_k \to l[V,\lambda][Z] \Longrightarrow x_k \to l(R_\lambda[Z])$.

Proof. Suppose that $x \in [V,\lambda]^\infty[Z]$ and $x_k \to l[V,\lambda][Z]$. Since $sup_k|\frac{1}{2}(x_k + x_{k-1} - l)| < \infty$, there exists a constant $T > 0$ such that $|\frac{1}{2}(x_k + x_{k-1}) - l| < T$ for all k. Then we have, for every $\epsilon > 0$ that

$$\frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]}|\frac{1}{2}(x_k+x_{k-1})-l|$$

$$= \frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]^*}|\frac{1}{2}(x_k+x_{k-1})-l|$$

$$+\frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]^{**}}|\frac{1}{2}(x_k+x_{k-1})-l|$$

$$\leq \frac{T}{\lambda_n}|£M_\lambda(\epsilon)| + \epsilon,$$

taking limit as $\epsilon \to 0$. Thus $x_k \to l([V,\lambda]^\infty[Z])$. □

Theorem 2.5. *If* $x \in [V,\lambda]^\infty[Z]$ *then* $[V,\lambda][Z] = R_\lambda[Z]$.

Proof. Proof follows from Theorem 2.3 and Theorem 2.4. □

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