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On H–Dichotomy for Skew-Evolution Semiflows in Banach Spaces

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Abstract

The aim of this paper is to define and characterize a particular case of dichotomy for skew-evolution semiflows, called the H–dichotomy, as a useful tool in describing the behaviors for the solutions of evolution equations that describe phenomena from engineering or economics. The paper emphasizes also other asymptotic properties, as ω–growth and ω–decay, H–stability and H–instability, as well as the classic concept of exponential dichotomy.

Keywords: Evolution semiflow, evolution cocycle, skew-evolution semiflow, ω–growth, ω–decay, H–dichotomy.
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1. Preliminaries

The study of the behaviors of the solutions of evolution equations by means of associated operator families has allowed to obtain answers to some previously open problems by involving techniques of functional analysis and operator theory.

In the qualitative theory of evolution equations, the exponential dichotomy is one of the most important asymptotic properties, and in the last years it was treated from various perspectives.

The notion of exponential dichotomy for linear differential equations was introduced by O. Perron in 1930. The classic paper (Perron, 1930) of Perron served as a starting point for many works on the stability theory. The property of exponential dichotomy for linear differential equations has gained prominence since the appearance of two fundamental monographs due to J.L. Daleckii and M.G. Krein (see (Daleckii & Krein, 1974)) and J.L. Massera and J.J. Schäffer (see (Massera & Schäffer, 1966)).

Diverse and important concepts of dichotomy for linear skew-product semiflows were studied by C. Chicone and Y. Latushkin in (Chicone & Latushkin, 1999), S.N. Chow and H. Leiva in (Chow & Leiva, 1995), R.J. Sacker and G.R. Sell in (Sacker & Sell, 1994) as well as G.R. Sell and Y.You in (Sell & You, 2002).

The exponential stability and exponential instability for nonautonomous differential equations are studied by L. Barreira and C. Valls in (Barreira & Valls, 2008), and, in particular, for linear skew-product semiflows, by M. Megan, A.L. Sasu and B. Sasu in (Megan *et al.*, 2004).

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We have reconsidered the definitions of asymptotic properties by means of skew-evolution semiflow on a Banach space, introduced in (Megan & Stoica, 2008*a*), as an important tool in the stability theory and as a natural generalization for semigroups of operators, evolution operators and skew-product semiflows.

A skew-evolution semiflow depends on three variables t, t_0 and x, while the classic concept of cocycle depends only on t and x, thus justifying a further study of asymptotic behaviors for skew-evolution semiflows in a more general case, the nonuniform setting (relative to the third variable).

The notion of linear skew-evolution semiflow arises naturally when considering the linearization along an invariant manifold of a dynamical system generated by a nonlinear differential equation. The notion has proved itself of interest in the development of the stability theory, in a uniform as well as in a nonuniform setting, being already adopted by some researchers, as, for example, P. Viet Hai in (Viet Hai, 2010) and A.J.G. Bento and C.M. Silva in (Bento & Silva, 2012). Some results concerning the asymptotic properties for skew-evolution semiflows were published in (Megan & Stoica, 2008*b*), (Megan & Stoica, 2010), (Stoica & Megan, 2010) and (Stoica, 2010).

In what follows, we will consider a more general case for asymptotic behaviors that does not involve necessarily exponentials, but, instead, properly defined functions, which allows a non restrained approach. The aim of this paper is to define and characterize a more general case of dichotomy for skew-evolution semiflows, called the H–dichotomy, as a tool in the study the behaviors for the solutions of differential equations that describe processes from engineering, physics or economics, and to emphasize connections with the classic concept.

The motivation for the approach of the H–dichotomy is due to the fact that the characterizations in this case do not impose restrictions neither on the matrix A, which defines the system of differential equations, nor on the solutions, such as bounded growth or decay.

2. Notations. Definitions

Let us denote by X a metric space, by V a Banach space, by V^* its dual, and by $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. We consider the set $T = \left\{(t, t_0) \in \mathbb{R}^2_+,\ t \geq t_0\right\}$. Let I be the identity operator on V. We denote $Y = X \times V$ and $Y_x = \{x\} \times V$, where $x \in X$.

Let us define the sets

$$\mathcal{H} = \{H : \mathbb{R}_+ \to \mathbb{R}^*_+ |\ H \text{ continuous}\}$$

and

$$\mathcal{F} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ |\ \exists \mu \in \mathbb{R} \text{ such that } f(t) = e^{\mu t}, \forall t \geq 0\}$$

with the subsets $\mathcal{F}_+$ and $\mathcal{F}_-$ for positive, respectively negative values of μ.

We will denote by $\mathcal{K}$ the set of all continuous functions $h : \mathbb{R}_+ \to [1, \infty)$ such that, for all $H \in \mathcal{H}$, there exist a function $f \in \mathcal{F}$ and a constant $k > 0$ with the properties

$$h(s) \leq k f(t - s) H(t), \text{ and } h(2t) h(2s) \leq H(t + s),\ \forall t, s \geq 0.$$

Remark. As we can consider $h(t) = f(t) = e^{\nu t}$ and $H(t) = e^{2\nu t}$, $\nu > 0$, $t \geq 0$, it follows that the set $\mathcal{K}$ is not empty.

Definition 2.1. The mapping $C : T \times Y \to Y$ defined by the relation

$$C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x) v),$$

where $\varphi : T \times X \to X$ has the properties

(s_1) $\varphi(t,t,x) = x,\ \forall (t,x) \in \mathbb{R}_+ \times X$;
(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$
and $\Phi : T \times X \to \mathcal{B}(V)$ satisfy
(c_1) $\Phi(t,t,x) = I, \forall (t,x) \in \mathbb{R}_+ \times X$;
(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$,
is called *skew-evolution semiflow* on Y.

Remark. φ is called *evolution semiflow* and Φ *evolution cocycle* over the evolution semiflow φ.

Remark. If $C = (\varphi, \Phi)$ denotes a skew-evolution semiflow and $\alpha \in \mathbb{R}$ a parameter, then $C_\alpha = (\varphi, \Phi_\alpha)$, where

$$\Phi_\alpha : T \times X \to \mathcal{B}(V),\ \Phi_\alpha(t,t_0,x) = e^{\alpha(t-t_0)}\Phi(t,t_0,x), \tag{2.1}$$

is the *α-shifted skew-evolution semiflow*.

Example 2.1. Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+,\ \varphi(t,s,x) = t - s + x$ is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : T \to \mathcal{B}(V)$ we obtain that

$$\Phi_E : T \times \mathbb{R}_+ \to \mathcal{B}(V),\ \Phi_E(t,s,x) = E(t - s + x, x)$$

is an evolution cocycle on V over the evolution semiflow φ. Hence, an evolution operator on V is generating a skew-evolution semiflow on Y.

Example 2.2. Let $f : \mathbb{R}_+ \to (0,\infty)$ be a decreasing function. We denote by X the closure in C, the set of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, of the set $\{f_t,\ t \in \mathbb{R}_+\}$, where $f_t(\tau) = f(t+\tau), \forall \tau \in \mathbb{R}_+$. The mapping $\varphi_0 : \mathbb{R}_+ \times X \to X,\ \varphi_0(t,x) = x_t$, where $x_t(\tau) = x(t+\tau), \forall \tau \geq 0$, is a semiflow on X. Let $V = \mathcal{L}^2(0,1)$ be a separable Hilbert space with the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ defined by $e_0 = 1$ and $e_n(y) = \sqrt{2}\cos n\pi y$, where $y \in (0,1)$ and $n \in \mathbb{N}$. Let us consider the Cauchy problem

$$\begin{cases} \dot{v}(t) = A(\varphi_0(t,x))v(t), & t > 0 \\ v(0) = v_0. \end{cases} \tag{2.2}$$

where $A : X \to \mathcal{B}(V)$ is a continuous mapping. We consider a C_0-semigroup S given by the relation

$$S(t)v = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}\langle v, e_n\rangle e_n,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in V. The mapping

$$\Phi_0 : \mathbb{R}_+ \times X \to \mathcal{B}(V),\ \Phi_0(t,x)v = S\left(\int_0^t x(s)ds\right)v$$

is a cocycle over the semiflow φ_0 and $C_0 = (\varphi_0, \Phi_0)$ is a linear skew-product semiflow on Y. Also, for all $v_0 \in D(A)$, we have that $v(t) = \Phi_0(t,x)v_0$, $t \geq 0$, is a strong solution of system (2.2). Then the mapping

$$C : T \times Y \to Y,\ C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),$$

where

$$\varphi(t,s,x) = \varphi_0(t-s,x) \text{ and } \Phi(t,s,x) = \Phi_0(t-s,x),\ \forall (t,s,x) \in T \times X$$

is a skew-evolution semiflow on Y. Hence, the skew-evolution semiflows are generalizations of skew-product semiflows.

Other examples of skew-evolution semiflows are given in (Stoica & Megan, 2010).

Definition 2.2. $C = (\varphi, \Phi)$ has *ω–growth* if there exists a nondecreasing function $\omega : \mathbb{R}_+ \to [1, \infty)$ with the property $\lim_{t\to\infty} \omega(t) = \infty$ such that:

$$\|\Phi(t, t_0, x)v\| \leq \omega(t - s) \|\Phi(s, t_0, x)v\| ,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark. If C has ω–growth, then the $-\alpha$-shifted skew-evolution semiflow $C_{-\alpha} = (\varphi, \Phi_{-\alpha})$, $\alpha > 0$, has also ω–growth.

Remark. The property of ω–growth is equivalent with the property of exponential growth (see (Stoica, 2010)).

Definition 2.3. $C = (\varphi, \Phi)$ has *ω–decay* if there exists a nondecreasing function $\omega : \mathbb{R}_+ \to [1, \infty)$ with the property $\lim_{t\to\infty} \omega(t) = \infty$ such that:

$$\|\Phi(s, t_0, x)v\| \leq \omega(t - s) \|\Phi(t, t_0, x)v\| ,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark. If C has ω–decay, then the α-shifted skew-evolution semiflow $C_\alpha = (\varphi, \Phi_\alpha)$, $\alpha > 0$, has also ω–decay.

Remark. The property of ω–decay is equivalent with the property of exponential decay (see (Stoica, 2010)).

3. Concepts of dichotomy

Definition 3.1. A continuous mapping $P : Y \to Y$ defined by

$$P(x, v) = (x, P(x)v), \ \forall (x, v) \in Y, \tag{3.1}$$

where $P(x)$ is a linear projection on Y_x, is called *projector* on Y.

Definition 3.2. A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C = (\varphi, \Phi)$ if following relation holds:

$$P(\varphi(t, s, x))\Phi(t, s, x) = \Phi(t, s, x)P(x), \tag{3.2}$$

for all $(t, s) \in T$ and all $x \in X$.

Definition 3.3. Two projectors P and Q, defined by (3.1), are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

(t_1) each of the projectors is invariant on Y, as in (3.2);

(t_2) $\forall x \in X$, the projections $P(x)$ and $Q(x)$ verify the relations

$$P(x) + Q(x) = I \text{ and } P(x)Q(x) = 0.$$

Definition 3.4. $C = (\varphi, \Phi)$ is *exponentially dichotomic* relative to the compatible projectors P and Q if there exist $\alpha > 0$ and two nondecreasing mappings $N_1, N_2 : \mathbb{R}_+ \to [1, \infty)$ such that:

(ed_1)

$$e^{\alpha(t-s)} \|\Phi_P(t, t_0, x)v\| \le N_1(s) \|\Phi_P(s, t_0, x)v\| ; \tag{3.3}$$

(ed_2)

$$e^{\alpha(t-s)} \left\|\Phi_Q(s, t_0, x)v\right\| \le N_2(t) \left\|\Phi_Q(t, t_0, x)v\right\| , \tag{3.4}$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark. In Definition 3.4, relation (3.3) is the definition for the exponential stability and relation (3.4) for the exponential instability.

A more general concept of dichotomy is given by

Definition 3.5. $C = (\varphi, \Phi)$ is *H–dichotomic* relative to the compatible projectors P and Q if there exist two nondecreasing mappings $N_1, N_2 : \mathbb{R}_+ \to [1, \infty)$ such that:

(Hed_1)

$$H(t) \|\Phi_P(t, t_0, x)v\| \le N_1(t_0) \|P(x)v\| ; \tag{3.5}$$

(Hed_2)

$$H(s) \left\|\Phi_Q(s, t_0, x)v\right\| \le N_2(t) \left\|\Phi_Q(t, t_0, x)v\right\| , \tag{3.6}$$

for all $(t, s), (s, t_0) \in T$, all $(x, v) \in Y$ and all $H \in \mathcal{H}$.

Remark. For $H(t) = e^{\nu t},\ t \ge 0,\ \nu > 0$ the exponential dichotomy for skew-evolution semiflows is obtained.

Example 3.1. Let us consider the system of differential equations

$$\begin{cases} \dot{u} = (-2t \sin t - 3)u \\ \dot{w} = (t \cos t + 2)w \end{cases}$$

Let $X = \mathbb{R}_+$ and $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|,\ v = (v_1, v_2) \in \mathbb{R}^2$. Then the mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi(t, s, x) = x_{t-s}$$

is an evolution semiflow and the mapping $\Phi : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}^2)$ given by

$$\Phi(t, s, x)(v_1, v_2) = (U(t, s)v_1, W(t, s)v_2) =$$

$$= (e^{2t \cos t - 2s \cos s - 2 \sin t + 2 \sin s - 3t + 3s} v_1, e^{t \sin t - s \sin s + \cos t - \cos s + 2t - 2s} v_2),$$

where $U(t, s) = u(t)u^{-1}(s)$, $W(t, s) = w(t)w^{-1}(s)$, $(t, s) \in T$, and $u(t)$, $w(t)$, $t \in \mathbb{R}_+$, are the solutions of the given differential equations, is an evolution cocycle. We obtain that the skew-evolution semiflow $C = (\varphi, \Phi)$ is H–dichotomic relative to the compatible projectors P, $Q : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $P(x, v) = (v_1, 0)$ and $Q(x, v) = (0, v_2)$, where $v = (v_1, v_2)$, with

$$H(u) = e^u,\ N_1(s) = e^{5s+4} \text{ and } N_2(s) = e^{-t+2}.$$

In what follows, if P is a given projector, we will denote for every $(t, s, x) \in T \times X$

$$\Phi_P(t, s, x) = \Phi(t, s, x)P(x) \text{ and } C_P = (\varphi, \Phi_P).$$

We remark that

(*i*) $\Phi_P(t, t, x) = P(x)$, for all $(t, x) \in \mathbb{R}_+ \times X$;

(*ii*) $\Phi_P(t, s, \varphi(s, t_0, x))\Phi_P(s, t_0, x) = \Phi_P(t, t_0, x)$, for all $(t, s), (s, t_0) \in T$, $x \in X$.

The following result is an integral characterization for the concept of H-dichotomy.

Theorem 3.2. *Let P, $Q : \mathbb{R}_+ \to \mathcal{B}(V)$ be two projectors compatible with $C = (\varphi, \Phi)$ with the property that C_P has ω–growth and C_Q has ω–decay. Let $H \in \mathcal{H}$ and $h \in \mathcal{K}$. Then C is H–dichotomic if and only if there exist two mappings $M_1, M_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that:*

(*i*)

$$\int_{t_0}^{t} h(\tau) \left\|\Phi_P(t, \tau, x)^* v^*\right\| d\tau \le M_1(t_0)H(t) \left\|P(x)v^*\right\|, \tag{3.7}$$

(*ii*)

$$h(t_0) \int_0^t \frac{1}{H(\tau)} \left\|\Phi_Q(\tau, t_0, x)v\right\| d\tau \le M_2(t) \left\|\Phi_Q(t, t_0, x)v\right\|, \tag{3.8}$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$, $v^* \in V^*$ with $\|v^*\| \le 1$.

Proof. Necessity. (*i*) As the skew-evolution semiflow C is H-dichotomic, it implies that the relation (3.5) of Definition 3.5 holds. There exist a function $f \in \mathcal{F}_-$ and a constant $k > 0$ such that

$$h(s) \le kf(t - s)H(t), \ \forall (t, s) \in T.$$

Let us denote $f(t) = e^{-\nu t}$, $\nu > 0$. We obtain the inequalities

$$\|\Phi_P(t, t_0, x)v\| \le \frac{N_1(t)}{H(t)} \|\Phi_P(s, t_0, x)v\| \le k\frac{N_1(s)}{h(s)} e^{-\nu(t-s)} \|\Phi_P(s, t_0, x)v\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$. Further we have

$$\int_{t_0}^{t} h(\tau) \left\|\Phi_P(t, \tau, x)^* v^*\right\| d\tau \le kH(t) \int_{t_0}^{t} h(\tau)e^{-\nu(t-\tau)} \left\|\Phi_P(t, \tau, x)^* v^*\right\| d\tau \le M_1(t_0)H(t) \left\|P(x)v^*\right\|,$$

where we have denoted $M_1(t) = k\nu^{-1}N_1(t)$, $t \ge 0$.

(*ii*) We have that the relation (3.6) of Definition 3.5 takes place. There exist a function $f \in \mathcal{F}_-$ and a constant $k > 0$ such that

$$h(t_0) \le kf(s - t_0)H(s), \ \forall (s, t_0) \in T.$$

Let us consider $f(t) = e^{-\nu t}$, $\nu > 0$. We have

$$\left\|\Phi_Q(s, t_0, x)v\right\| \le \frac{N_2(t)}{H(s)} \left\|\Phi_Q(t, t_0, x)v\right\| \le k\frac{N_2(t)}{h(t_0)} e^{-\nu(s-t_0)} \left\|\Phi_Q(t, t_0, x)v\right\| \le$$

$$\le k\frac{N_2(t)}{h(t_0)} e^{\nu t} e^{-\nu(s-t_0)} e^{-\nu(2s-t_0)} \left\|\Phi_Q(t, t_0, x)v\right\| \le kN_2(t)e^{\nu t} e^{-\nu(t-s)} \left\|\Phi_Q(t, t_0, x)v\right\|,$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$. Further we have

$$h(t_0)\int_{t_0}^{t}\frac{1}{H(\tau)}\left\|\Phi_Q(t,t_0,x)v\right\|d\tau\le kM\int_{t_0}^{t}e^{-\nu(\tau-t_0)}e^{\delta(t-\tau)}\left\|\Phi_Q(t,t_0,x)v\right\|d\tau\le M_2(t)\left\|P(x)v\right\|,$$

where we have denoted $M_2(t)=\dfrac{kM}{\nu+\delta}e^{(\nu+\delta)t}$, $t\ge 0$, and where we have defined in Definition 2.3 the function $\omega(t)=Me^{\delta t}$, $M\ge 1$ and $\delta>0$.

Sufficiency. (i) We suppose that relation (3.7) takes place. Let us first consider the case $t\in[t_0,t_0+1)$. We have, as $0\le t-t_0<1$,

$$\left\|\Phi_P(t,t_0,x)v\right\|\le Me^{\alpha+\delta}e^{-\alpha(t-t_0)}\left\|P(x)v\right\|,$$

for all $(x,v)\in Y$, where we have considered in Definition 2.2 the function $\omega(t)=Me^{\delta t}$, $M\ge 1$ and $\delta>0$.

On the second hand, we consider the case $t\ge t_0+1$ and $s\in[t_0,t_0+1]$. As $H\in\mathcal{H}$ and $h\in\mathcal{K}$, there exists a constant $\alpha>0$ such that $h(s)\ge e^{-\alpha(t-s)}H(t)$, for all $(t,s\in T)$. We have

$$e^{-(\alpha+\delta)}\left|\langle v^*,e^{\alpha(t-t_0)}\Phi_P(t,t_0,x)v\rangle\right|\le e^{-(\alpha+\delta)(\tau-t_0)}\left|\langle v^*,e^{\alpha(t-t_0)}\Phi_P(t,t_0,x)v\rangle\right|=$$

$$=e^{-(\alpha+\delta)(\tau-t_0)}\int_{t_0}^{t_0+1}\left|\langle\Phi_P(t,\tau,\varphi(\tau,t_0,x))^*v^*,e^{\alpha(t-t_0)}\Phi_P(\tau,t_0,x)v\rangle\right|d\tau\le$$

$$\le\int_{t_0}^{t_0+1}e^{\alpha(t-\tau)}\left\|\Phi_P(t,\tau,\varphi(\tau,t_0,x))^*v^*\right\|e^{-\delta(\tau-t_0)}\left\|\Phi_P(\tau,t_0,x)v\right\|d\tau\le$$

$$\le M\left\|P(x)v\right\|\int_{t_0}^{t}e^{\alpha(t-\tau)}\left\|\Phi_P(t,\tau,\varphi(\tau,t_0,x))^*v^*\right\|d\tau\le$$

$$\le MM_1(t_0)\left\|P(x)v\right\|\left\|P(x)v^*\right\|.$$

By taking supremum relative to $\|v^*\|\le 1$ it follows that

$$\left\|\Phi_P(t,t_0,x)v\right\|\le Me^{\alpha+\delta}M_1(t_0)e^{-\alpha(t-t_0)}\left\|P(x)v\right\|$$

Thus, we obtain

$$\left\|\Phi_P(t,t_0,x)v\right\|\le Me^{\alpha+\delta}\left[M_1(t_0)+1\right]e^{-\alpha(t-t_0)}\left\|P(x)v\right\|,$$

for all $(t,t_0)\in T$ and $(x,v)\in Y$. Let us now define $H(t)=e^{\alpha t}$ and $N_1(t_0)=Me^{\alpha+\delta}\left[M_1(t_0)+1\right]e^{\alpha t_0}$. We obtain thus relation (3.5).

(ii) For $H\in\mathcal{H}$ and $h\in\mathcal{K}$, there exists a constant $\beta>0$ such that $h(s)\le e^{-\beta(t-s)}H(t)$, $\forall(t,s)\in T$. Let us denote

$$K=\int_0^1 e^{-\beta\tau}\omega(\tau)d\tau,$$

where the function ω is given by Definition 2.3. We have

$$K\left\|Q(x)v\right\|=\int_{t_0}^{t_0+1}e^{-\beta(\tau-t_0)}\omega(\tau-t_0)\left\|\Phi_Q(t_0,t_0,x)v\right\|d\tau\le$$

$$\le\int_{t_0}^{t_0+1}e^{-\beta(\tau-t_0)}\left\|\Phi_Q(\tau,t_0,x)v\right\|d\tau\le M_2(t)e^{\beta(t-t_0)}\left\|\Phi_Q(t,t_0,x)v\right\|,$$

for all $(t, t_0) \in T$ and all $(x, v) \in Y$. This relation implies

$$\left\| \Phi_Q(s, t_0, x)v \right\| \leq \frac{1}{K} M_2(t) e^{\beta(t-s)} \left\| \Phi_Q(t, t_0, x)v \right\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$. Let us define $H(s) = e^{\beta s}$ and $N_2(t) = \frac{1}{K} M_2(t) e^{\beta t}$. Relation (3.6) is thus obtained.

□

Remark. In Definition 3.5, relation (3.5) gives the definition for the H-stability and relation (3.6) for the H-instability, characterized, respectively, by the relations (3.7) and (3.8) of Theorem 3.2.

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