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On Multiset Topologies

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Abstract

In this paper an attempt is made to extend the concept of topological spaces in the context of multisets (mset, for short). The paper begins with basic definitions and operations on msets. The mset space $[X]^w$ is the collection of msets whose elements are from X such that no element in the mset occurs more than finite number (w) of times. Different types of collections of msets such as power msets, power whole msets and power full msets which are submsets of the mset space and operations under such collections are defined. The notion of M-topological space and the concept of open msets are introduced. More precisely, an M-topology is defined as a set of msets as points. Furthermore the notions of basis, sub basis, closed sets, closure and interior in topological spaces are extended to M-topological spaces and many related theorems have been proved. The paper concludes with the definition of continuous mset functions and related properties, in particular the comparison of discrete topology and discrete M-topology are established.

Keywords: Multisets, Power Multisets, Multiset Relations, Multiset Functions, M-Topology, M-Basis and Sub M-Basis, Continuous Mset Functions.
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1. Introduction

The notion of a multiset (bag) is well established both in mathematics and computer science (Clements, 1988; Conder *et al.*, 2007; Galton, 2003; Singh *et al.*, 2011; Skowron, 1988; Šlapal, 1993). In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained (Singh, 1994; Singh *et al.*, 2007; Singh & Singh, 2003; Wildberger, 2003). In various counting arguments it is convenient to distinguish between a set like $\{a, b, c\}$ and a collection like $\{a, a, a, b, c, c\}$. The latter, if viewed as a set, will be identical to the former. However, it has some of its elements purposely listed several times. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer.

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Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

Topology, as a branch of mathematics, can be formally defined as the study of qualitative properties of certain objects called topological spaces that are invariant under certain kinds of transformations called continuous maps (Galton, 2003; Skowron, 1988; Šlapal, 1993). There are many occasions, however, when one encounters collections of non-distinct objects. In such situations the term 'multiset' is used instead of 'set'. In this paper topologies on multisets are provided and they can be useful for measuring the similarities and dissimilarities between the universes of the objects which are multisets. Moreover, topologies on multisets can be associated to IC-bags or n^k-bags introduced by K. Chakrabarthy (Chakrabarty, 2000; Chakrabarty & Despi, 2007) with the help of rough set theory. The association of rough seth theory and topologies on multisets through bags with interval counts (Chakrabarty & Despi, 2007) can be used to develop theoretical study of covering based rough sets with respect to universe as multisets.

The mset space $[X]^w$ is the collection of finite msets whose elements are from X such that no member of an element of $[X]^w$ occurs more than finite number (w) of times. i.e., every msets in the collection $[X]^w$ are finite cardinality with each element having multiplicity atmost w. Different types of collections of msets such as power msets, power whole msets and power full msets which are submsets of the mset space and operations under such collections of msets are defined. The notion of M-topological space and the concept of open multisets are introduced. More precisely, a multiset topology is defined as a set of multisets as points. The notion of basis, sub basis, closed sets, closure and interior in topological spaces are extended to M-topological spaces and many related theorems have been proved. The paper concludes with the definition of continuous mset functions and related properties.

2. Preliminaries and Basic Definitions

In this section some basic definitions, results and notations as introduced by V. G. Cerf et al. (Gostelow *et al.*, 1972) in 1972, J. L. Peterson (Peterson, 1976) in 1976, R. R. Yager (Yager, 1987, 1986) in 1986, W. D. Blizard (Blizard, 1989*a*, 1990, 1989*b*, 1991) in 1989, K. Chakrabarty et al. (Chakrabarty & Despi, 2007; Chakrabarty *et al.*, 1999*b*,*a*; Chakrabarty & Despi, 2007) in 1999, S. P. Jena et al. (Jena *et al.*, 2001) in 2001 and the authors concepts in (Girish & John, 2009*a*, 2012, 2009*b*; Girish & Jacob, 2011; Girish & John, 2011) are presented.

Definition 2.1. (Girish & John, 2012) An mset M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We present the mset M drawn from the set $X = \{x_1, x_2, \ldots, x_n\}$ as $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, \ldots, n$ in the mset M. However those elements which are not included in the mset M have zero count.

Example 2.1. (Girish & John, 2012) Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Let M and N be two msets drawn from a set X. Then, the following are defined in (Girish & John, 2012):

(i) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
(ii) $M \subseteq N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.
(iii) $P = M \cup N$ if $C_P(x) = \text{Max}\ \{C_M(x), C_N(x)\}$ for all $x \in X$.
(iv) $P = M \cap N$ if $C_P(x) = \text{Min}\ \{C_M(x), C_N(x)\}$ for all $x \in X$.
(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.
(vi) $P = M \ominus N$ if $C_P(x) = \text{Max}\{C_M(x) - C_N(x), 0\}$ for all $x \in X$ where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.

Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set. M^* is also called root set.

An mset M is said to be an empty mset if for all $x \in X$, $C_M(x) = 0$.

The cardinality of an mset M drawn from a set X is denoted by Card (M) or M and is given by Card $M = \sum_{x \in X} C_M(x)$.

Definition 2.2. (Girish & John, 2012) A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times.

The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset.

If $X = \{x_1, x_2, \ldots, x_k\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, \ldots, m_k/x_k\} : \text{for } i = 1, 2, \ldots k;\ m_i \in \{0, 1, 2, \ldots w\}\}$.

Definition 2.3. (Girish & John, 2012) Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C^c_M(x) = w - C_M(x)$ for all $x \in X$.

Remark 2.1. Using Definition 2.3, the mset sum can be modified as follows:

$$C_{M_1 \oplus M_2}(x) = \min\{w, C_{M1}(x) + C_{M2}(x)\} \text{ for all } x \in X.$$

Notation 2.1. *(Girish & John, 2012) Let M be an mset from X with x appearing n times in M. It is denoted by* $x \in^n M$. $M = \{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ *where M is an mset with* x_1 *appearing* k_1 *times,* x_2 *appearing* k_2 *times and so on.* $[M]_x$ *denotes that the element x belongs to the mset M and* $|[M]_x|$ *denotes the cardinality of an element x in M.*

A new notation can be introduced for the purpose of defining Cartesian product, Relation and its domain and co-domain. The entry of the form $(m/x, n/y)/k$ denotes that x is repeated m-times, y is repeated n-times and the pair (x, y) is repeated k times. The counts of the members of the domain and co-domain vary in relation to the counts of the x co-ordinate and y co-ordinate in $(m/x, n/y)/k$. For this purpose we introduce the notation $C_1(x, y)$ and $C_2(x, y)$. $C_1(x, y)$ denotes the count of the first co-ordinate in the ordered pair (x, y) and $C_2(x, y)$ denotes the count of the second co-ordinate in the ordered pair (x, y).

Throughout this paper M stands for a multiset drawn from the multiset space $[X]^w$. We can define the following types of submets of M and collection of submsets from the mset space $[X]^w$.

Definition 2.4. (Girish & John, 2012) (Whole submset) A submset N of M is a whole submset of M with each element in N having full multiplicity as in M. i.e., $C_N(x) = C_M(x)$ for every x in N.

Definition 2.5. (Girish & John, 2012) (Partial Whole submset) A submset N of M is a partial whole submset of M with at least one element in N having full multiplicity as in M. i.e., $C_N(x) = C_M(x)$ for some x in N.

Definition 2.6. (Girish & John, 2012) (Full submset) A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in M. i.e., $M^* = N^*$ with $C_N(x) \leq C_M(x)$ for every x in N.

Note 2.1. *(Girish & John, 2012) Empty set $\emptyset$ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset M.*

Example 2.2. (Girish & John, 2012) Let $M = \{2/x, 3/y, 5/z\}$ be an mset. Following are the some of the submsets of M which are whole submsets, partial whole submsets and full submsets.

(a) A submset $\{2/x, 3/y\}$ is a whole submset and partial whole submset of M but it is not full subset of M.
(b) A submset $\{1/x, 3/y, 2/z\}$ is a partial whole submset and full submset of M but it is not a whole submset of M.
(c) A submset $\{1/x, 3/y\}$ is partial whole submset of M which is neither whole submset nor full submset of M.

Definition 2.7. (Girish & John, 2012) (Power Whole Mset) Let $M \in [X]^w$ be an mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M. i. e., for constructing power whole submsets of M, every element of M with its full multiplicity behaves like an element in a classical set. The cardinality of $PW(M)$ is 2^n where n is the cardinality of the support set (root set) of M.

Definition 2.8. (Girish & John, 2012) (Power Full Mset) Let $M \in [X]^w$ be an mset. Then the power full mset of M, $PF(M)$, is defined as the set of all full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

Note 2.2. *$PW(M)$ and $PF(M)$ are ordinary sets whose elements are msets.*

If M is an ordinary set with n distinct elements, then the power set $P(M)$ of M contains exactly 2^n elements. If M is a multiset with n elements (repetitions counted), then the power set $P(M)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(M)$. In the classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 2.9. (Girish & John, 2012) (Power Mset) Let $M \in [X]^w$ be an mset. The power mset $P(M)$ of M is the set of all sub msets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \Phi$, then $N \in^1 P(M)$; and if $N = \Phi$, then $N \in^k P(M)$ where $k = \prod_z \binom{[M]_z}{[N]_z}$, the product $\prod_z$ is taken over by distinct elements of z of the mset N and $|[M]_z| = m$ iff $z \in^m M$, $|[N]_z| = n$ iff $z \in^n N$, then

$$\begin{pmatrix} |[M]_z| \\ |[N]_z| \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}.$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$. The following theorem shows the cardinality of the power set of an mset.

Theorem 2.1 (23)**.** *Let $P(M)$ be a power mset drawn from the mset $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ and $P^*(M)$ be the power set of an mset M. Then $Card(P^*(M)) = \prod_{i=1}^{n}(1 + m_i)$.*

Example 2.3. (Girish & John, 2012) Let $M = \{2/x, 3/y\}$ be an mset.
The collection $PW(M) = \{\{2/x\}, \{3/y\}, M, \emptyset\}\}$ is a power whole submset of M.
The collection $PF(M) = \{\{2/x, 1/y\}, \{2/x, 2/y\}, \{2/x, 3/y\}, \{1/x, 1/y\}, \{1/x, 2/y\}, \{1/x, 3/y\}\}$ is a power full submset of M.
The collection $P(M) = \{3/\{2/x, 1/y\}, 3/\{2/x, 2/y\}, 6/\{1/x, 1/y\}, 6/\{1/x, 2/y\}, 2/\{1/x, 3/y\}, 1/\{2/x\}, 1/\{3/y\}, 2/\{1/x\}, 3/\{1/y\}, 3/\{2/y\}, M, \emptyset\}\}$ is the power mset of M.
The collection $P^*(M) = \{\{2/x, 1/y\}, \{2/x, 2/y\}, \{1/x, 1/y\}, \{1/x, 2/y\}, \{1/x, 3/y\}, \{2/x\}, \{3/y\}, \{1/x\}, \{1/y\}, \{2/y\}, M, \emptyset\}\}$ is the support set of $P(M)$.

Note 2.3. *Power mset is an mset but its support set is an ordinary set whose elements are msets.*

Definition 2.10. (Girish & John, 2012) The maximum mset is defined as Z where $C_Z(x) = \text{Max}\ \{C_M(x) : x \in^k M, M \in [X]^w$ and $k \le w\}$.

Operations under collection of msets. (Girish & John, 2012) Let $[X]^w$ be an mset space and $\{M_1, M_2, \ldots\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection of msets.

(i) The union
$$\coprod_{i \in I} M_i = \{C_{M_i}(x)/x : C_{M_i}(x) = \max\{C_{M_i}(x) : x \in X\}\}.$$

(ii) The intersection
$$\cap_{i \in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = \min\{C_{M_i}(x) : x \in X\}\}.$$

(iii) The mset addition
$$\oplus_{i \in I} M_i = \{C_{\oplus M_1}(x)/x : C_{\oplus M_i}(x) = \sum_{i \in I} C_{M_i}(x), x \in X\}.$$

(iv) The mset complement
$$M^c = Z \ominus M = \{C_M c(x)/x : C_M c(x) = C_Z(x) - C_M(x), x \in X\}.$$

Remark 2.2. Every nonempty set of real numbers that has an upper bound has a supremum and that have a lower bound has an infimum. Thus, the arbitrary union and arbitrary intersection defined in 2.20 are closed under the collection $\{M_i\}_{i \in I}$, because the collection $\{M_i\}_{i \in I}$ drawn from $[X]^m$ contains elements with finite cardinality and multiplicity of each element x_i in M_i is always less than or equal to m.

Definition 2.11. (Girish & John, 2012) Let M_1 and M_2 be two msets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1, y \in^n M_2\}$.

We can define the Cartesian product of three or more nonempty msets by generalizing the definition of the Cartesian product of two msets.

Definition 2.12. (Girish & John, 2012) A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count, product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/x\ R\ n/y$.

The Domain and Range of the mset relation R on M is defined as follows:

Dom $R = \{x \in^r M : \exists y \in^s M$ such that $r/xRs/y\}$ where $C_{DomR}(x) = \sup\{C_1(x, y) : x \in^r M\}$.
Ran $R = \{y \in^s M : \exists x \in^r M$ such that $r/xRs/y\}$ where $C_{RanR}(y) = \sup\{C_2(x, y) : y \in^s M\}$.

Example 2.4. (Girish & John, 2012) Let M=$\{8/x, 11/y, 15/z\}$ be an mset. Then $R = \{(2/x, 4/y)/8, (5/x, 3/x)/15, (7/x, 11/z)/77, (8/y, 6/x)/48, (11/y, 13/z)/143, (7/z, 7/z)/49, (12/z, 10/y)/120, (14/z, 5/x)/70\}$ is an mset relation defined on M. Here Dom $R = \{7/x, 11/y, 14/z\}$ and Ran $R = \{6/x, 10/y, 13/z\}$. Also $S = \{(2/x, 4/y)/5, (5/x, 3/x)/10, (7/x, 11/z)/77, (8/y, 6/x)/48, (11/y, 13/z)/143, (7/z, 7/z)/49, (12/z, 10/y)/120, (14/z, 5/x)/70\}$ is a submset of $M \times M$ but S is not an mset relation on M because $C_S((x, y)) = 5 \neq 2 \times 4$ and $C_S((x, x)) = 10 \neq 5 \times 3$, i.e., count of some elements in S is not a product of $C_1(x, y)$ and $C_2(x, y)$.

Definition 2.13. (Girish & John, 2012)

(i) An mset relation R on an mset M is reflexive if $m/xRm/x$ for all m/x in M.
(ii) An mset relation R on an mset M is symmetric if $m/xRn/y$ implies $n/yRm/x$.
(iii) An mset relation R on an mset M is transitive if $m/xRn/y$, $n/yRk/z$ then $m/xRk/z$.

An mset relation R on an mset M is called an equivalence mset relation if it is reflexive, symmetric and transitive.

Example 2.5. (Girish & John, 2012) Let $M = \{3/x, 5/y, 3/z, 7/r\}$ be an mset. Then the mset relation given by $R = \{(3/x, 3/x)/9,$
$(3/z, 3/z)/9, (3/x, 7/r)/21, (7/r, 3/x)/21, (5/y, 5/y)/25, (3/z, 3/z)/9, (7/r, 7/r)/49, (3/z, 3/x)/9, (3/z, 7/r)/21, (7/r, 3/z)/21\}$ is an equivalence mset relation.

Definition 2.14. (Girish & John, 2012) An mset relation f is called an mset function if for every element m/x in Dom f, there is exactly one n/y in Ran f such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x, y)$ and $C_2(x, y)$.

For functions between arbitrary msets it is essential that images of indistinguishable elements of the domain must be indistinguishable elements of the range but the images of the distinct elements of the domain need not be distinct elements of the range.

Example 2.6. (Girish & John, 2012) Let $M_1 = \{8/x, 6/y\}$ and $M_2 = \{3/a, 7/b\}$ be two msets. Then an mset function from M_1 to M_2 may be defined as $f = \{(8/x, 3/a)/24, (6/y, 7/b)/42\}$.

3. Multiset Topology

This section gives the basic definitions and examples introduced in (Girish & John, 2012).

Definition 3.1. (Girish & John, 2012) Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties.

1. The mset M and the empty mset $\emptyset$ are in τ.
2. The mset union of the elements of any sub collection of τ is in τ.
3. The mset intersection of the elements of any finite sub collection of τ is in τ.

Mathematically a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Note that τ is an ordinary set whose elements are msets. Multiset Topology is abbreviated as an M-topology.

General topology is defined as a set of sets but multiset topology is defined as a set of multisets. Moreover in general topology τ is a subset of the power set but in M-topology τ is a subset of support set of the power mset. If M is an M-topological space with M-topology τ, we say that a submset U of M is an open mset of M if U belongs to the collection τ. Using this terminology, one can say that an M-topological space is an mset M together with a collection of submsets of M, called open msets, such that $\emptyset$ and M are both open and the arbitrary mset unions and finite mset intersections of open msets are open.

Example 3.1. (Girish & John, 2012) Let M be any mset in $[X]^w$. The collection $P^*(M)$, the support set of the power mset of M is an M-topology on M and is called the discrete M-topology.

In general topology, discrete topology is the power set but in M-topology, discrete M-topology is the support set of the power mset.

Example 3.2. (Girish & John, 2012) The collection consisting of M and $\emptyset$ only, is an M-topology called indiscrete M-topology, or trivial M-topology.

Example 3.3. (Girish & John, 2012) If M is any mset in $[X]^w$, then the collection $PW(M)$ is an M-topology on M.

Example 3.4. (Girish & John, 2012) The collection $PF(M)$ is not an M-topology on M, because $\emptyset$ does not belong to $PF(M)$, but $PF(M) \cup \{\emptyset\}$ is an M-topology on M.

Example 3.5. (Girish & John, 2012) The collection τ of partial whole submsets of M is not an M-topology. Let $M = \{2/x, 3/y\}$. Then $A = \{2/x, 1/y\}$ and $B = \{1/x, 3/y\}$ are partial whole submsets of M. Now $A \cap B = \{1/x, 1/y\}$, but it is not a partial whole submset of M. Thus τ is not closed under finite intersection.

4. *M*-Basis and Sub *M*-Basis

Definition 4.1. (Girish & John, 2012) If M is an mset, then the M-basis for an M- topology on M in $[X]^w$ is a collection $\mathcal{B}$ of submsets of M (called M basis elements) such that

1. For each $x \in^m M$, for some $m > 0$, there is at least one M-basis element $B \in \mathcal{B}$ containing m/x. i.e., for each indistinguishable element in M, there is at least one M-basis element in $\mathcal{B}$ having that element with same multiplicity as in M.
2. If m/x belongs to the intersection of two M-basis elements M and N, then there exists an M-basis element P containing m/x such that $P \subseteq M \cap N$ with $C_P(x) = C_M \cap N(x)$ and $C_P(y) \leq C_{M\cap N}(y)$ for all $y \neq x$.

Remark 4.1. (Girish & John, 2012) If a collection $\mathcal{B}$ satisfies the conditions of M-basis, then the M-topology τ generated by $\mathcal{B}$ can be defined as follows. A submset U of M is said to be an open mset in M (i.e., to be an element of τ) if for each $x \in^k U$, there is an M-Basis element $B \in \mathcal{B}$ such that $x \in^k B$ and $C_B(y) \leq C_U(y)$ for all $y \neq x$.

Note that each M-basis element is itself an element of τ.

Theorem 4.1. *The collection τ generated by an M-basis* **B** *is an M-topology on M in $[X]^w$.*

Proof. 1. Clearly $\emptyset$ and M are in τ.

2. Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed family of elements of τ. Then $* = \prod_{\alpha\in J} U_\alpha$ belongs to τ. For, given $x \in^m \mathcal{U}$, $m = \max_\alpha\{C_{U_\alpha}(x)\}$, there is an index α such that U_α containing m/x. Since U_α is an open mset, there is an M-basis element B containing m/x such that $B \subseteq U_\alpha$. Then $x \in^m B$ and $B \subseteq \mathcal{U}$, so that $\mathcal{U}$ is an open mset, by definition.
3. If U_1 and U_2 are two elements of τ, to prove $U_1 \cap U_2$ belongs to τ. Given $x \in^k U_1 \cap U_2$, $k = \min\{C_{U_1}(x), C_{U_2}(x)\}$. By definition of M-basis, there exists an element B_1 containing k/x, such that $B_1 \subseteq U_1$ and another M-basis element B_2 containing k/x such that $B_2 \subseteq U_2$. The second condition for an M-basis enables us to choose an M-basis element B_3 containing k/x such that $B_3 \subseteq B_1 \cap B_2$. Then $x \in^k B_3$ and $B_3 \subseteq U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ, by definition.

Finally, by induction it follows that any finite intersection $U_1 \cap U_2 \cap \cdots \cap U_k$ of elements of τ is in τ. This fact is trivial for $k = 1$ and to be proved for $k = n$. Now $U_1 \cap U_2 \cap \cdots \cap U_n = (U_1 \cap U_2 \cdots \cap U_{n-1}) \cap U_n$. By hypothesis, $U_1 \cap U_2 \cap \cdots \cap U_{n-1}$ belongs to τ and by the result proved above, the intersection of $U_1 \cap U_2 \cap \cdots \cap U_{n-1}$ and U_n also belongs to τ. Thus the collection of open msets generated by an M-basis $\mathcal{B}$ is, in fact, an M-topology. □

Theorem 4.2. *Let M be an mset in $[X]^w$ and $\mathcal{B}$ be an M-basis for an M-topology τ on M. Then τ equals the collection of all mset unions of elements of the M-basis $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, which are also elements of τ, because τ is an M-topology, their union is in τ. Conversely, given $U \in \tau$, for each m/x in U, there is an element B of $\mathcal{B}$ containing m/x, denoted by $B_{m/x}$, such that $B_{m/x} \subseteq U$. Then $U = \cup B_{m/x}$, so U equals a union of elements of $\mathcal{B}$. □

Lemma 4.3. *Let $M \in [X]^w$ be an M-topological space. Suppose $\mathcal{M}$ is a collection of open msets of M such that for each open mset U of M and each element m/x in U, there is an element N of $\mathcal{M}$ containing m/x such that $C_N(x) \leq C_U(x)$. Then $\mathcal{M}$ is an M-basis for the M-topology of M.*

Proof. Given $x \in^m M$, since M itself is an open mset, by hypothesis there is an element N of $\mathcal{M}$ containing m/x such that $N \subseteq M$. To check the second condition, let m/x be in $N_1 \cap N_2$, N_1 and N_2 are elements of $\mathcal{M}$. Since N_1 and N_2 are open msets, so is its intersection $N_1 \cap N_2$. Therefore, by hypothesis there exists an element N_3 in C containing m/x such that $N_3 \subseteq N_1 \cap N_2$. Hence the collection $\mathcal{M}$ is an M-basis.

Let τ be the collection of open msets of M. Then the M-topology τ' generated by $\mathcal{M}$ equals the M-topology τ. If U belongs to τ and $x \in^m U$, then by hypothesis there is an element N of $\mathcal{M}$ containing m/x such that $N \subseteq U$. By definition, it follows that U belongs to the M-topology τ'. Conversely, if W belongs to the M-topology τ', then W equals a union of elements of $\mathcal{M}$, by theorem 4.4. Since each element $\mathcal{M}$ belongs to τ and τ is an M-topology, W also belongs to τ. Thus the M-topology generated by the M-basis and M-topology on M are the same. □

Definition 4.2. Suppose τ and τ' are two M-topologies on a given mset M in $[X]^w$. If $\tau' \subset \tau$, then we say that τ' is finer than τ or τ is coarser than τ'. If $\tau' \subset \tau$, then τ' is strictly finer than τ or τ is strictly coarser than τ'. Thus τ is comparable with τ' if either $\tau' \supseteq \tau$ or $\tau \supseteq \tau'$.

The next theorem gives a criterion for determining whether an M-topology on M is finer than another in terms of M-basis.

Theorem 4.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ are M-basis for the M-topologies τ and τ' on M in $[X]^w$ respectively. Then the following are equivalent:*

1. *τ' is finer than τ.*

2. *For each $x \in^m M$ and each M-basis element $B \in \mathcal{B}$ containing m/x, there is an M-basis element $B' \in \mathcal{B}'$ containing m/x such that $C_{B'}(x) \leq C_B(x)$.*

Proof. (1) ⇒ (2). Given an element m/x in M and $B \in \mathcal{B}$ containing m/x, B belongs to τ by definition and $\tau \subseteq \tau'$ by (1). Therefore $B \in \tau'$. Since τ' is generated by $\mathcal{B}'$, there is an M-basis element $B' \in \mathcal{B}$ containing m/x such that $C_{B'}(x) \leq C_B(x)$.

(2) ⇒ (1). Given an element U of τ, we show that $U \in \tau'$. Let $x \in^m U$, since $\mathcal{B}$ generates τ, there is an M-basis element $B \in \mathcal{B}$ containing m/x such that $B \subseteq U$. From (2), there exists an M-basis element $B' \in \mathcal{B}'$ containing m/x such that $B' \subseteq B$. Then $B' \subseteq U$ and $U \in \tau'$. □

Example 4.1. The collection $\{\{m/x\} : x \in^m M\}$ is an M-basis for the M-topology $PW(M)$.

In general topology $\{\{x\} : x \in X\}$ is a basis for the discrete topology, but in the case of M-topology the collection $\{\{m/x\} : x \in^m M\}$ is not an M-basis for the discrete M-topology.

Definition 4.3. Let (M, τ) be an M-topological space and N is a submset of M. The collection $\tau_N = \{U' = N \cap U; U \in \tau\}$ is an M-topology on N, called the subspace M-topology. With this M-topology, N is called a subspace of M and its open msets consisting of all mset intersections of open msets of M with N.

Theorem 4.5. *If $\mathcal{B}$ is an M-basis for the M-topology of M in $[X]^w$, then the collection $\mathcal{B}_N = \{B \cap N : B \in \mathcal{B}\}$ is an M-basis for the subspace M-topology on a submset N of M.*

Proof. Given U open in M and $y \in^m U \cap N$, we can choose an element B of $\mathcal{B}$ such that $y \in^m B \subseteq U$. Then, $y \in^m B \cap N \subseteq U \cap N$. It follows from Lemma 4.5 that $\mathcal{B}_N$ is an M-basis for the subspace M-topology on N. □

Example 4.2. Let $M = \{3/a, 4/b, 2/c, 5/d\}$ and $\tau = \{\emptyset, M, \{2/c\}, \{2/a\}, \{3/a, 2/b\}, \{2/a, 3/d\}, \{2/a, 2/c\}, \{3/a, 3/b, 3/d\}, \{3/a, 4/b, 2/c\}, \{2/a, 2/c, 3/d\}\}$ is an M-topology on M. If $N = \{2/a, 2/b, 3/d\} \subseteq M$, then $\tau' = \{\emptyset, \{2/a, 2/b, 3/d\}, \{2/a\}, \{2/a, 2/b\}, \{2/a, 3/d\}\}$ is an M-topology on N and it is the subspace M-topology on N.

Definition 4.4. A sub collection $\mathcal{P}$ of τ on M is called a sub M-basis for τ, if the collection of all finite mset intersections of elements of $\mathcal{P}$ is an M-basis for τ. The M-topology generated by the sub M-basis $\mathcal{P}$ is defined to be the collection τ of mset union of all finite mset intersections of elements of $\mathcal{P}$.

Note 4.1. *The empty mset intersection of the members of sub M-basis is the universal mset.*

Theorem 4.6. *Let (M, τ) be an M-topological space and $\mathcal{P}$ be a collection of submsets of M. Then $\mathcal{P}$ is a sub M-basis for τ if and only if $\mathcal{P}$ generates τ.*

Proof. Let $\mathcal{B}$ be the family of finite intersections of members of $\mathcal{P}$ and $\mathcal{P}$ be a sub M-basis for τ. It can be shown that τ is the smallest M-topology on M containing $\mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \tau, \subseteq \tau$. Suppose τ^* is some other M-topology on M such that $\mathcal{P} \subseteq \tau^*$. We have to show that $\tau \subseteq \tau^*$. Since $\mathcal{P} \subseteq \tau^*$, τ^* contains all finite intersections of members of $\mathcal{P}$, i.e., $\mathcal{B} \subseteq \tau^*$. Since $\mathcal{B}$ is an M-basis, each member of τ can be written as the union of some members of $\mathcal{B}$ and it follows that $\tau \subseteq \tau^*$.

Conversely suppose τ is the smallest M-topology containing $\mathcal{P}$. We have to show that $\mathcal{P}$ is a sub M-basis for τ. i.e., $\mathcal{B}$ is an M-basis for τ. Suppose there is an M-topology τ^* on M such that $\mathcal{B}$ is an M-basis for τ^*. Then every member of τ^* can be expressed as a union of the sub family of $\mathcal{B}$ and so it is in τ since $\mathcal{B} \subseteq \tau$. This means $\tau^* \subseteq \tau$ and consequently $\tau^* = \tau$. Since τ is the smallest M-topology containing $\mathcal{P}$, it can be shown that $\mathcal{B}$ is an M-basis for τ and $\mathcal{P}$ is a sub M-basis for τ. □

Example 4.3. Let $M = \{3/a, 5/b, 4/c\}$. If the collection $\mathcal{P} = \{\{3/a, 5/b\}, \{5/b, 4/c\}\}$ is a sub M-basis, then the collection $\mathcal{B} = \{\{5/b\}, \{3/a, 5/b\}, \{5/b, 4/c\}\}$ is the corresponding M-basis and $\tau = \{M, \emptyset, \{5/b\}, \{3/a, 5/b\}, \{5/b, 4/c\}\}$ is the M-topology generated by the M-basis.

If we assume the empty mset intersection of the members of sub M-basis is the universal mset, then we can give the following example.

Example 4.4. Let $M = \{3/a, 4/b, 2/c, 5/d\}$. If the collection
$\mathcal{P} = \{\{3/a, 3/b\}, \{4/d\}, \{2/a\}\}$ is a sub M-basis, then the collection
$\mathcal{B} = \{\{3/a, 3/b\}, \{4/d\}, \{2/a\}, \emptyset, M\}$ is the corresponding M-basis and
$\tau = \{\emptyset, M, \{2/a\}, \{4/d\}, \{3/a, 3/b\}, \{2/a, 4/d\}, \{3/a, 3/b, 4/d\}\}$
is the M-topology generated by the M-basis.

5. Closed Multisets

Definition 5.1. A sub mset N of an M-topological space M in $[X]^w$ is said to be closed if the mset $M \ominus N$ is open.

In discrete M-topology every mset is an open mset as well as a closed mset. In the M-topology $PF(M) \cup \{\emptyset\}$, every mset is an open mset as well as a closed mset.

Theorem 5.1. *Let (M, t) be an M-topological space. Then the following conditions hold:*

1. *The mset M and the empty mset $\emptyset$ are closed msets.*
2. *Arbitrary mset intersection of closed msets is a closed mset.*
3. *Finite mset union of closed msets is a closed mset.*

Proof.
1. $\emptyset$ and M are closed msets because they are the complements of the open msets M and $\emptyset$ respectively.
2. Given a collection of closed msets $\{N_\alpha\}_{\alpha \in J}$, we have

$$C_{M \oplus \cap_\alpha N_\alpha}(x) = C_M(x) - \min_{\alpha \in J}\{C_{N_\alpha}(x)\} = \max_{\alpha \in J}\{C_M(x) - C_{N_\alpha}(x)\}$$
$$= C_{\cap_\alpha (M \ominus) N_\alpha)}(x)$$

From this
$$M \ominus \cap_{\alpha \in J} N_\alpha = cap_\alpha (M \ominus) N_\alpha)$$

By definition the msets $M \ominus N_\alpha$'s are open. Since the arbitrary union open of msets are open, $M \ominus \cap_{\alpha \in J} N_\alpha$ is an open mset and therefore $\cap_{\alpha \in J} N_\alpha$ is a closed mset.

3. Similarly, if N_i is closed, for $i = 1, 2, \ldots, n$, consider

$$C_{M \ominus \prod_i N_i}(x) = C_M(x) - \max_i \{C_{N_i}(x)\} = \min_i \{C_M(x) - C_{N_i}(x)\} = C_{\cap_i (M \oplus N_i)(x)}.$$

Thus
$$M \ominus \prod_{i=1}^{n} N_i = \cap_{i=1}^{n} (M \ominus N_i).$$

Since finite mset intersections of open msets are open, $\prod_{i=1}^{n} N_i$ is a closed mset. □

Theorem 5.2. *Let N be a subspace of an M-topological space M in $[X]^w$. Then an mset A is a closed mset in N if and only if it equals the intersection of a closed mset of M with N.*

Proof. Assume $A = C \cap N$ where C is a closed mset in M. By the definition of subspace M-topology, $M \ominus C$ is an open mset in M, so that $(M \ominus C) \cap N$ is an open mset in N. But $(M \ominus C) \cap N = N \ominus A$. Hence $N \ominus A$ is an open mset in N, so that A is a closed mset in N. Conversely, assume that A is closed mset in N. Then $N \ominus A$ is open mset in N, so that by definition it equals the intersection of an open mset U of M with N. The mset $M \ominus U$ is a closed mset in M and $A = N \cap (M \ominus U)$, so that A equals the intersection of the closed mset of M with N, as desired. □

Theorem 5.3. *Let N be a subspace of an M-topological space M in $[X]^w$. If A is a closed mset in N and N is a closed mset in M, then A is a closed mset in M.*

Proof. Proof directly follows from Theorem 5.3. □

6. Closure, Interior and Limit Point

Definition 6.1. Given a submset A of an M-topological space M in $[X]^w$, the interior of A is defined as the mset union of all open msets contained in A and is denoted by Int (A).

i.e., Int $(A) = \cup\{G \subseteq M : G$ is an open mset and $G \subseteq A\}$ and $C_{\mathrm{Int}(A)}(x) = \max\{C_G(x) : G \subseteq A\}$.

Definition 6.2. Given a submset A of an M-topological space M in $[X]^w$, the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by Cl (A).

i.e., $\mathrm{Cl}(A) = \cap\{K \subseteq M : K$ is a closed mset and $A \subseteq K\}$ and $C_{\mathrm{Cl}}(A)(x) = \min\{C_K(x) : A \subseteq K\}$.

Definition 6.3. Let (M, τ) be an M-topological space, let $x \in^k M$ and $N \subseteq M$. Then N is said to be a neighborhood of k/x if there is an open mset V in τ such that $x \in^k V$ and $C_V(y) \leq C_N(y)$ for all $y \neq x$.

i.e., a neighborhood of k/x in M means any open mset containing k/x. Here k/x is said to be an interior point of N.

Definition 6.4. Let A be a submset of the M-topological space M in $[X]^w$. If k/x is an element of M, then k/x is a limit point of an mset A when every neighborhood of k/x intersects A in some point (point with non zero multiplicity) other than k/x itself. A' denotes the mset of all limit points of A.

Theorem 6.1. *Let N be a subspace of an M-topological space M in $[X]^w$ and A be a submset of an mset N and $Cl(A)$ denote the closure of an mset A in M. Then the closure of an mset A in N equals $Cl(A) \cap N$.*

Proof. Let B denote the closure of an mset A in N. If mset $\mathrm{Cl}(A)$ is a closed mset in M, then by Theorem 5.3 $\mathrm{Cl}(A) \cap N$ is a closed mset in N. Since $\mathrm{Cl}(A) \cap N$ contains A, and since by definition, B equals the intersection of all closed submsets of N containing A, we get $B \subseteq \mathrm{Cl}(A) \cap N$.

On the other hand, B is a closed mset in N. Hence by Theorem 4.4.3, $B = C \cap N$ for some mset C, a closed mset in M. Then C is a closed mset of M containing A, because $\mathrm{Cl}(A)$ is the intersection of all such closed msets. We conclude that $\mathrm{Cl}(A) \subseteq C$. Therefore $\mathrm{Cl}(A) \cap N \subseteq C \cap N = B$. □

Theorem 6.2. *Let (M, τ) be an M-topological space, $x \in^k M$ and $A \subseteq M$, then*
1. $x \in^k cl(A)$ if and only if every open mset U containing k/x intersects A.
2. If the M-topology (M, τ) is given by an M-basis $\mathcal{B}$, then, $x \in^k Cl(A)$ if and only if every M-basis element $B \in \mathcal{B}$ containing k/x intersects A.

Proof. 1. If k/x is not in Cl(A), then the mset $U = M \ominus \text{Cl}(A)$ is an open mset containing k/x that does not intersect A. Conversely, if there exists an open mset U containing k/x which does not intersect A, then the mset $M \ominus U$ is a closed mset containing A. By the definition of the closure Cl(A), the mset $M \ominus U$ must contain Cl(A). Therefore k/x cannot be in Cl(A).

2. If every open mset containing k/x intersects A, so does every M-basis element B containing k/x, because B is an open mset.

Conversely, if every M-basis element containing k/x intersects A, so does every open mset U containing k/x, because U contains an M-basis element that contains k/x. □

Theorem 6.3. *A submset of an M-topological space is an open mset if and only if it is a neighborhood of each of its elements with some multiplicity.*

Proof. Let M be an M-topological space and $N \subseteq M$. First suppose N is an open mset. Then clearly N is a neighborhood of each of its points with some multiplicity. Conversely suppose N is a neighborhood of each of its points, then for each k/x in N, there is an open mset $V_{k/x}$ such that $x \in^k V_{k/x}$ and $V_{k/x} \subseteq N$. Clearly,

$$N = \prod_{x \in^k N} V_{k/x},\ k = \max\{C_{V_{k/x}(x)}\}.$$

Since each $V_{k/x}$ is an open mset so is N. □

Theorem 6.4. *Let A be a submset of the M-topological space M and A′ be the mset of all limit points of A. Then* $C_{Cl\,(A)}(x) = \max\{C_A(x), C_{A'}(x)\}$.

Proof. If k/x is in A', then every neighborhood of k/x intersects A. Therefore, by Theorem 4.5.6 k/x belongs to Cl(A). Hence $A' \subseteq \text{Cl}\,(A)$. Since by definition $A \subseteq \text{Cl}\,(A)$, it follows that $A \cup A' = \text{Cl}\,(A)$.

Conversely suppose k/x is a point of Cl(A), then $x \in^k A \cup A'$. If k/x is in A, it is clear that $x \in^k A \cup A'$. Suppose k/x does not belonging to A, since $x \in^k \text{Cl}\,(A)$, we know that every neighborhood U of k/x intersects A. Thus the mset U must intersect A in a point different from k/x. Hence $x \in^k A'$ and $x \in^k A \cup A'$. □

Corollary 6.5. *A submset of an M-topological space is a closed mset if and only if it contains all its limit points.*

Proof. The mset A is a closed mset:

- if and only if $A = \text{Cl}\,(A)$,
- if and only if $A = A \cup A'$,
- if and only if $A' \subseteq A$.

□

Theorem 6.6. *If A and B are submsets of the M-topological space M in* $[X]^w$*, then the following properties hold:*

1. *If* $C_A(x) \leq C_B(x)$*, then* $C_{A'}(x) \leq C_{B'}(x)$.
2. *If* $C_A(x) \leq C_B(x)$*, then* $C_{Int\,(A)}(x) \leq C_{Int\,(B)}(x)$.
3. *If* $C_A(x) \leq C_B(x)$*, then* $C_{Cl\,(A)}(x) \leq C_{Cl\,(B)}(x)$

4. $C_{Int(A\cap B)}(x) = \min\{C_{Int(A)}(x), C_{Int(B)}(x)\}$.
5. $C_{Cl(A\cup B)}(x) = \max\{C_{Cl(A)}(x), C_{Cl(B)}(x)\}$.

Proof. 1. $x \in^k A'$ if and only if $(N \ominus \{k/x\}) \cap A \neq \emptyset$, for all open mset N containing k/x. Since $B \supseteq A$, $(N \ominus \{k/x\}) \cap B \supseteq (N \ominus \{k/x\}) \cap A \neq \emptyset$. So $x \in^k A'$ implies $x \in^k B'$. Thus $A' \subseteq B'$ and $C_{A'}(x) \leq C_{B'}(x)$.

2. We have $C_{\text{Int}(A)}(x) \leq C_A(x)$ and $C_{\text{Int}(B)}(x) \leq C_B(x)$. Since $A \subseteq B$ and $C_A(x) \leq C_B(x)$, we get $C_{\text{Int}(A)}(x) \leq C_B(x)$ and Int $(A) \subseteq B$. Thus Int (A) is an open mset contained in B, but Int (B) is the largest open mset contained in B. Hence $C_{\text{Int}(A)}(x) \leq C_{\text{Int}(B)}(x)$ and Int$(A) \subseteq$ Int (B).
3. We have

$$\begin{aligned} C_{\text{Cl}(A)}(x) &= \max\{C_A(x), C_{A'}(x)\}, \text{ from Theorem 6.8} \\ &\leq \max\{C_B(x), C_{B'}(x)\}, \text{ by (1)} \\ &= C_{\text{Cl}(B)}(x) \end{aligned}$$

Thus Cl$(A) \subseteq$ Cl (B).

4. We have $C_{\text{Int}(A\cap B)}(x) \leq C_{\text{Int}(A)}(x)$ and $C_{\text{Int}(A\cap B)}(x) \leq C_{\text{Int}(B)}(x)$.
Therefore $C_{textInt(A\cap B)}(x) \leq \min\{C_{\text{Int}(A)}(x), C_{\text{Int}(B)}(x)\}$. Thus

$$\text{Int}(A \cap B) \subseteq \text{Int}(A) \cap \text{Int}(B) \tag{i}$$

Also $C_{\text{Int}(A)}(x) \leq C_A(x)$ and $C_{\text{Int}(B)}(x) \leq C_B(x)$.
Therefore $\min\{C_{\text{Int}(A)}(x), C_{\text{Int}(B)}(x)\} \leq \min\{C_A(x), C_B(x)\}$.
Thus Int$(A)\cap$ Int$(B) \subseteq A \cap B$, but Int $(A \cap B)$ is the largest open mset contained in $A \cap B$, i.e., $C_{\text{Int}(A\cap B)}(x)$ is that largest integer which is less than or equal to $C_{A\cap B}(x)$.
Therefore $\min\{C_{\text{Int}(A)}(x), C_{\text{Int}(B)}(x)\} \leq C_{\text{Int}(A\cap B)}(x)$. Thus

$$\text{Int}(A) \cap \text{Int}(B) \subseteq \text{Int}(A \cap B) \tag{ii}$$

From (i) and (ii) it follows that Int $(A \cap B) =$ Int$(A)\cap$ Int(B).

5. We have $C_{\text{Cl}(A)}(x) \leq C_{\text{Cl}(A\cup B)}(x)$ and $C_{\text{Cl}(B)}(x) \leq C_{\text{Cl}(A\cup B)}(x)$. Therefore

$$\max\{C_{\text{Cl}(A)}(x), C_{\text{Cl}(B)}(x)\} \leq C_{\text{Cl}(A\cup B)}(x) \tag{i}$$

But $C_A(x) \leq C_{\text{Cl}(A)}(x)$ and $C_B(x) \leq C_{\text{Cl}(B)}(x)$.
Therefore $\max\{C_A(x), C_B(x)\} \leq \max\{C_{\text{Cl}(A)}(x), C_{\text{Cl}(B)}(x)\}$. Hence

$$C_{\text{Cl}(A\cup B)}(x) \leq \max\{C_{\text{Cl}(A)}(x), C_{\text{Cl}(B)}(x)\} \tag{ii}$$

From (i) and (ii) it follows that $C_{\text{Cl}(A\cup B)}(x) = \max\{C_{\text{Cl}(A)}(x), C_{\text{Cl}(B)}(x)\}$.

Thus Cl$(A \cup B) =$ Cl $(A) \cup$ Cl (B). □

7. Continuous Multiset Functions

Definition 7.1. Let M and N be two M-topological spaces. The mset function $f : M \to N$ is said to be continuous if for each open submset V of N, the mset $f^{-1}(V)$ is an open submset of M, where $f^{-1}(V)$ is the mset of all points m/x in M for which $f(m/x) \in^n V$ for some n.

Example 7.1. Let $M = \{5/a, 4/b, 4/c, 3/d\}$ and $N = \{7/x, 5/y, 6/z, 4/w\}$ be two msets, $\tau = \{M, \emptyset, \{5/a\}, \{5/a, 4/b\}, \{5/a, 4/b, 4/c\}\}$ and $\tau' = \{N, \emptyset, \{7/x\}, \{5/y\}, \{7/x, 5/y\}, \{5/y, 6/z, 4/w\}\}$ be two M-topologies on M and N respectively.

Consider the mset functions $f : M \to N$ and $g : M \to N$ are given by

$$f = \{(5/a, 5/y)/25, (4/b, 6/z)/24, (4/c, 4/w)/16, (3/d, 6/z)/18\},$$
$$g = \{(5/a, 7/x)/35, (4/b, 7/x)/28, (4/c, 6/z)/24, (3/d, 4/w)/12\}.$$

The mset function f is continuous since the inverse of each member of the M-topology τ' on N is a member of the M-topology τ on M. The mset function g is not continuous since $\{5/y, 6/z, 4/w\} \in \tau'$, i.e., an open mset of N, but its inverse image $g^{-1}(\{5/y, 6/z, 4/w\}) = \{4/c, 3/d\}$ is not an open submset of M, because the mset $\{4/c, 3/d\}$ does not belong to τ.

Example 7.2. Let $f : M \to N$ be an mset function and $\tau = P^*(M)$, the support set of the power mset of M, the M-topology on M. Then every mset function $f : M \to N$ is continuous for any M-topology on N.

Example 7.3. Let $f : M \to N$ be an mset function and $\tau' = PF(N) \cup \{\emptyset\}$ be an M-topology on N, then every mset function $f : M \to N$ is continuous for any M-topology τ on M, because open msets in τ' are submsets of N whose support set is N^*. Let H and $\emptyset$ be open in τ', then $f^{-1}(H) = M$ and $f^{-1}(\emptyset) = \emptyset$. Hence f is continuous for any τ.

Theorem 7.1. *Let M and N be two M-topological spaces and $f : M \to N$ be an mset function. Then the following are equivalent:*

1. *The mset function f is continuous,*
2. *For every submset A of M, $C_{f(Cl(A))}(x) \leq C_{Cl(f(A))}(x)$,*
3. *For every closed mset B of N, the mset $f^{-1}(B)$ is a closed mset in M,*
4. *For each $x \in^k M$ and each neighborhood V of $f(k/x)$, there is a neighborhood U of k/x such that $C_{f(U)}(x) \leq C_V(x)$.*

Proof. (1) ⇒ (2) Assume that the mset function f is continuous. Let A be a submset of M. We show that if $x \in^k \mathrm{Cl}(A)$, then $f(k/x) \in^r \mathrm{Cl}(f(A))$ for some r. If V is a neighborhood of $f(k/x)$, then $f^{-1}(V)$ is an open mset of M containing k/x which intersects A in some point n/y. Then V intersects $f(A)$ in the point $f(n/y)$ and $f(k/x) \in^r \mathrm{Cl}(f(A))$ for some r.

(2) ⇒ (3) Let B be a closed mset in N and let $A = f^{-1}(B)$. We wish to prove that A is a closed mset in M; we show that $\mathrm{Cl}(A) = A$. We have $f(A) = f(f^{-1}(B)) \subseteq B$. Therefore, if $x \in^k \mathrm{Cl}(A)$, then $f(k/x) \in^r f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A)) \subseteq \mathrm{Cl}(B) = B$. So that $x \in^k f^{-1}(B) = A$. Thus $\mathrm{Cl}(A) \subseteq A$, so that $\mathrm{Cl}(A) = A$.

(3) ⇒ (1) Let V be an open mset of N. Set $B = N \ominus V$. Then $f^{-1}(B) = f^{-1}(N) \ominus f^{-1}(V) = M \ominus f^{-1}(V)$. Now since B is a closed mset of N, $f^{-1}(B)$ is a closed mset in M by hypothesis so that $f^{-1}(V)$ is an open mset in M.

(1) ⇒ (4) Let $x \in^k M$ and let V be a neighborhood of $f(k/x)$. Then the mset $U = f^{-1}(V)$ is a neighborhood of k/x such that $f(U) \subseteq V$.

(4) ⇒ (1) Let V be an open mset of N and k/x be a point of $f^{-1}(V)$. Then $f(k/x) \in^r V$ for some r, so by hypothesis there is a neighborhood U_x of k/x such that $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open msets U_x. Thus $f^{-1}(V)$ is an open mset of M and f is continuous. □

Theorem 7.2. *If M, N and P are M-topological spaces and $f : M \to N$ and $g : N \to P$ are continuous mset functions, then its composition $g \circ f : M \to P$ is a continuous mset function.*

Proof. If H is an open mset in P, then $g^{-1}(H)$ is an open mset in N by continuity of g. Now again by continuity of $f, f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is an open mset in M. Thus $g \circ f$ is a continuous mset function. □

Remark 7.1.

1. In general topology, discrete topology is the set of all subsets of X and clearly it contains 2^n elements where n is the cardinality of X. But in an M-topology, discrete M-topology $P^*(M)$ is the support set of the power mset of M in $[X]^w$ and it contains $\prod_{i=1}^{n}(1 + m_i) < 2^n$ elements where m_i is the occurrence of an element x_i in the mset M and n is the cardinality of the mset M.
2. In general topology any function $f : X \to Y$ is continuous if X has the discrete topology and Y has any topology. But in the case of M-topological spaces, every mset function $f : M \to N$ is continuous whenever M-topology of M in $[X]^w$ contains $\prod_{i=1}^{n}(1 + m_i) < 2^n$ elements and for any M-topology of N in $[X]^w$ where m_i is the occurrence of an element x_i in the multiset M and n is the cardinality of the multiset M.

8. Conclusion and Future Work

In this paper the authors focus on topology of multisets. This work extends the theory of general topology on general sets to multisets. It begins with a brief survey of the notion of msets introduced by Yager, different types of collections of msets and operations under such collections. It also gives the definition of mset relation and mset function introduced by the authors. After presenting the preliminaries and basic definitions the authors introduced the notion of M-topological space. Basis, sub basis, closure, interior and limit points of multisets are defined and some of the existing theorems are proved in the context of multisets. Finally the authors have established the relationship between continuous function and discrete topology in the context of M-topological space.

The concept of topological structures and their generalizations is one of the most powerful notions in branches of science such as chemistry, physics and information systems. In most applications the topology is employed out of a need to handle the qualitative information. In any information system, some situations may occur, where the respective counts of objects in the universe of discourse are not single. In such situations we have to deal with collections of information in which duplicates are significant. In such cases multisets play an important role in processing the information. The information system dealing with multisets is said to be an information multisystem. Thus, information multisystems are more compact when compared to the original information system. In fact, topological structures on multisets are generalized methods for measuring the similarity and dissimilarity between the objects in multisets as universes. The theoretical study of general topology on general sets in the context of multisets can be a very useful theory for analyzing an information multisystem.

Most of the theoretical concepts of multisets come from combinatorics. Combinatorial topology is the branch of topology that deals with the properties of geometric figures by considering the figures as being composed of elementary geometric figures. The combinatorial method is used not only to construct complicated figures from simple ones but also to deduce the properties of the complicated from the simple. In combinatorial topology it is remarkable that the only machinery to make deductions is the elementary process of counting. In such situations we may deal with collections of elements with duplicates. The theory of M-topology may be useful for studying combinatorial topology with collections of elements with duplicates.

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