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GPS Satellite Range and Relative Velocity Computation

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Abstract

In this work the estimation of a Global Positioning System satellite orbit is considered. The range and relative velocity of the satellite is computed in the observer's local reference frame (topocentric system of coordinates) by including the Earth gravitational perturbations (up to J_3 term) and the solar radiation pressure. Gauss perturbation equations are used to obtain the orbital elements as a function of time, from which the position vector is derived.

Keywords: GPS Satellite, Gauss Equations, Solar Radiation Pressure, Range.

1. Introduction

Global Positioning System (GPS) satellites are used in a variety of applications such as wireless locations, navigation, GPS/INS integrations, as well in attitude and orbit estimation (Mikhailov & Vasilév, 2011). GPS satellite orbits are at an altitude of 25,000 km, with eccentricity ranging from 0.001 to 0.02, and inclined at 55°. At such high altitude the atmospheric drag can be disregarded and the dominant forces affecting the orbital motion are the gravitational and the Solar Radiation Pressure (SRP). Reference (Stelian, 2007) has used fourth-order Runge-Kutta algorithm to numerically integrate the GPS satellite perturbed orbit showing that the most dominant orbital perturbation is the Earth oblateness, namely the so called J_2 term of the Earth gravitational potential.

In this work the J_2 and J_3 orbital gravitational perturbations are considered as well as the solar radiation pressure. Gaussian planetary differential equations are integrated to quantify the effects of the perturbations in the orbital elements. The time-varying orbital elements are obtained by rewriting the Gaussian planetary equations in the orbital coordinate system. Then, from the ephemerides the GPS satellite position and velocity can be evaluated at any time and in any reference coordinate system. In particular, position and velocity vectors can be computed in a ground station reference frame, from where the satellite is observed. This transformation implies the evaluation of the geodetic latitude to consider the Earth an oblate spheroid. The GPS satellite position and velocity are then evaluated in the Earth-Centered-Inertial (ECI) reference

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frame and then transformed into the topo-centric ground station coordinate system. The final purpose of this study is to quantify the variation in the GPS satellite range (as seen by an observer in the ground station) due to the J_2 and J_3 orbital gravitational and solar pressure perturbations.

2. Coordinate system used

To quantify the range rate effect due to orbital perturbation in the ground reference frame, four coordinates systems are adopted. There are shown in figure 1.

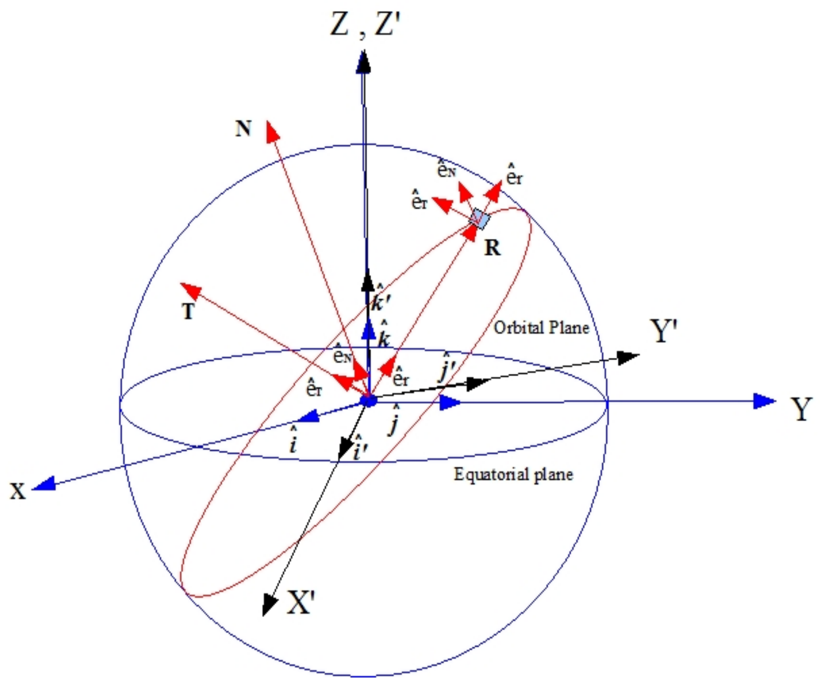

Figure 1. Coordinate systems.

(i) The Earth Centered Inertial ECI coordinate system $OXYZ$. In this system the X-axis is directed toward the vernal Equinox, the Y-axis is in the equatorial plane and normal to the X-axis, and the Z-axis is directed along the rotation axis of the Earth (i.e. normal to the equatorial plane). The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are taken in the directions of the X-axis, Y-axis and Z-axis respectively. (ii) The Earth Centered Earth Fixed $ECEF$ coordinate system $OXYZ$. In this system the X-axis is directed to- ward Greenwich, the Y-axis is in the equatorial plane and normal to the X-axis, and the Z-axis is directed along the rotation axis of the Earth. The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are taken in the directions of the X-axis, Y-axis and Z-axis respectively. (iii) The Orbital Coordinate $ORTN$ coordinate system . In this system the R -axis is directed along the radius vector of the satellite, the T -axis is in the local orbital plane and normal to the R-axis, and the N-axis is normal to the orbital plane. The unit vectors $\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_N$ are taken in the directions of the R-axis, T-axis and N-axis respectively. (iv) The Topocentric Horizon (SEZ) coordinate system. In this system the fundamental plane is the observer's horizon plane, the positive x-axis is directed in the south direction, the y-axis is directed toward the East and z-axis is directed toward the observer's zenith.

3. Earth's oblateness

Earth is an oblate spheroid. A truncated gravitational potential up J_3 is given by:

$$U_g = -\frac{\mu R_\oplus^2}{r^3}\left[J_2\left(1-\frac{3}{2}\sin^2\phi\right)+J_3\frac{R_\oplus}{r}\left(5\sin^3\phi-3\sin\phi\right)\right], \tag{3.1}$$

where μ is the gravitational constant and ϕ is the angle between the Earth's spin axis and satellite radius.

The gradient of this potential gives the perturbing gravitational force in ECI (see (Schaub & Junkins, 2009))

$$\begin{aligned}\mathbf{F}_g &= -\frac{3}{2}J_2\left(\frac{\mu}{r^3}\right)\left(\frac{R_\oplus}{r}\right)\begin{Bmatrix}(1-5\sin^2\phi)x\\(1-5\sin^2\phi)y\\(3-5\sin^2\phi)z\end{Bmatrix}+\\ &\quad -\frac{1}{2}J_3\left(\frac{\mu}{r^3}\right)\left(\frac{R_\oplus}{r}\right)^3\begin{Bmatrix}5(7\sin^3\phi-3\sin\phi)x\\5(7\sin^3\phi-3\sin\phi)y\\(-105\sin^4\phi+30\sin^2\phi-3)z\end{Bmatrix}\end{aligned} \tag{3.2}$$

and this force is expressed in the orbital frame as

$$\mathbf{F}_g = F_R\,\hat{\mathbf{e}}_R + F_T\,\hat{\mathbf{e}}_T + F_N\,\hat{\mathbf{e}}_N, \tag{3.3}$$

where, by setting $S_\bullet = \sin(\bullet)$, and $C_\bullet = \cos(\bullet)$, and $\theta = \omega + f$, the expressions of F_R, F_T, and F_N are

$$F_R = -3\mu\frac{R_\oplus^2}{r^4}\left[\frac{J_2}{2}(1-3S_i^2S_\theta^2)+J_3\frac{R_\oplus}{r}(-15S_iS_\theta-3S_i^2S_\theta^2+40S_i^3S_\theta^3+30S_i^4S_\theta^4-70S_i^5S_\theta^5)\right]$$

$$F_T = -3\mu\frac{R_\oplus^2}{r^4}\left\{J_2S_i^2S_\theta C_\theta+J_3\frac{R_\oplus}{r}\left[5S_i(C_i^2+S_i^2S_\theta^2)(-3+7S_i^2S_\theta^2)-S_i^2C_\theta^2(3-30S_i^2S_\theta^2+35S_i^4S_\theta^4)\right]\right\}$$

$$F_N = -3\mu\frac{R_\oplus^2}{r^4}\left\{J_2S_iS_\theta C_i+J_3\frac{R_\oplus}{r}\left[S_i^2(-15S_i^3S_\theta^3)+C_i^2(-3+30S_i^2S_\theta^2-35S_i^4S_\theta^4)\right]\right\},$$

where i is the orbit inclination, ω the argument of perigee, f the true anomaly, and $\hat{\mathbf{e}}_R$, $\hat{\mathbf{e}}_T$, and $\hat{\mathbf{e}}_N$ are the unit-vectors of the orbital reference frame axes.

4. Solar radiation pressure

A simplified expression for SRP acceleration vector was given in (Schaub & Junkins, 2009) by

$$\mathbf{a} = -C_R\,P_\odot\,S\,m\,\frac{\mathbf{r}_{s\odot}}{r_{s\odot}^3} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{i}} + a_z\,\hat{\mathbf{k}},$$

where $P_\odot \approx 4.56\cdot 10^{-6}$ Nm^{-2} is the solar radiation pressure coefficient, S is the surface area, and m the satellite mass, $\mathbf{r}_{s\odot} = \mathbf{r}_\odot - \mathbf{r}$ is the position vector of the Sun with respect to the satellite, and C_R is the radiation pressure coefficient, which is a function of the reflectivity coefficient, ϵ. The reflectivity coefficient becomes $\epsilon = 0$ when the satellite surface absorbs all the solar radiation while it becomes $\epsilon = 1$ when it reflects all the solar radiation.

Using a pseudo potential function, The acceleration components of the SRP can be expressed in the ECI frame as

$$\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix} = -\frac{C_RP_\odot S}{m\,r_{s\odot}}\begin{Bmatrix}x_\odot-x\\y_\odot-y\\z_\odot-z\end{Bmatrix}\qquad\text{where}\qquad\begin{Bmatrix}x_\odot\\y_\odot\\z_\odot\end{Bmatrix} = r_{s\odot}\begin{Bmatrix}\cos\lambda_\odot\\\sin\lambda_\odot\,\cos\varepsilon\\\sin\lambda_\odot\,\sin\varepsilon\end{Bmatrix}, \tag{4.1}$$

where $\lambda_\odot$ is the sun ecliptic longitude and ε is the obliquity of the ecliptic.

Equations (4.1) are transformed to orbital coordinate system using the transformation

$$\begin{bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_T \\ \hat{\mathbf{e}}_N \end{bmatrix}^{\mathbf{T}} = R_{313} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix}^{\mathbf{T}},$$

where R_{313} is the transformation matrix that can be expressed by the "3-1-3" Euler sequence

$$R_{313} = R_3(\omega + f)\, R_1(i)\, R_2(\Omega).$$

So that the SRP force can be expressed as three components in the directions of $(\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_N)$ coordinate system as a_R, a_T, a_N.

5. Perturbed motion

In case of unperturbed motion, the angles ω, Ω, and i are constant. These angles are used in the transformation equations between coordinate systems and also can be used to determine the position and velocity of the satellite at any given time. The orbit of the satellite undergoes perturbations from several environmental forces resulting in changes in the elements of the orbits.

The rates of change of the orbital elements $(a, e, i, \omega, \Omega, M)$ due to a perturbing force

$$\mathbf{F} = F_R\,\hat{\mathbf{e}}_R + F_T\,\hat{\mathbf{e}}_T + F_N\,\hat{\mathbf{e}}_N \tag{5.1}$$

are given in (Guochang, 2008) and called Gaussian planetary equations. These equations are:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}[e\cos f\, F_R + (1 + e\cos f)\, F_T)] \tag{5.2}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}[\sin f\, F_R + (\cos E + \cos f)\, F_T] \tag{5.3}$$

$$\frac{di}{dt} = \frac{(1 - e\cos E)\cos(\omega + f)}{na\sqrt{1-e^2}}\, F_N \tag{5.4}$$

$$\frac{d\Omega}{dt} = \frac{(1 - e\cos E)\sin(\omega + f)}{na\sqrt{1-e^2}\sin i}\, F_N \tag{5.5}$$

and

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left(-\cos f\, F_R + \sin f\frac{2 + e\cos f}{1 + e\cos f}F_T\right) - \cos i\frac{d\Omega}{dt}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{nae}\left[-\left(\cos f - \frac{2e}{1 + e\cos f}\right)F_R + \sin f\frac{2 + e\cos f}{1 + e\cos f}F_T\right],$$

where a is the semi-major axis, e is the eccentricity of the orbit, n is the mean motion, E is the eccentric anomaly, and M is the mean anomaly. We solve this system of differential equations to get the elements $(a, e, i, \Omega, \omega, M)$ as functions of time. Having these elements one can find the position and velocity at any time. The angles (i, Ω, ω) are needed for the transformations between coordinate systems. We need to compute the radius vector $\mathbf{r}$ in the ECI reference frame.

6. Position vector of the ground station

6.1. Position of the ground station in ECI frame

Assuming the Earth is an oblate spheroid, the position vector of the station in the *ECI* frame has the components :

$$\begin{aligned} R_i &= (N + H) \cos\lambda_E \cos\theta, \\ R_j &= (N + H) \cos\lambda_E \sin\theta, \\ R_k &= (N(1 - e_E^2) + H) \sin\lambda_E, \end{aligned}$$

where $N = \dfrac{a_E}{\sqrt{1 - e_E^2 \sin^2\lambda_E}}$, is the Earth's mean radius, λ_E is the geodetic longitude of the station and H is the height of the station.

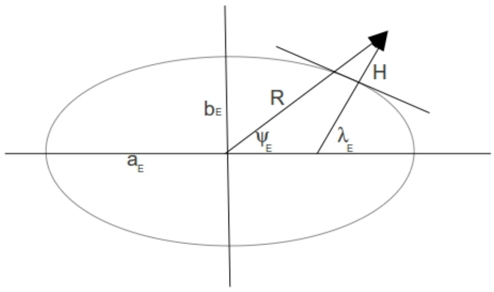

Figure 2. Ground Station Geodetic Coodinates.

Earth rotates around the $\hat{\mathbf{k}}$-axis with angular velocity $\omega_\oplus = 7.2921158553 \cdot 10^{-5}$ rad/s. The angle θ between the $\hat{\mathbf{i}}$-axis and the $\hat{\mathbf{i}}'$-axis is a function of time and is related to $\omega_\oplus$ by

$$\alpha(t) = \alpha_0 + \omega_\oplus(t - t_0).$$

The angle α, called *Greenwich hour angle*, is the right ascension of the Greenwich meridian.

6.2. Satellite range

The range of the satellite is given by

$$\boldsymbol{\rho} = \mathbf{r}_{sat} - \mathbf{R}_{station}.$$

We have described both $\mathbf{r}_{sat}$ and $\mathbf{R}_{station}$ in the *ECI* frame. Now we need to have an expression of this range as seen in the observer's Topocentric Horizon coordinate system (local, on the Earth surface). In this reference rame the fundamental plane is the observer's horizon plane, the positive $\hat{\mathbf{x}}$-axis is taken in the South direction, the $\hat{\mathbf{y}}$-axis is pointing toward the East, and $\hat{\mathbf{z}}$-axis pointing toward the observer's Zenith. The frame is referred to as SEZ frame.

The transformation of the range vector from the *ECI* frame to the SEZ frame is done using the transformation equation

$$\boldsymbol{\rho}_{SEZ} = A_{tp}\, \boldsymbol{\rho}_{ECI}$$

where the transformation matrix is given as

$$A_{tp} = \begin{bmatrix} \sin\psi_E \, \cos\theta & \sin\psi_E \, \sin\theta & -\cos\psi_E \\ -\sin\theta & \cos\theta & 0 \\ \cos\psi_E \, \cos\theta & \cos\psi_E \, \sin\theta & \sin\psi_E \end{bmatrix},$$

where ψ_E is the angle between the radius vector of the station and the semi-major axis of the spheroidal Earth. The magnitude of the range is given by

$$\rho = \sqrt{\rho_S^2 + \rho_E^2 + \rho_Z^2}.$$

The time derivative of the range gives the relative velocity magnitude of the satellite with respect to the station in the observer's local frame (SEZ frame). We have from the above equation

$$v_R = \dot{\rho} = \frac{1}{\rho}(\rho_S \, \dot{\rho}_S + \rho_E \, \dot{\rho}_E + \rho_Z \, \dot{\rho}_Z).$$

7. Numerical example

Considering a GPS satellite with initial values $a = 26,550$ km, $e = 0.02$, $i = 55°$, $s/m = 0.02$ m^2/kg, $\Omega = 0°$, $\omega = 0°$, and $M = 0°$. The period of this GPS Satellite is 12 hours and data are computed for 4 days. Figures 3 through 8 show the perturbation in the elements of the orbit. Figure 9 shows the change in range as seen from the ECI coordinate system and Figure 10 shows the change in range as seen from the topo-centric coordinate system.

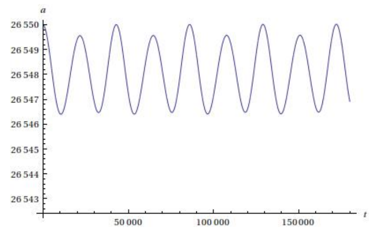

Figure 3. Perturbation in the semi major axis of a GPS satellite.

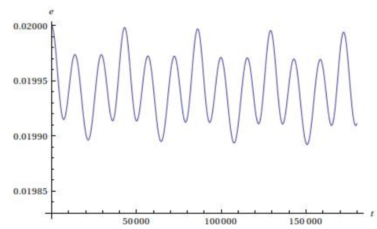

Figure 4. Perturbation in the eccentricity of a GPS satellite.

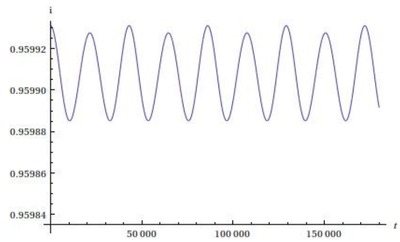

Figure 5. Perturbation in the inclination of a GPS satellite.

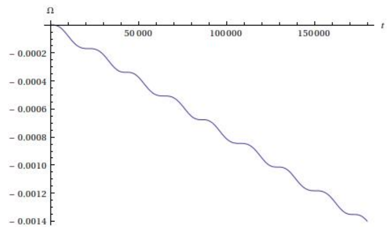

Figure 6. Perturbation in the longitude of ascending node of a GPS satellite.

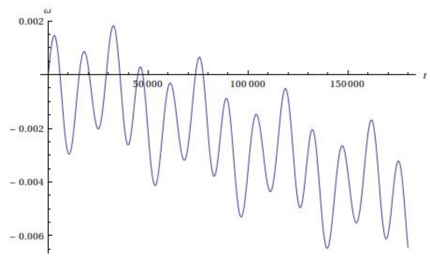

Figure 7. Perturbation in the argument of perigee of a GPS satellite.

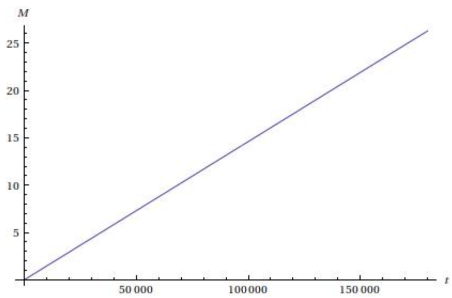

Figure 8. Perturbation in the mean anomaly of a GPS satellite.

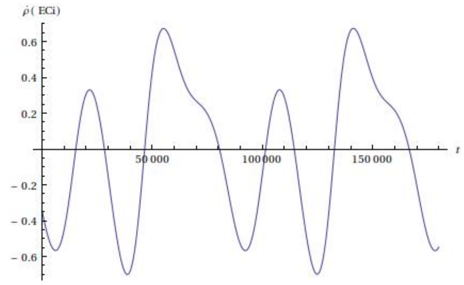

Figure 9. Change in the range as it is seen from the ECI coordinate system.

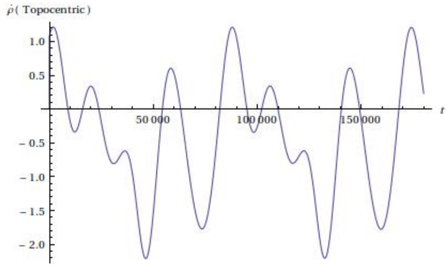

Figure 10. Change in the range as it is seen from the topocentric coordinate system.

8. Conclusion

In this paper we have computed the range and the change in range (relative velocity) of a GPS satellite as seen by an observer in the ground station. The relative motion of the satellite with respect to the ground station is affected by the rotation of the Earth and by the perturbation of the satellite. The GPS satellite's motion was under the effect of the perturbation of the oblateness of the Earth up to J_3 and the perturbation of the Solar Radiation Pressure force.

9. Acknowledgment

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