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A_σ-Double Sequence Spaces and Double Statistical Convergence in 2-Normed Spaces Defined by Orlicz Functions

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Abstract

The main aim of this paper is to introduce a new class of sequence spaces which arise from the notion of invariant means, de la Valee-Pousin means and double lacunary sequence with respect to an Orlicz function in 2-normed space. Some properties of the resulting sequence space were also examined. Further we study the concept of uniformly $(\bar{\lambda}, \sigma)$-statistical convergence and establish natural characterization for the underline sequence spaces.

Keywords: Double sequence spaces, 2-normed space, Double statistical convergence, Orlicz function.
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1. Introduction

Let l_∞ and c denote the Banach spaces of bounded and convergent sequences $x = (x_i)$, with complex terms respectively, normed by $\|x\|_\infty = \sup_i |x_i|$, where $i \in \mathbb{N}$. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits that is to say, if and only if, for all $i = 0, j = 0, \sigma^j(i) \neq i$ and T be the operator defined on l_∞ by $(T(x_i)_{i=1}^{\infty}) = (x_{\sigma(i)})_{i=1}^{\infty}$.

A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if

1. $\phi(x) \geq 0$, when the sequence $x = (x_i)$ has $x_i \geq 0$ for all i,
2. $\phi(e) = 1$, where $e = \{1, 1, 1, ...\}$ and
3. $\phi(x_{\sigma(i)}) = \phi(x)$ for all $x \in l_\infty$.

The space $[V_\sigma]$ of strongly $\sigma-$convergent sequence was introduced by Mursaleen in (Mursaleen, 1983). A sequence $x = (x_k)$ is said to be strongly $\sigma-$convergent if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k} |x_{\sigma^j}(m) - L| \to \infty$$

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as $k \to \infty$ uniformly in m. If we take $\sigma(m) = m + 1$ then $[V_\sigma] = [\hat{c}]$, which was defined by Maddox in (Maddox, 1967).

If $x = (x_i)$ write $Tx = (Tx_i) = (x_{\sigma(i)})$. It can be shown that

$$V_\sigma = \left\{x = (x_i) : \sum_{m=1}^{\infty} t_{m,i}(x) = L \text{ uniformly in i, } L = \sigma - \lim x\right\} \tag{1.1}$$

where $m \geq o, i > 0$.

$$t_{m,i}(x) = \frac{x_i + x_{\sigma(i)} + ... + x_{\sigma^m(i)}}{m+1} \text{ and } t_{-1,i} = 0, \tag{1.2}$$

where, $\sigma^m(i)$ denote the mth iterate of $\sigma(i)$ at i. In the case σ is the translation mapping, $\sigma(i) = i + 1$ is often called a Banach limit and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequence(see (Móricz & Rhoades, 1988)). Subsequently invariant means have been studied by Ahmad and Mursaleen in (Ahmad & Mursaleen, 1988), (Raimi, 1963) and many others.

The concept of 2-normed spaces was initially introduced by (Gähler, 1963) in the mid of 1960's. Since then, many researchers have studied this concept and obtained various results, see for instance (Gähler, 1965, 1964; Gunawan & Mashadi, 2001).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies the following four conditions (Khan & Tabassum, 2011*b*, 2010):

- **(i)** $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent;
- **(ii)** $\|x_1, x_2\| = \|x_2, x_1\|$;
- **(iii)** $\|\alpha x_1, x_2\| = \alpha\|x_1, x_2\|$, for any $\alpha \in R$;
- **(iv)** $\|x + x', x_2\| \leq \|x, x_2\| + \|x', x_2\|$.

The pair $(X, \|.,.\|)$ is then called a 2-normed space.

Exemple 1.1. A standard example of a 2-normed space is R^2 equipped with the following 2-norm: $\|x, y\| :=$ the area of the triangle having vertices $0, x, y$.

Exemple 1.2. Let Y be a space of all bounded real-valued functions on R. For f, g in Y, define $\|f, g\| = 0$ if f, g are linearly dependent, $\|f, g\| = \sup_{t \in R} |f(t).g(t)|$, if f, g are linearly independent. Then $\|.,.\|$ is a 2-norm on Y.

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M satisfies the $\Delta_2 - condition$ ($M \in \Delta_2$ for short) if there exist constant $K \geq 2$ and $u_0 > 0$ such that $M(2u) \leq KM(u)$ whenever $|u| \leq u_0$.

An Orlicz function M can always be represented in the integral form $M(x) = \int_0^x q(t)dt$, where q known as the kernel of M, is right differentiable for $t \geq 0, q(t) > 0$ for $t > 0, q$ is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1,$$

since M is convex and $M(0) = 0$.

Lindesstrauss and Tzafriri in (Lindenstrauss & Tzafriri, 1971) used the idea of Orlicz sequence space;

$$l_M := \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\},$$

which is Banach space with the norm

$$\|x\|_M = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

Throughout $x = (x_{jk})$ is a double sequence that is a double infinite array of elements x_{jk}, for $j, k \in \mathbb{N}$.

Double sequence have been studied by Vakeel A. Khan and S. Tabassum in (Khan, 2010; Khan & Tabassum, 2012, 2011*b*,*a*, 2010) and many others.

The following inequality will be used throughout

$$|x_{jk} + y_{jk}|^{p_{jk}} \le D(|x_{jk}|^{p_{jk}} + |y_{jk}|^{p_{jk}}), \tag{1.3}$$

where x_{jk} and y_{jk} are complex numbers, $D = \max(1, 2^{H-1})$ and $H = \sup_{j,k} p_{jk} < \infty$.

Definition 1.1. A double sequence $x = (x_{jk})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \epsilon$ whenever $j, k > N$. We shall describe such an x more briefly as $P - convergent$.

Definition 1.2. (Savaş & Patterson, 2007) The four dimensional matrix $A = (a_{m,n,j,k})$ is said to be RH-regular if it maps every bounded P-convergent sequences into a P-convergent sequence with the same P-limit.

Theorem 1.3. *(Savaş & Patterson, 2007) The four dimensional matrix $A = (a_{m,n,j,k})$ is said to be RH-regular if and only if*

- **(i)** $P - \lim_{m,n} a_{m,n,j,k} = 0$ *for each* j, k;
- **(ii)** $P - \lim_{m,n} \sum_{j,k=1}^{\infty} a_{m,n,j,k} = 1$;
- **(iii)** $P - \lim_{m,n} \sum_{j=1}^{\infty} |a_{m,n,j,k}| = 0$; *for each* k;
- **(iv)** $P - \lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,j,k}| = 0$; *for each* j;
- **(v)** $\sum_{j,k=1}^{\infty} |a_{m,n,j,k}|$ *is P-convergent and*
- **(vi)** *there exist positive numbers A and B such that* $\sum_{j,k>B} |a_{m,n,j,k}| < A$.

2. Main Results

Let M be an Orlicz function $P = (p_{jk})$ be any factorable double sequence of strictly positive real numbers. Let $A = (a_{m,n,j,k})$ be a non negative RH-regular summabilty matrix method, $(X, \|., .\|)$ be 2-norm space, σ be an injection of the set of positive integers $\mathbb{N}$ into itself and $p,\ q \in \mathbb{N}$. We define the following double sequence spaces:

$${}_2W_o(A_\sigma, M, p, \|., .\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

$$_2W(A_\sigma, M, p, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X\Big\}$$

$$_2W_\infty(A_\sigma, M, p, \|.,.\|) = \Big\{x = (x_{jk}) : \sup_{m,n,j,k} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} < \infty,$$

$$\text{uniformly in } p,q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

Let us consider a few special cases of above definitions:
(**i**) In particular, when $\sigma(p,q) = (p+1, q+1)$, we have

$$_2W_o(A, M, p, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{j+p,k+q}, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

$$_2W(A, M, p, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{j+p,k+q} - L, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X\Big\}$$

$$_2W_\infty(A, M, p, \|.,.\|) = \Big\{x = (x_{jk}) : \sup_{m,n,j,k} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{j+p,k+q}, z\|}{\rho}\Big)\Big]^{p_{jk}} < \infty,$$

$$\text{uniformly in } p,q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

(**ii**) If $M(x) = x$ then we have

$$_2W_o(A_\sigma, p, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\|x_{\sigma^j(p),\sigma^k(q)}, z\|^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q, \text{ and } z \in X\Big\}$$

$$_2W(A_\sigma, p, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q \text{ and } L > 0, z \in X\Big\}$$

$$_2W_\infty(A_\sigma, p, \|.,.\|) = \Big\{x = (x_{jk}) : \sup_{m,n,j,k} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \|x_{\sigma^j(p),\sigma^k(q)}, z\|^{p_{jk}} < \infty, \text{ uniformly in } p, q, \text{ and } z \in X\Big\}$$

(iii) If $p_{jk} = 1$ for all (j,k), we have

$$_2W_o(A_\sigma, M, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big] = 0, \text{ uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

$$_2W(A_\sigma, M, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\Big)\Big] 0, \text{ uniformly in } p, q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X\Big\}$$

$$_2W_\infty(A_\sigma, M, \|.,.\|) = \Big\{x = (x_{jk}) : \sup_{m,n,j,k} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big] < \infty, \text{ uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}.$$

Definition 2.1. (Savaş & Patterson, 2008) A bounded double sequence $x = (x_{jk})$ of real number is said to be $(\bar{\lambda}, \sigma)$-convergent to L provided that

$$P - \lim_{r,s} T_{r,s}^{p,q} = L \text{ uniformly in } (p, q),$$

where

$$T_{p,q}^{r,s} = \frac{1}{\bar{\lambda}_{r,s}} \sum_{(j,k)\in \bar{I}_{r,s}} x_{\sigma^j(p),\sigma^k(q)}.$$

In this case we write $(\bar{\lambda}, \sigma) - \lim x = L$.

One can see that in contrast to the case for single sequences, a P-convergent sequences need not be $(\bar{\lambda}, \sigma)$−convergent. But it is easy to see that every bounded P-convergent double sequence is $(\bar{\lambda}, \sigma)$−convergent. In addition, if we let $\sigma(p) = p+1, \sigma(q) = q+1$, and $\bar{\lambda}_{r,s} = rs$ in the above definition then $(\bar{\lambda}, \sigma)$−convergence reduces to almost P-convergence which was defined by Moricz and Rhoades in (Móricz & Rhoades, 1988).

Definition 2.2. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non decreasing sequences of positive real numbers both of which tends to ∞ as r, s approach ∞, respectively. Also let $\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 0$ and $\mu_{s+1} \leq \mu_s + 1, \mu_1 = 0$. We write the generalized double de la Valee-Pousin mean by

$$t_{r,s}(x) = \frac{1}{\lambda_r \mu_s} \sum_{j\in I_r} \sum_{k\in I_s} x_{j,k},$$

where $I_r = [r - \lambda_r + 1, r]$ and $I_s = [s - \mu_s + 1, s]$.

We shall denote $\lambda_r\mu_s$ by $\bar{\lambda}rs$ and $(j \in I_r,\ k \in I_s)$ by $(j,k) \in \bar{I}_{r,s}$. Let M be an Orlicz function, x_{jk} be double sequence space and $p = (p_{jk})$ be any factorable double sequence of strictly positive real numbers. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be the same as defined above and $(X, \|.,.\|)$ be 2-norm space. If we take

$$a_{r,s,j,k} = \begin{cases} \frac{1}{\bar{\lambda}rs} \text{ if } & (j,k) \in \bar{I}_{r,s}, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_o = \Big\{x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

$$[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|] = \Big\{x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p, q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X\Big\}$$

$$[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_\infty = \Big\{x = (x_{jk}) : \sup_{r,s,p,q} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} < \infty,$$

$$\text{for some } \rho > 0 \text{ and } z \in X\Big\}.$$

Definition 2.3. The double lacunary sequence was defined by E. Savaş and R. F. Patterson (Savaş & Patterson, 1994) as follows:
The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations : $k_{r,s} = k_r l_s,\ \ h_{r,s} = h_r \bar{h}_s$.
The following intervals are determined by θ :

$$I_r = \{(k_r) : k_{r-1} < k < k_r\}, I_s = \{(l) : l_{s-1} < l < l_s\},$$

$$I_{r,s} = \{(k,l) : k_{r-1} < k < k_r \text{ and } l_{s-1} < l < l_s\},$$

$q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r\bar{q}_s$. We will denote the set of all double lacunary sequences by $N_{\theta_{r,s}}$. The space of double lacunary strongly convergent sequence is defined as follows

$$N_{\theta_{r,s}} = \Big\{x = (x_{k,l}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k,l} - L| = 0 \text{ for some } L\Big\}$$

see (Savaş & Patterson, 1994).
If we take

$$a_{r,s,j,k} = \begin{cases} \frac{1}{\bar{h}_{rs}} & \text{if } (j,k) \in I_{r,s}, \\ 0 & \text{otherwise }. \end{cases}$$

We have

$$[{}_2W_\sigma, \theta, M, p, \|.,.\|]_o = \Big\{ x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{h}rs} \sum_{(j,k)\in I_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho} \Big) \Big]^{p_{jk}} = 0,$$
$$\text{uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X \Big\}$$

$$[{}_2W_\sigma, \theta, M, p, \|.,.\|] = \Big\{ x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{h}rs} \sum_{(j,k)\in I_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \Big) \Big]^{p_{jk}} = 0,$$
$$\text{uniformly in } p, q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X \Big\}$$

$$[{}_2W_\sigma, M, \theta, p, \|.,.\|]_o = \Big\{ x = (x_{jk}) : \sup_{r,s,p,q} \frac{1}{\bar{h}rs} \sum_{(j,k)\in I_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho} \Big) \Big]^{p_{jk}} < \infty,$$
$$\text{for some } \rho > 0 \text{ and } z \in X \Big\}.$$

Theorem 2.1. *Let $P = p_{jk}$ be bounded. Then ${}_2W(A_\sigma, M, p, \|.,.\|)$, ${}_2W_o(A_\sigma, M, p, \|.,.\|)$ and ${}_2W_\infty(A_\sigma, M, p, \|.,.\|)$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.2. *Let $P = p_{jk}$ be bounded. Then $[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_o$, $[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]$ and $[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_{jk})$ and $y = (y_{jk}) \in [{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_o$ and $\alpha, \beta \in \mathbb{C}$ then there exist two positive numbers ρ_1, ρ_2 such that

$$P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_1} \Big) \Big]^{p_{jk}} = 0,$$

$$P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_2} \Big) \Big]^{p_{jk}} = 0,$$

uniformly in (p, q). Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex, we have

$$\frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|\alpha x_{\sigma^j(p),\sigma^k(q)} + \beta y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_3} \Big) \Big]^{p_{jk}} = \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|\alpha x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_3} + \frac{|\beta y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_3} \Big) \Big]^{p_{jk}}$$

$$\leq \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \frac{1}{2^{p_{jk}}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_1} \Big) + M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_2} \Big) \Big]^{p_{jk}}$$

$$\leq \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_1} \Big) + M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_2} \Big) \Big]^{p_{jk}}$$

$$\leq D\frac{1}{\bar{\lambda}rs}\sum_{(j,k)\in \bar{I}_{r,s}}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)},z\|}{\rho_1}\Big)\Big]^{p_{jk}}+D\frac{1}{\bar{\lambda}rs}\sum_{(j,k)\in \bar{I}_{r,s}}\Big[M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)},z\|}{\rho_2}\Big)\Big]^{p_{jk}}.$$

(From equation (1.1).)

Now since the last inequality tends to zero as (r,s) approaches in Pringsheim sense, uniformly in (p,q), $[{}_2V_\sigma,\bar{\lambda},M,p,\|.,.\|]_o$ is linear. The proof of others follow in similar manner. □

Theorem 2.3. *Let $P = p_{jk}$ be bounded. Then $[{}_2W_\sigma,\theta,M,p,\|.,.\|]_o$, $[{}_2W_\sigma,\theta,M,p,\|.,.\|]$ and $[{}_2W_\sigma,\theta,M,p,\|.,.\|]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.4. *Let A be non negative RH regular summability matrix method and M be an Orlicz function which satisfies $\triangle_2$ condition. Then ${}_2W_o(A_\sigma,p,\|.,.\|)\subset {}_2W_o(A_\sigma,M,p,\|.,.\|)$, ${}_2W(A_\sigma,p,\|.,.\|)\subset {}_2W(A_\sigma,M,p,\|.,.\|)$ and ${}_2W(A_\sigma,p,\|.,.\|)_\infty\subset {}_2W(A_\sigma,M,p,\|.,.\|)_\infty$.*

Proof. Let $x=(x_{jk})\in {}_2W(A_\sigma,p,\|.,.\|)$, then

$$P-\lim_{m,n}\sum_{j,k=0}^{\infty}a_{m,n,j,k}\|x_{\sigma^j(p),\sigma^k(q)},z\|^{p_{jk}}\to 0, \tag{2.1}$$

as $m,n\to\infty$ uniformly in (p,q). Let $\epsilon>0$ and choose $0<\delta<1$ such that $M(t)<\frac{\epsilon}{2}$ for $0\leq t\leq\delta$. Write $y_{jk}=\|x_{\sigma^j(p),\sigma^k(q)},z\|$ and consider

$$\sum_{j,k=0}^{\infty}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}=\sum_{1}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}+\sum_{2}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}.$$

Where the first summation is over $y_{jk}\leq\delta$ and the second summation is over $y_{jk}>\delta$. Since M is continuous, we have

$$\sum_{1}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}\leq\epsilon^H\sum_{j,k=0}^{\infty}a_{m,n,j,k}.$$

For $y_{jk}>\delta$, we use the fact that

$$y_{jk}<\frac{y_{jk}}{\delta}\leq 1+\Big(\frac{y_{jk}}{\delta}\Big).$$

Since M is non decreasing and convex, it follows that

$$M(y_{jk})<M(1+\delta^{-1}y_{jk})=M\Big(\frac{2}{2}+\frac{2}{2}\delta^{-1}y_{jk}\Big)<\frac{1}{2}M(2)+\frac{1}{2}M(2\delta^{-1}y_{jk}).$$

Since M satisfies Δ_2-condition, there is a constant $K>2$ such that

$$M(2\delta^{-1}y_{jk})\leq\frac{1}{2}K\delta^{-1}y_{jk}M(2).$$

Hence

$$\sum_{2}[M(y_{jk})]^{p_{jk}}<\max(1,(K\delta^{-1}M(2)))\sum_{j,k=0}^{\infty}[M(y_{jk})]^{p_{jk}}.$$

Thus we have

$$\sum_{(j,k=0)}^{\infty}[M(y_{jk})]^{p_{jk}}<\max(1,\epsilon^H)\sum_{j,k=0}^{\infty}a_{m,n,j,k}+\max(1,(K\delta^{-1}M(2)))\sum_{j,k=0}^{\infty}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}.$$

Thus (2.1) and R-H Regularity of A grants us ${}_2W(A_\sigma,p,\|.,.\|)\subset {}_2W(A_\sigma,M,p,\|.,.\|)$. Similarly we can prove the other two inclusion relations. □

3. Double Statistical Convergence

The concept of statistical convergence was first introduced by Fast in (Fast, 1951) and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Šalát (Tibor, 1980), Fridy in (Fridy, 1985) and many others.

Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is called statistically convergent to L if

$$\lim_n \frac{1}{n} |k : |x_k - L| \geq \epsilon, k \leq n| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$.

The following definition was presented by Mursaleen in (Mursaleen, 2000). A sequence x is said to be $\lambda-$statistical convergent to L, if for $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} |k \in I_n : |x_k - L| \geq \epsilon, k \leq n| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set and $I_n = [n - \lambda_n + 1, n]$. In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$.

Savaş (Savaş, 2000) presented and studied the concepts of uniformly $\lambda-$statistical convergence as follows: A sequence x is said to uniformly $\lambda-$statistical convergent to L , if for $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \max_m |k \in I_n : |x_{k+m} - L| \geq \epsilon| = 0.$$

In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(_\lambda)$.

A double sequence (x_{jk}) is called statistically convergent to L if

$$\lim_{m,n\to\infty} \frac{1}{mn} |(j,k) : |x_{jk} - L| \geq \epsilon, j \leq m, k \leq n| = 0,$$

where the vertical bars indicate the number of elements in the set.(see[10])

Definition 3.1. (Savaş & Patterson, 2008) A double sequence $x = (x_{jk})$ is said to be uniformly $(\bar{\lambda}, \sigma)$-statistical convergent to L, provided that for every $\epsilon > o$

$$P - \lim_{r,s} \frac{1}{\lambda_{rs}} \max_{p,q} |\{(j,k) \in \bar{I}_{r,s} : |x_{\sigma^j(p),\sigma^k(q)} - L| \geq \epsilon\}| = 0.$$

In this case we write ${}_2S_{(\bar{\lambda},\sigma)} - \lim x = L$ or $x_{jk} \to L({}_2S_{(\bar{\lambda},\sigma)})$.

Theorem 3.1. *Let M be an Orlicz Function and* $0 < h = \inf p_{jk} \leq p_{jk} \leq \sup_{j,k} p_{jk} = H < \infty$ *then* $[{}_2V_\sigma, \bar{\lambda}, M, p, ||.,.||] \subset {}_2S_{(\bar{\lambda},\sigma)}$.

Proof. Let $x = (x_{jk}) \in [{}_2V_\sigma, \bar{\lambda}, M, p, ||.,.||]$. Then there exists $\rho > 0$ such that

$$\frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in\bar{I}_{r,s}} \left[M\left(\frac{||x_{\sigma^j(p),\sigma^k(q)} - L, z||}{\rho} \right) \right]^{p_{jk}} = 0,$$

as $r, s \to \infty$ in the Pringsheim sense uniformly in (p, q).

If $\epsilon > 0$ and let $\epsilon_1 = \frac{\epsilon}{\rho}$, then we obtain the following:

$$\frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}} = \frac{1}{\bar{\lambda}rs} \sum_{1} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}}$$

$$+ \frac{1}{\bar{\lambda}rs} \sum_{2} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}}$$

Where the first summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| \geq \epsilon$ and the second summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| < \epsilon$

$$\geq \frac{1}{\bar{\lambda}rs} \sum_{1} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}} \geq \frac{1}{\bar{\lambda}rs} \sum_{1} [M(\epsilon_1)]^{p_{jk}} \geq \frac{1}{\bar{\lambda}rs} \sum_{1} \min\{[M(\epsilon_1)]^{p_{jk}}, [M(\epsilon_1)]^{H}\}$$

$$\geq \frac{1}{\bar{\lambda}rs} |\{(j,k) \in \bar{I}_{r,s} : |x_{\sigma^j(p),\sigma^k(q)} - L, z| \geq \epsilon\}| \min\{[M(\epsilon_1)]^{h}, [M(\epsilon_1)]^{H}\}.$$

This implies that $x \in {}_2S_{(\bar{\lambda},\sigma)}$. □

Theorem 3.2. *Let* M *be a bounded Orlicz function and* $0 < h = \inf p_{jk} \leq p_{jk} \leq \sup_{j,k} p_{jk} = H < \infty$ *then* ${}_2S_{(\bar{\lambda},\sigma)} \subset [{}_2V_\sigma, \bar{\lambda}, M, p, \|., .\|]$

Proof. Since M is bounded there exists an integer K such that $M(x) < K$ for $x > 0$. Thus

$$\frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}} = \frac{1}{\bar{\lambda}rs} \sum_{1} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}}$$

$$+ \frac{1}{\bar{\lambda}rs} \sum_{2} \left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \right) \right]^{p_{jk}}.$$

Where the first summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| \geq \epsilon$ and the second summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| < \epsilon \leq \frac{1}{\bar{\lambda}rs} \sum_{1} \max\{K^h, K^H\} + \frac{1}{\bar{\lambda}rs} \sum_{2} \left[M\left(\frac{\epsilon}{\rho} \right) \right]^{p_{jk}}$

$$\leq \max\{K^h, K^H\} \frac{1}{\bar{\lambda}rs} |\{(j,k) \in \bar{I}_{r,s} : |x_{\sigma^j(p),\sigma^k(q)} - L, z| \geq \epsilon\}| + \max\left\{ \left[M\left(\frac{\epsilon}{\rho} \right) \right]^{h}, \left[M\left(\frac{\epsilon}{\rho} \right) \right]^{H} \right\}.$$

Hence $x \in [{}_2V_\sigma, \bar{\lambda}, M, p, \|., .\|]$. □

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