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On Some Generalized I-Convergent Sequence Spaces Defined by a Modulus Function

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Abstract

In this article we introduce the sequence spaces $c_0^I(f,p)$, $c^I(f,p)$ and $l_\infty^I(f,p)$ for a modulus function f, $p = (p_k)$ is a sequence of positive reals and study some of the properties of these spaces.

Keywords: Ideal, filter, paranorm, modulus function, I-convergent sequence spaces, Lipschitz function, I-convergence field.
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1. Introduction

Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and ω denotes the set of natural,real,complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H. Fast (Fast, 1951). Later on it was studied by J. A. Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát and Wilczyński (P. Kostyrko & Wilczyński, 2000). Later on it was studied by Šalát, Tripathy and Ziman (T. Šalát & Ziman, 2004, 2005), Esi and Ozdemir (Esi & Ozdemir, 2012), Hazarika and Esi (Esi & Hazarika, 2012) and Demirci (Demirci, 2001).

Here we give some preliminaries about the notion of I-convergence.

Let N be a non empty set. Then a family of sets I$\subseteq 2^N$(power set of N) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

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A non-empty family of sets £$(I) \subseteq 2^N$ is said to be filter on N if and only if $\Phi \notin$ £(I),for $A, B \in$ £(I) we have $A \cap B \in$ £(I) and for each $A \in$ £(I) and $A \subseteq B$ implies $B \in$ £(I).

An Ideal I$\subseteq 2^N$ is called non-trivial if I$\neq 2^N$.

A non-trivial ideal I$\subseteq 2^N$ is called admissible if $\{x : \{x\} \in N\} \subseteq$I.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset. For each ideal I, there is a filter £(I) corresponding to I, i.e £(I) = $\{K \subseteq N : K^c \in I\}$,where K^c = N-K.

Definition 1.1. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write I-lim $x_k = L$.

The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq\} \in I,\ for\ some\ L \in \mathbb{C}\}.$$

Definition 1.2. A sequence $(x_k) \in \omega$ is said to be I-null if L = 0 .In this case we write I-lim $x_k = 0$.

Definition 1.3. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in$ I.

Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in \mathbb{N} : |x_k| > M\}$.

Definition 1.5. Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for *almost all k relative to I (a.a.k.r.I)*, if $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$

Definition 1.6. For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \ \text{ for all } \ x \in E\}$$ (see (Simons, 1965)).

Definition 1.7. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \rightarrow \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$. (Tripathy & Hazarika, 2011).

Definition 1.8. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (See (T. Šalát & Ziman, 2005)).

Define a function $\hbar : F(I) \rightarrow \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \rightarrow \mathbb{R}$ is a Lipschitz function (c.f (Dems, 2005; T. Šalát & Ziman, 2004; Gurdal, 2004; Tripathy & Hazarika, 2009, 2011; Khan, 2005; Khan & Ebadullah, 2011*b*,*a*, 2012; Vakeel. A. Khan & Ahmad, 2012).

Definition 1.9. The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value.
Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,
(PI) $g(x) = 0 \ if \ x = \theta$,
(P2) $g(-x) = g(x)$,
(P3) $g(x + y) \leq g(x) + g(y)$,
(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a \ \ (n \to \infty)$, in the sense that $g(x_n - a) \to 0 \ \ (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0 \ \ (n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space. See (Maddox, 1969).
The idea of modulus was structured in 1953 by Nakano. See (Nakano, 1953).

A function f : [0,∞)⟶[0,∞) is called a modulus if:

(1) f(t) = 0 if and only if t = 0,
(2) f(t+u)≤ f(t)+ f(u) for all t, u≥0,
(3) f is increasing, and
(4) f is continuous from the right at zero.

Ruckle [17-19] used the idea of a modulus function f to construct the sequence space:

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space,and Ruckle[19 - 21] proved that that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[19- 21] proved that, for any modulus f,

$$X(f) \subset l_1 \text{ and } X(f)^{\alpha} = l_{\infty}$$

where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty. \text{ (See[17-19])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch in (Gramsch, n.d.). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D. J. H

Garling (Garling, 1966, 1968),G. Köthe (Köthe, 1970) and W. H. Ruckle ((Ruckle, 1968), (Ruckle, 1967), (Ruckle, 1973)).

The following subspaces of ω were first introduced and discussed by Maddox ((Maddox, 1986), (Maddox, 1969)):

$$l(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\},$$

$$l_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\},$$

$$c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \ for\ some\ l \in \mathbb{C}\},$$

$$c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0\},$$

where $p = (p_k)$ is a sequence of strictly positive real numbers.

After then Lascarides ((Lascarides, 1971, 1983)) defined the following sequence spaces:

$$l_\infty\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty\},$$

$$c_0\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0\ \},$$

$$l\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\},$$

where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

We need the following lemmas in order to establish some results of this article.

Lemma 1.1. *Let* $h = \inf_k p_k$ *and* $H = \sup_k p_k$. *Then the following conditions are equivalent.(See (Lascarides, 1983)).*
(a) $H < \infty$ *and* $h > 0$,
(b) $c_0(p) = c_0$ *or* $l_\infty(p) = l_\infty$,
(c) $l_\infty\{p\} = l_\infty(p)$,
(d) $c_0\{p\} = c_0(p)$,
(e) $l\{p\} = l(p)$.

Lemma 1.2. *Let* $K \in$ £(I) *and* $M \subseteq N$. *If* $M \notin I$, *then* $M \cap K \notin I$. *(See (T. Šalát & Ziman, 2004), (Tripathy & Hazarika, 2011)).*

Lemma 1.3. *If* $I \subset 2^N$ *and* $M \subseteq N$. *If* $M \notin I$, *then* $M \cap K \notin I$. *(See (T. Šalát & Ziman, 2004), (Tripathy & Hazarika, 2011)).*

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded, I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

$$c^I(f,p) = \{(x_k) \in \omega : f(|x_k - L|^{p_k}) \geq \epsilon \text{ for some L}\} \in I,$$

$$c_0^I(f,p) = \{(x_k) \in \omega : f(|x_k|^{p_k}) \geq \epsilon\} \in I,$$

$$l_\infty^I(f,p) = \{(x_k) \in \omega : \sup_k f(|x_k|^{p_k}) < \infty\} \in I.$$

Also we write

$$m^I(f,p) = c^I(f,p) \cap l_\infty(f,p)$$

and

$$m_0^I(f,p) = c_0^I(f,p) \cap l_\infty(f,p).$$

2. Main Results

Theorem 2.1. *Let $(p_k) \in l_\infty$. Then $c^I(f,p), c_0^I(f,p), m^I(f,p)$ and $m_0^I(f,p)$ are linear spaces.*

Proof. Let $(x_k), (y_k) \in c^I(f,p)$ and α, β be two scalars. Then for a given $\epsilon > 0$ we have:

$$\left\{k \in \mathbb{N} : f(|x_k - L_1|^{p_k}) \geq \frac{\epsilon}{2M_1}, for\ some\ L_1 \in \mathbb{C}\right\} \in I$$

$$\left\{k \in \mathbb{N} : f(|y_k - L_2|^{p_k}) \geq \frac{\epsilon}{2M_2}, for\ some\ L_2 \in \mathbb{C}\right\} \in I$$

where

$$M_1 = D.max\{1, \sup_k |\alpha|^{p_k}\}$$

$$M_2 = D.max\{1, \sup_k |\beta|^{p_k}\},$$

and

$$D = max\{1, 2^{H-1}\} \text{ where } H = \sup_k p_k \geq 0.$$

Let

$$A_1 = \left\{k \in \mathbb{N} : f(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}, for\ some\ L_1 \in \mathbb{C}\right\} \in I,$$

$$A_2 = \left\{k \in \mathbb{N} : f(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}, for\ some\ L_2 \in \mathbb{C}\right\} \in I$$

be such that $A_1^c, A_2^c \in I$.
Then

$$
\begin{aligned}
A_3 &= \{k \in \mathbb{N} : f(|(\alpha x_k + \beta y_k) - f(\alpha L_1 + \beta L_2)|^{p_k}) < \epsilon\} \\
&\supseteq \{k \in \mathbb{N} : |\alpha|^{p_k} f(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}|\alpha|^{p_k}.D\} \\
&\cap \{k \in \mathbb{N} : |\beta|^{p_k} f(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}|\beta|^{p_k}.D\}.
\end{aligned}
$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in c^I(f, p)$. Therefore $c^I(f, p)$ is a linear space. The rest of the result follows simililarly. □

Theorem 2.2. *Let $(p_k) \in l_\infty$. Then $m^I(f, p)$ and $m_0^I(f, p)$ are paranormed spaces, paranormaed by $g(x_k) = \sup_k f(|x_k|^{\frac{p_k}{M}})$ where $M = max\{1, \sup_k p_k\}$*

Proof. Let $x = (x_k), y = (y_k) \in m^I(f, p)$.
(1) Clearly, $g(x) = 0$ if and only if $x = 0$.
(2) $g(x) = g(-x)$ is obvious.
(3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality and the definition of f we have:

$$\sup_k f(|x_k + y_k|^{\frac{p_k}{M}}) \leq \sup_k f(|x_k|^{\frac{p_k}{M}}) + \sup_k f(|y_k|^{\frac{p_k}{M}}).$$

(4) Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$.
Let $x_k \in m^I(f, p)$ such that $f(|x_k - L|^{p_k}) \geq \epsilon$.
Therefore, $g(x_k - L) = \sup_k f(|x_k - L|^{\frac{p_k}{M}}) \leq \sup_k f(|x_k|^{\frac{p_k}{M}}) + \sup_k f(|L|^{\frac{p_k}{M}})$.
Hence $g(\lambda_n x_k - \lambda L) \leq g(\lambda_n x_k) + g(\lambda L) = \lambda_n g(x_k) + \lambda g(L)$ as $(k \to \infty)$.
Hence $m^I(f, p)$ is a paranormed space.
The rest of the result follows similarly. □

Theorem 2.3. *A sequence $x = (x_k) \in m^I(f, p)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p). \tag{2.1}$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L|^{p_k} < \frac{\epsilon}{2}\} \in m^I(f, p). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_k|^{p_k} \leq |x_{N_\epsilon} - L|^{p_k} + |L - x_k|^{p_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which holds for all $k \in B_\epsilon$.

Hence $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p)$.

Conversely, suppose that $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(f, p)$ for all $\epsilon > 0$.

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(f, p)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(f, p)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(f, p)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I(f, p)$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J.

In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2} diamI_{k-1}$ for (k = 2, 3, 4, ...) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(f, p)$ for (k=1,2,3,4,......).

Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f(\xi) = I - \lim f(x)$, that is $L = I - \lim f(x)$.

□

Theorem 2.4. *Let $H = \sup_k p_k < \infty$ and I an admissible ideal. Then the following are equivalent.*

(a) $(x_k) \in c^I(f, p)$;

(b) there exists$(y_k) \in c(f, p)$ such that $x_k = y_k$, for a.a.k.r.I;

(c) there exists$(y_k) \in c(f, p)$ and $(x_k) \in c_0^I(f, p)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f(|y_k - L|^{p_k}) \geq \epsilon\} \in I$;

(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} f(|x_{k_n} - L|^{p_{k_n}}) = 0$.

Proof. (a) implies (b). Let $(x_k) \in c^I(f, p)$. Then there exists $L \in \mathbb{C}$ such that

$$\{k \in \mathbb{N} : f(|x_k - L|^{p_k}) \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : f(|x_k - L|^{p_k}) \geq \epsilon\} \in I.$$

Define a sequence(y_k) as

$$y_k = x_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, & \text{if } |x_k - L|^{p_k} < t^{-1}, \\ & L, otherwise. \end{cases}$$

Then $(y_k) \in c(f, p)$ and form the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : f(|x_k - L|^{p_k}) \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.
(b) implies (c).For $(x_k) \in c^I(f, p)$. Then there exists $(y_k) \in c(f, p)$ such that $x_k = y_k$, for a.a.k.r.I.
Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $k \in I$.
Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, otherwise. \end{cases}$$

Then $z_k \in c_0^I(f, p)$ and $y_k \in c(f, p)$.
(c) implies (d).Let $P_1 = \{k \in \mathbb{N} : f(|x_k|^{p_k}) \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have $\lim_{n\to\infty} f(|x_{k_n} - L|^{p_{k_n}}) = 0$.
(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim_{n\to\infty} f(|x_{k_n} - L|^{p_{k_n}}) = 0$.
Then for any $\epsilon > 0$, and Lemma 1.10, we have

$$\{k \in \mathbb{N} : f(|x_k - L|^{p_k}) \geq \epsilon\} \subseteq K^c \cup \{k \in K : f(|x_k - L|^{p_k}) \geq \epsilon\}.$$

Thus $(x_k) \in c^I(f, p)$. □

Theorem 2.5. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(f, p) \supseteq m_0^I(f, q)$ if and only if $\lim_{k\in K}\inf \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. Let $\lim_{k\in K}\inf \frac{p_k}{q_k} > 0$. and $(x_k) \in m_0^I(f, q)$. Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$. Since $(x_k) \in m_0^I(f, q)$,for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : f(|x_k|^{q_k}) \geq \epsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : f(|x_k|^{p_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : f(|x_k|^{\beta q_k}) \geq \epsilon\} \in I.$$

Therefore $(x_k) \in m_0^I(f, p)$. □

Theorem 2.6. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(f, q) \supseteq m_0^I(f, p)$ if and only if $\lim_{k\in K}\inf \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. The proof follows similarly as the proof of Theorem 2.5. □

Theorem 2.7. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(f, q) = m_0^I(f, p)$ if and only if $\lim_{k\in K}\inf \frac{p_k}{q_k} > 0$, and $\lim_{k\in K}\inf \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.*

Proof. On combining Theorem 2.5 and 2.6 we get the required result.

□

Theorem 2.8. *Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then the following results are equivalent.*
(a) $H < \infty$ and $h > 0$.
(b) $c_0^I(f, p) = c_0^I$.

Proof. Suppose that $H < \infty$ and $h > 0$, then the inequalities $min\{1, s^h\} \leq s^{p_k} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$. Therefore the equivalent of (a) and (b) is obvious. □

Theorem 2.9. *Let f be a modulus function. Then $c_0^I(f, p) \subset c^I(f, p) \subset l_\infty^I(f, p)$ and the inclusions are proper.*

Proof. Let $(x_k) \in c^I(f, p)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f(|x_k - L|^{p_k}) = 0$. We have $f(|x_k|^{p_k}) \leq \frac{1}{2} f(|x_k - L|^{p_k}) + \frac{1}{2} f(|L|^{p_k})$. Taking supremum over k both sides we get $(x_k) \in l_\infty^I(f, p)$ and the inclusion $c_0^I(f, p) \subset c^I(f, p)$ is obvious. Hence $c_0^I(f, p) \subset c^I(f, p) \subset l_\infty^I(f, p)$ and the inclusions are proper. □

Theorem 2.10. *If $H = \sup_k p_k < \infty$, then for any modulus f, we have $l_\infty^I \subset M(m^I(f, p))$, where the inclusion may be proper.*

Proof. Let $a \in l_\infty^I$.This implies that $\sup_k |a_k| < 1 + K$. for some $K > 0$ and all k. Therefore $x \in m^I(f, p)$ implies $\sup_k f(|a_k x_k|^{p_k}) \leq (1 + K)^H \sup_k f(|x_k|^{p_k}) < \infty$. which gives $l_\infty^I \subset M(m^I(f, p))$.

To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all k. Take $a_k = k$ for all k. Therefore $x \in m^I(f, p)$ implies $\sup_k f(|a_k x_k|^{p_k}) \leq \sup_k f(|k|^{\frac{1}{k}}) \sup_k f(|x_k|^{p_k}) < \infty$. Thus in this case $a = (a_k) \in M(m^I(f, p))$ while $a \notin l_\infty^I$. □

Theorem 2.11. *The function $\hbar : m^I(f, p) \to \mathbb{R}$ is the Lipschitz function,where $m^I(f, p) = c^I(f, p) \cap l_\infty(f, p)$, and hence uniformly continuous.*

Proof. Let $x, y \in m^I(f, p)$,$x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)|^{p_k} \geq \|x - y\|\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)|^{p_k} \geq \|x - y\|\} \in I.$$

Here $\|x - y\| = \sup_k f(|x_k - y_k|^{\frac{p_k}{M}})$ where $M = max\{1, \sup_k p_k\}$
Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)|^{p_k} < \|x - y\|\} \in m^I(f, p),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)|^{p_k} < \|x - y\|\} \in m^I(f, p).$$

Hence also $B = B_x \cap B_y \in m^I(f, p)$,so that $B \neq \phi$.
Now taking k in B,

$$|\hbar(x) - \hbar(y)|^{p_k} \leq |\hbar(x) - x_k|^{p_k} + |x_k - y_k|^{p_k} + |y_k - \hbar(y)|^{p_k} \leq 3\|x - y\|.$$

Thus $\hbar$ is a Lipschitz function. For $m_0^I(f, p)$ the result can be proved similarly. □

Theorem 2.12. *If $x, y \in m^I(f,p)$,then $(x.y) \in m^I(f,p)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)|^{p_k} < \epsilon\} \in m^I(f,p),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)|^{p_k} < \epsilon\} \in m^I(f,p).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)|^{p_k} &= |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)|^{p_k} \\ &\leq |x_k|^{p_k}|y_k - \hbar(y)|^{p_k} + |\hbar(y)|^{p_k}|x_k - \hbar(x)|^{p_k}. \end{aligned} \tag{2.2}$$

As $m^I(f,p) \subseteq l_\infty(f,p)$,there exists an $M \in \mathbb{R}$ such that $|x_k|^{p_k} < M$ and $|\hbar(y)|^{p_k} < M$.
Using (2.2) we get

$$|x_k y_k - \hbar(x)\hbar(y)|^{p_k} \leq M\epsilon + M\epsilon = 2M\epsilon,$$

for all $k \in B_x \cap B_y \in m^I(f.p)$. Hence $(x.y) \in m^I(f,p)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For $m_0^I(f,p)$ the result can be proved similarly. □

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