Theory and Applications of
Mathematics & Computer Science
(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (2) (2012) 12–19

A Variant of Classical Von Kármán Flow for a Second Grade Fluid due to a Rotating Disk

A. A. Farooq[a], A. R. Ansari[b,*], A. M. Siddiqui[c]

[a]*COMSATS Institute of Information Technology, University Road, Tobe Camp, Post Code 22060, Abbottabad, NWFP, Pakistan*
[b]*Department of Mathematics & Natural Sciences, Gulf University for Science & Technology, P.O. Box 7207, Hawally 32093, Kuwait*
[c]*Department of Mathematics, York Campus, Pennsylvania State University, York,PA 17403, USA*

Abstract

An attempt is made to examine the classical Von Kármán flow problem for a second grade fluid by using a generalized non-similarity transformation. This approach is different from that of Von Kármán's evolution of the flow in such a way that the physical quantities are allowed to develop non-axisymmetrically. The three-dimensional equations of motion for the second grade fluid are treated analytically yielding the derivation of the exact solutions for the velocity components. The physical interpretation of the velocity components, vorticity components, shear stresses and boundary layer thickness are also presented.

Keywords: Non-axisymmetric flow, rotating disk, second grade fluid.
2010 MSC: 76-XX.

1. Introduction

The theoretical study of the flow near a rotating disk of infinite extent can be traced back to Von Kármán's similarity analysis. That is why the flow is widely known as Von Kármán flow. He assumed that the flow possessed axial symmetry, and introduced a similarity transformation which reduced the Navier-Stokes equation into a system of coupled nonlinear ordinary differential equations. These equations have been used as a test problem for numerical methods, and in the study of matched asymptotic expansions. This problem has received considerable attention over the years and different extensions of Von Kármánn's swirling flow problem have been made to address various applications, see for instance (Benton, 1966; Kuiken, 1971; Riley, 1964; Sahoo, 2009; Ariel, 2003). However, the possibility of an exact solution for the flow due to a rotating

*Corresponding author
Email addresses: (A. A. Farooq), ansari.a@gust.edu.kw (A. R. Ansari), (A. M. Siddiqui)

disk in a fluid which is at infinity and is rotating rigidly has been implied by Berker (Berker, 1982). Parter and Rajagopal (Parter & Rajagopal, 1984) have established the existence of solutions which do not possess axial symmetry, to the Navier-Stokes equations for the problem governing the flow of two infinite disks rotating about a common axis. Based on that work, Huilgol and Rajagopal(Huilgol & Rajagopal, 1987) have shown that in the case of certain non-Newtonian fluid models, solutions that lack axisymmetry are possible. Recently, Turkyilmazoglu (Turkyilmazoglu, 2009) has obtained exact solutions to the Navier-Stokes equations for the swirling flow problem in such a way that the physical quantities are allowed to develop non-axisymmetrically over a rotating disk.

It is a well-known fact that the Navier-Stokes equations seem to be a weak model for a class of real fluids, called non-Newtonian fluids. During the last few decades, considerable efforts have been devoted to the study of flow of non-Newtonian fluids because of their technological applications. A vast amount of literature is now available for the flow problems associated with non-Newtonian fluids in a variety of situations. One important and simple model of non-Newtonian fluids for which one can reasonably hope to obtain analytical solutions is the second grade fluid. Keeping this in mind, the aim of this work is to extend the analysis of (Turkyilmazoglu, 2009) for a second grade fluid. Undoubtedly, the equations of motion for a second grade fluid are more complicated with highly non-linear terms which make the question of well-posedness extremely difficult to address. Here, it is shown that by using a generalized transformation, the governing equations for the second grade fluid are transformed into a well posed second order system of ODEs whose exact solution is straightforward. In solving this problem we have relaxed the axisymmetric condition of the traditional Von Kármán flow. This analysis is important, not only from a mathematical point of view, but mainly as an essential test for the underlying physical model. The practical applications that can be envisaged for this problem are in the design of thrust bearings, radial diffusers etc., used in the defence industry for instance. We note that a similar problem of a Jeffrey Fluid, has been addressed by (Siddiqui *et al.*, 2013).

The following structure is pursued in the rest of the paper. In section two mathematical formulation of the problem is given. Section three concerns with the flow analysis and section four contains some concluding remarks.

2. Formulation of the problem

Consider the three dimensional flow of an incompressible second grade fluid due to an infinite disk which rotates in the plane $z = 0$ about its axis of rotation z with a constant angular velocity Ω. In cylindrical coordinates (r, θ, z) which rotates with the disk, the governing equations of motion of the second grade fluid are the laws of conservation of mass and momentum which are

$$\nabla.\mathbf{V} = 0, \tag{2.1}$$

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla P + \nabla.\sigma, \tag{2.2}$$

where $\mathbf{V} = (u, v, w)$ is the velocity vector, $\frac{d}{dt}$ is the material time derivative, ρ is the fluid density and P is the pressure. For the second grade fluid the extra stress tensor σ is given by (Ariel, 1997)

$$\sigma = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.3}$$

in which μ is the dynamic viscosity, $\alpha_i (i = 1, 2)$ are material constants satisfying $\alpha_1 \geq 0$, and $\alpha_1 + \alpha_2 = 0$, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T, \quad \mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 (\nabla\mathbf{V}) + (\nabla\mathbf{V})^{\mathrm{T}}\mathbf{A}_1, \tag{2.4}$$

where the superscript T is the transpose of the matrix. In the present analysis the flow is assumed to take place in the semi-infinite space $z \geq 0$. Boundary conditions accompanying (2.1)-(2.2) are such that the fluid adheres to the wall at $z = 0$ with a given axial velocity and the velocities are bounded at far distances from the disk.

The flow in this analysis is such that the physical quantities are allowed to develop non-axisymmetrically and we assume that there is no flow along the normal, thus the velocity field can be taken in the form

$$\mathbf{V} = [u(r, \theta, z), v(r, \theta, z), 0] \tag{2.5}$$

We introduce the following dimensionless variables:

$$r^* = \frac{r}{L}, z^* = \frac{z}{L}, u^* = \frac{u}{U}, v^* = \frac{v}{U}, P^* = \frac{P}{\rho U^2}, \sigma_{ij} = \frac{\sigma_{ij}}{\frac{\mu U}{L}}, i, j = r, \theta, z, \tag{2.6}$$

where L is the length scale and $U = L\Omega$. Hence, the dimensionless form of the continuity and the equations of motion, after dropping the *s are given by

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = 0, \tag{2.7}$$

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{\partial P}{\partial r} + \frac{1}{\mathrm{Re}}\left[\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right] \tag{2.8}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{1}{\mathrm{Re}}\left[\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\mathbf{2}\sigma_{\theta\theta}}{r}\right] \tag{2.9}$$

$$0 = -\frac{\partial P}{\partial z} + \frac{1}{\mathrm{Re}}\left[\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}\right] \tag{2.10}$$

In equations (2.7)-(2.10), σ_{rr}, $\sigma_{r\theta}$, σ_{rz}, $\sigma_{\theta z}$, $\sigma_{\theta\theta}$ and σ_{zz}, are the components of the stress tensor σ in (2.3), and are given by

$$c\sigma_{rr} = 2\frac{\partial u}{\partial r} + 2\lambda_1\left\{\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\frac{\partial u}{\partial r} + \left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + 2\left(\frac{\partial u}{\partial r}\right)^2\right\}$$
$$+\lambda_2\left\{4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 + \left(\frac{\partial \mathrm{u}}{\partial z}\right)^2\right\}, \tag{2.11}$$

$$c\sigma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \lambda_1\left\{\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + 2\frac{\partial u}{\partial r}\left(\frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r}\right)\right\}$$
$$+\lambda_2\left\{\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right)\right\}, \tag{2.12}$$

$$c\sigma_{rz} = \frac{\partial u}{\partial z} + \lambda_1\left\{\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + 3\frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}\left(2\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{2v}{r}\right)\right\}$$
$$+\lambda_2\left\{2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)\right\}, \tag{2.13}$$

$$c\sigma_{\theta z} = \frac{\partial v}{\partial z} + \lambda_1\left\{\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\left(\frac{\partial v}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) - 3\frac{\partial u}{\partial r}\frac{\partial v}{\partial z}\right\}$$
$$+\lambda_2\left\{\frac{\partial u}{\partial z}\left(\frac{\partial v}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) - 2\frac{\partial u}{\partial r}\frac{\partial v}{\partial z}\right\}, \tag{2.14}$$

$$c\sigma_{\theta\theta} = -2\left(\frac{\partial u}{\partial r}\right) + \lambda_1\left\{4\left(\frac{\partial u}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial u}{\partial \theta}\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) - 2\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial u}{\partial r}\right)\right\}$$
$$+\lambda_2\left\{4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}, \tag{2.15}$$

$$\sigma_{zz} = (2\lambda_1 + \lambda_2)\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right), \tag{2.16}$$

where $\lambda_1 = \frac{\alpha_{1U_c}}{\mu L}$ and $\lambda_2 = \frac{\alpha_{2U_c}}{\mu L}$ are the material parameters of the second grade fluid.

3. Flow analysis

In this section we restrict ourselves to the stationary mean flow relative to the rotating disk. Within this view, via a coordinate transformation $\zeta = \sqrt{\frac{\mathrm{Re}}{2}}z$, we assume a solution of the form (Turkyilmazoglu, 2009)

$$u = aF(\theta,\zeta),\ v = r + aW(\theta,\zeta), w = 0, P = \frac{r^2}{2} - ra\cos(\theta - \sigma) + a^2 p(\zeta), \tag{3.1}$$

such that, non-axisymmetric and periodic solutions with respect to θ of F and W are determined here, subjected to the pressure field given by (3.1). The parameters a and σ correspond to the polar representation of a fixed point on the disk surface and $p(\zeta)$ is some function of ζ.

The transformations (2.7)-(2.10) along with (2.11)-(2.16), satisfy the continuity equation directly, and for the momentum equations, the periodicity assumption of F and W with respect to θ, gives the set of ordinary differential equations

$$F_{\zeta\zeta} + \lambda_1 W_{\zeta\zeta} + 2W = -2\cos(\theta - \sigma), \tag{3.2}$$

$$W_{\zeta\zeta} - \lambda_1 F_{\zeta\zeta} - 2F = 2\sin(\theta - \sigma), \tag{3.3}$$

$$p(\zeta) = \frac{(2\lambda_1 + \lambda_2)}{2}(F_\zeta^2 + W_\zeta^2) + K, \tag{3.4}$$

where the constant K is determined from the pressure prescribed at the disk surface. The boundary conditions for the problem reduce to

$$F = 0, \quad W = 0 \text{ at } \zeta = 0, \quad F, W \text{ bounded}, \text{ as } \zeta \to \infty. \tag{3.5}$$

Introducing a new function of the form $V = F + iW$, transforms the pair of equations (3.2)-(3.3) into a single complex differential equation with real variables

$$(1 - i\lambda_1)V_{\zeta\zeta} - 2iV = -2((\cos(\theta - \sigma) - i\sin(\theta - \sigma)), \tag{3.6}$$

whose solution is bounded with respect to ζ and can be immediately expressed as

$$V = Ce^{m\zeta} - i\left(\cos((\theta - \sigma) - i\sin(\theta - \sigma)\right), \tag{3.7}$$

where, C is a complex integration constant depending on θ and is determined by using the no-slip condition on the wall and the constant $m = -\sqrt{\frac{2}{1+\lambda_1^2}(i - \lambda_1)}$. Equating real and imaginary parts of the solution given in (3.7), F and W are found to be

$$F(\zeta, \theta) = f(\zeta)\cos(\theta - \sigma) + g(\zeta)\sin(\theta - \sigma), \tag{3.8}$$

$$W(\zeta, \theta) = -f(\zeta)\sin(\theta - \sigma) + g(\zeta)\cos(\theta - \sigma), \tag{3.9}$$

where

$$f(\zeta) = \sin(d_2\zeta)e^{-d_1\zeta}, \quad g(\zeta) = -1 + \cos(d_2\zeta)e^{-d_1\zeta}, \tag{3.10}$$

and where

$$d_1 = \sqrt{\frac{\left(-\lambda_1 + \sqrt{(1 + \lambda_1^2)}\right)}{(1 + \lambda_1^2)}}, d_2 = \sqrt{\frac{\left(\lambda_1 + \sqrt{(1 + \lambda_1^2)}\right)}{(1 + \lambda_1^2)}}. \tag{3.11}$$

As $\zeta \to \infty$ we note from (3.10), that the velocities far away from the disk turn out to be $u = -a\sin(\theta - \sigma)$, $v = r - a\cos(\theta - \sigma)$ different from the no-slip velocities. In order to see the effects of the material parameter of the second grade fluid on the flow, graphs of f and $-g$ are plotted for various values of λ_1 in Figure 1. These graphs clearly indicate that the flow exhibits a boundary layer like behavior near the disk. It is also seen that when λ_1 increases, the oscillatory behavior of the flow becomes more prominent and can be seen up to a considerable distance from the disk. It should be noted that when $\lambda_1 = 0$, our results are in agreement with those of (Turkyilmazoglu, 2009) without suction and injection. Moreover, (3.10) shows that the velocity distribution is in the form of an Ekman spiral representing the flow over a disk in a rotating system similar to (Siddiqui *et al.*, 2013).

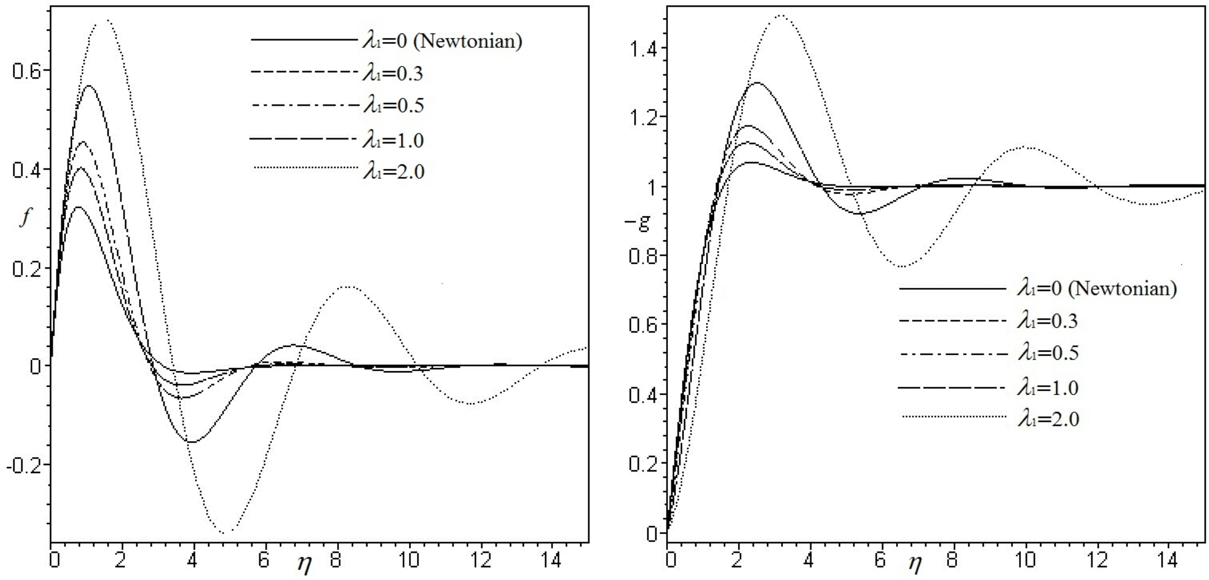

Figure 1. Variation of f and $-g$ with η for different values of λ_1.

The effects of viscosity in the fluid adjacent to the disk tends to develop some tangential shear stress which opposes the rotation of the disk. There is also a surface shear stress in the radial direction. The dimensionless expressions for the tangential and radial stresses are given as

$$\sigma_{\theta z} = a\left[W_\zeta - \lambda_1 F_\zeta\right]_{\zeta=0} = a\sqrt{\frac{\text{Re}}{2}}\left[(-d_2 + \lambda_1 d_1)\cos(\theta-\sigma) - (d_1 + \lambda_1 d_2)\sin(\theta-\sigma)\right], \quad (3.12)$$

$$\sigma_{rz} = a\left[F_\zeta + \lambda_1 W_\zeta\right]_{\zeta=0} = -a\sqrt{\frac{\text{Re}}{2}}\left[(-d_2 + \lambda_1 d_1)\sin(\theta-\sigma) + (d_1 + \lambda_1 d_2)\cos(\theta-\sigma)\right]. \quad (3.13)$$

In the particular case when $\theta = \sigma$, we obtain $\sigma_{\theta z} = a\sqrt{\frac{\text{Re}}{2}}(-d_2 + \lambda_1 d_1)$ and $\sigma_{rz} = -a\sqrt{\frac{\text{Re}}{2}}(d_1 + \lambda_1 d_2)$. Moreover, when $\sigma = 0$, the results obtained point out the fact that maximum resistance due to viscosity of the fluid will take place at the locations $\theta = \tan^{-1}\left(\frac{-d_2+\lambda_1 d_1}{d_1+\lambda_1 d_2}\right)$ and $\theta = \tan^{-1}\left(\frac{-d_2+\lambda_1 d_1}{d_1+\lambda_1 d_2}\right)+\pi$ for the tangential stress and at the locations $\theta = \tan^{-1}\left(\frac{d_1+\lambda_1 d_2}{d_2-\lambda_1 d_1}\right)$ and $\theta = \tan^{-1}\left(\frac{d_1+\lambda_1 d_2}{d_2-\lambda_1 d_1}\right)+\pi$ for the radial stress. From the above equations one can easily find out the locations at which the minimum and maximum skin friction occurs against the flow.

The fluid dynamic thickness in radial and tangential directions are evaluated as

$$\delta_r = \int_0^\infty f(\zeta)d\zeta = \frac{d_2}{d_1^2 + d_2^2}, \; \delta_\theta = \int_0^\infty (1 + g(\zeta))d\zeta = \frac{d_1}{d_1^2 + d_2^2}. \quad (3.14)$$

Hence, an increase in λ_1 results in an increase in the boundary layer thickness, this is clearly because as λ_1 increases d_1 and d_2 decrease and tend to zero.

The vorticity components $(\omega_r, \omega_\theta, \omega_z) = \nabla \times \mathbf{V}$ that exists within the fluid can be found out exactly with the help of equations (3.8)-(3.10), which are respectively

$$\omega_r = -\frac{\partial v}{\partial z} = -a\sqrt{\frac{\text{Re}}{2}}W_\zeta, \; \omega_\theta = \frac{\partial u}{\partial z} = a\sqrt{\frac{\text{Re}}{2}}F_\zeta, \omega_z = 2 \quad (3.15)$$

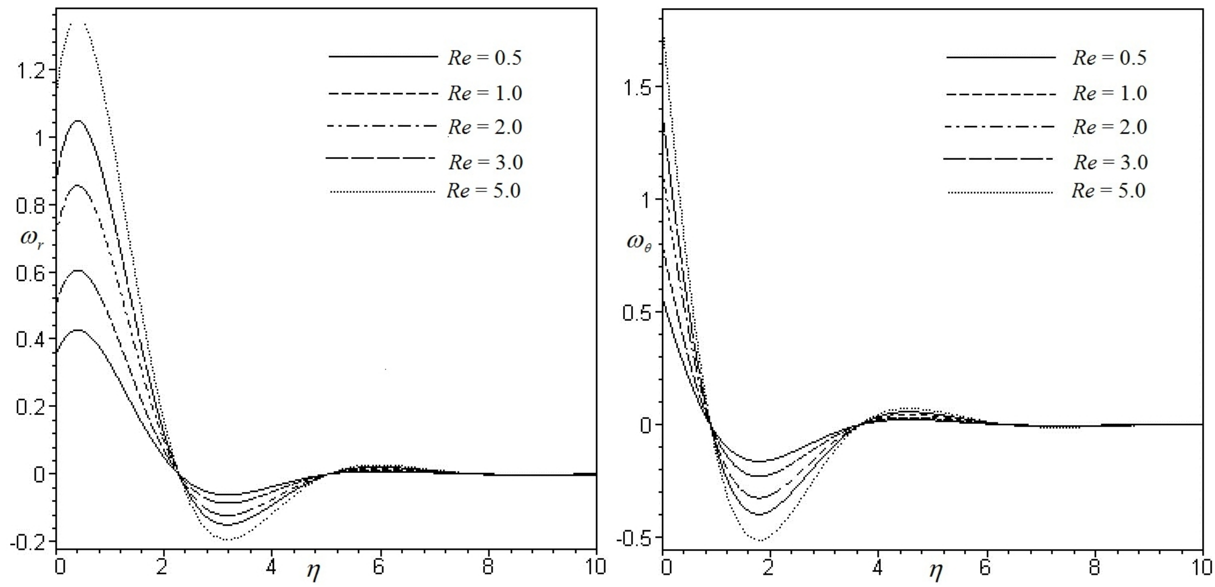

Figure 2. Variation of ω_r and ω_θ with η for different values of *Re* keeping $\lambda_1 = 0.5$.

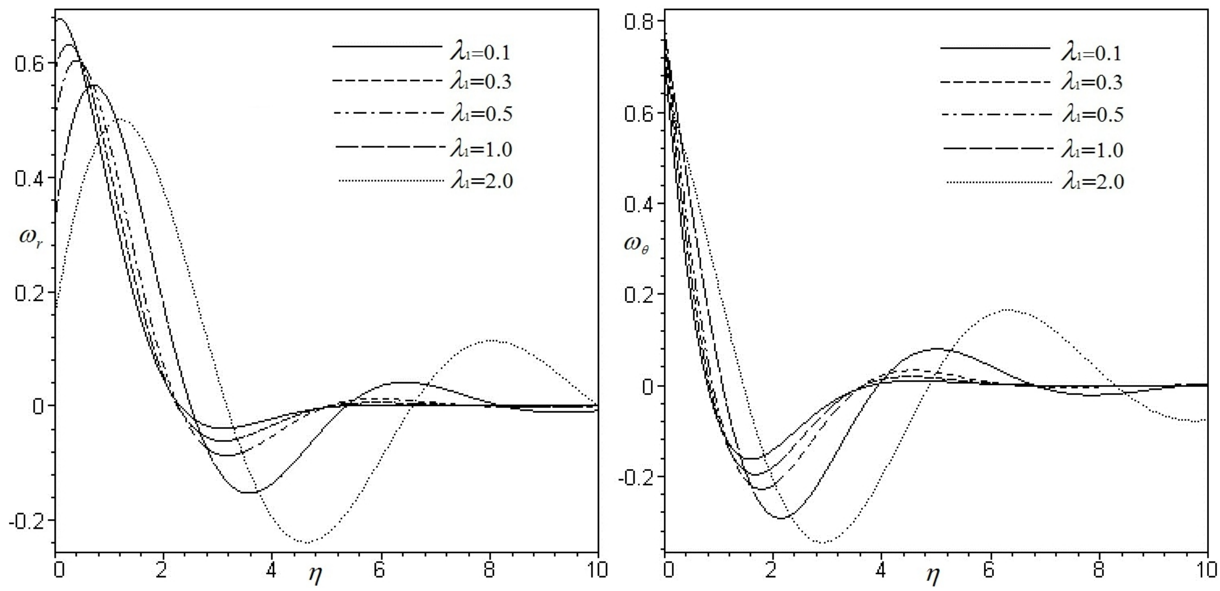

Figure 3. Variation of ω_r and ω_θ with η for different values of λ_1 when $Re = 1$.

In order to get the nature of the vorticity near the disk the expressions for ω_r and ω_θ are plotted for different values of Re and λ_1 when $\sigma = \theta$. It is observed from Figure 2 that both the components increase near the disk with increasing values of Re and show oscillatory behavior before approaching the asymptotic limits. Figure 3 is to demonstrate the effects of λ_1 on ω_r and ω_θ. It is noted that ω_r decreases whereas ω_θ increases near the disk with increasing values of λ_1. However, a large gradient is observed for ω_r near the wall. Actually, these vorticity components are responsible for driving the motion of fluid flow considered in the current study.

4. Concluding remarks

In this article, an exact solution for three-dimensional equations governing the incompressible second grade fluid flow over a single rotating disk has been obtained in such a way that the physical quantities are allowed to develop non-axisymmetrically within a no-normal flow assumption. We have worked through cylindrical coordinates which rotate with the disk, whose polar representation is (a, σ). The particular case $a = 0$ is associated with the rigid body rotation. The non-zero choice of a has enabled us to achieve the solutions bounded away from the disk. These results point out that a boundary layer structure develops near the surface of the disk whose far away behavior is distinct from the near wall solutions. It is observed that increases in λ_1 cause an increase in the boundary layer thickness. There is no effect of the material parameter λ_2 on the velocity field since both the disk and the fluid rotate with the same speed. We also note that this technique can also be applied to other non-Newtonian fluid flow problems successfully.

References

Ariel, P. Donald (2003). On the flow of an elastico-viscous fluid near a rotating disk. *J. Comput. Appl. Math.* **154**(1), 1–25.

Ariel, P.Donald (1997). Computation of flow of a second grade fluid near a rotating disk. *International Journal of Engineering Science* **35**(14), 1335 – 1357.

Benton, Edward R. (1966). On the flow due to a rotating disk. *Journal of Fluid Mechanics* **24**, 781–800.

Berker, R. (1982). An exact solution of the Navier-Stokes equation the vertex with curvilinear axis. *International Journal of Engineering Science* **20**, 217–230.

Huilgol, R.R. and K.R. Rajagopal (1987). Non-symmetric flow of a viscoelastic fluid between rotating disks. *Journal of non-Newtonian Fluid Mechanics* **27**, 423–432.

Kuiken, H. K. (1971). The effect of normal blowing on the flow near a rotating disk of infinite extent. *Journal of Fluid Mechanics* **47**, 789–798.

Parter, S. V. and K. R. Rajagopal (1984). Swirling flow between rotating plates. *Archive for Rational Mechanics and Analysis* **86**, 305–315.

Riley, N. (1964). The heat transfer from a rotating disk. *The Quarterly Journal of Mechanics and Applied Mathematics* **17**(3), 331–349.

Sahoo, Bikash (2009). Effects of partial slip, viscous dissipation and Joule heating on Von Kármán flow and heat transfer of an electrically conducting non-Newtonian fluid. *Communications in Nonlinear Science and Numerical Simulation* **14**(7), 2982 – 2998.

Siddiqui, A. M., A. A. Farooq, T. Haroon and B. S. Babcock (2013). A variant of the classical Von Kármán flow for a Jeffrey fluid. *Applied Mathematical Sciences* **7**(20), 983 – 991.

Turkyilmazoglu, M. (2009). Exact solutions for the incompressible viscous fluid of a porous rotating disk flow. *International Journal of Non-Linear Mechanics* **44**(4), 352 – 357.