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Second Order $(\Phi, \Psi, \rho, \eta, \theta)-$Invexity Frameworks and $\epsilon-$Efficiency Conditions for Multiobjective Fractional Programming

Ram U. Verma[a,*]

[a] *International Publications USA, 3400 S Brahma Blvd Suite 31B, Kingsville, TX 78363, USA*

Abstract

A generalized framework for a class of second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities is developed, and then some parametric sufficient efficiency conditions for multiobjective fractional programming problems are established. The obtained results generalize and unify a wider range of investigations in the literature on applications to other results on multiobjective fractional programming.

Keywords: Generalized invexity, multiobjective fractional programming, $\epsilon-$ efficient solutions, parametric sufficient $\epsilon-$ efficiency conditions.
2010 MSC: 90C30, 90C32, 90C34.

1. Introduction

Zalmai and Zhang (see (Zalmai & Zhang, 2007*a*)) have established a set of necessary efficiency conditions and a fairly large number of global nonparametric sufficient efficiency results under various frameworks for generalized $(\eta, \rho)-$invexity for semi-infinite discrete minimax fractional programming problems. Recently, Verma (see (Verma, 2013)) developed a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\epsilon-$efficient solutions. On the other hand, the work of Kim, Kim and Lee (see (Kim *et al.*, 2011)) extends the results of Kim and Lee (see (Kim & Lee, 2013)) on $\epsilon-$optimality theorems for a convex multiobjective optimization problem to a multiobjective fractional optimization problem, while this has been followed by other research advances. They also applied the generalized Abadie constraint qualification to the context of the optimal solvability of a semi-infinite discrete minimax fractional programming problems.

*Corresponding author
Email address: verma99@msn.com (Ram U. Verma)

Based on the recent advances in the study of $\epsilon-$optimality and weak $\epsilon-$optimality conditions for multiobjective fractional programming problems, we first generalize the $(\rho, \eta, \theta)-$invexities to second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities, and we introduce some parametric sufficient efficiency conditions for multiobjective fractional programming to achieve $\epsilon-$efficient solutions to multiobjective fractional programming problems. The results established in this communication, not only generalize the results on weak $\epsilon-$efficiency conditions for multiobjective fractional programming problems, but also generalize the second order invexity results in general setting. The notion of the second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities encompass most of the existing notions of the generalized invexities (see (Ben-Israel & Mond, 1986), (Caiping & Xinmin, 2009), (Hanson, 1981) (Jeyakumar, 1985), (Liu, 1999), (Mangasarian, 1975), (Mishra, 1997), (Mishra, 2000), (Mishra & Rueda, 2000), (Mishra & Rueda, 2006), (Mond & Weir, 1981-1983), (Mond & Zhang, 1995), (Mond & Zhang, 1998), (Patel, 1997), (Srivastava & Bhatia, 2006), (Srivastava & Govil, 2000), (Suneja *et al.*, 2003), (Vartak & Gupta, 1987), (Yang, 1995), (Yang, 2009),(Yang & Hou, 2001), (Yang *et al.*, 2004*a*), (Yang *et al.*, 2003), (Yang *et al.*, 2005), (Yang *et al.*, 2008), (Yang *et al.*, 2004*b*), (Yokoyama, 1996), (Zalmai, 2007), (Zalmai, 2007), (Zhang & Mond, 1996), (Zhang & Mond, 1997)). There exists a vast literature on higher order generalized invexity and duality models in mathematical programming. For more details, we refer the reader (see (Verma, 2012), (Verma, 2013), (Zalmai, 2012), (Zalmai & Zhang, 2007*b*), (Zeidler, 1985)).

We consider under the general framework of $(\Phi, \Psi, \rho, \eta, \theta)-$invexities of functions, the following multiobjective fractional programming problem:

(P)

$$Minimize\Big(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$,

where X is an open convex subset of $\mathfrak{R}^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:

(Pλ)

$$Minimize\Big(f_1(x) - \lambda_1 g_1(x), \cdots, f_p(x) - \lambda_p g_p(x)\Big)$$

subject to $x \in Q$ with

$$\lambda = \Big(\lambda_1, \lambda_2, \cdots, \lambda_p\Big) = \Big(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\Big),$$

where x^* is an efficient solution to (P).

General Mathematical programming problems serve a significant useful purpose, especially in terms of applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, robotics, and others.

2. Generalized second order invexities

In this section, we develop some concepts and notations for the problem on hand. Let X be an open convex subset of $\mathfrak{R}^n$ (n-dimensional Euclidean space). Let $\langle\cdot,\cdot\rangle$ denote the inner product, and let $\eta : X\times X \to \mathfrak{R}^n$ be a function. Suppose that f is a real-valued twice continuously differentiable function defined on X, and that $\triangledown f(y)$ and $\nabla^2 f(y)$ denote, respectively, the gradient and hessian of f at y.

Definition 2.1. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\Phi, \Psi, \rho, \eta, \theta)-$invex at x^* of second order if there exist a superlinear function $\Phi : \mathfrak{R}^n \to \mathfrak{R}$, a sublinear function $\Psi : \mathfrak{R}^n \to \mathfrak{R}$ and a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \geq \langle \triangledown f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2.$$

Definition 2.2. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exist a superlinear function $\Phi : \mathfrak{R}^n \to \mathfrak{R}$, a sublinear function $\Psi : \mathfrak{R}^n \to \mathfrak{R}$ and a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\langle \triangledown f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.3. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be strictly $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\langle \triangledown f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) > 0.$$

Definition 2.4. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be prestrictly $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\langle \triangledown f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 > 0 \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.5. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\Psi(f(x) - f(x^*)) \leq 0 \Rightarrow \langle \triangledown f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.$$

Definition 2.6. A twice differentiable function $f : X \to \Re$ is said to be strictly $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$,

$$\Psi(f(x) - f(x^*)) \le 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^r < 0.$$

Definition 2.7. A twice differentiable function $f : X \to \Re$ is said to be prestrictly $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$,

$$\Psi(f(x) - f(x^*)) < 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^r \le 0.$$

We observe that the second order generalized $(\Phi, \Psi, \rho, \eta, \theta)-$invexities can be specialized to second order $(\rho, \eta, \theta)-$invexities.

Definition 2.8. A twice differentiable function $f : X \to \Re$ is said to be $(\rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exist a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \ge 0 \Rightarrow f(x) - f(x^*) \ge 0.$$

Definition 2.9. A twice differentiable function $f : X \to \Re$ is said to be strictly $(\rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \ge 0 \Rightarrow f(x) - f(x^*) > 0.$$

Definition 2.10. A twice differentiable function $f : X \to \Re$ is said to be prestrictly $(\rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 > 0 \Rightarrow f(x) - f(x^*) \ge 0.$$

Definition 2.11. A twice differentiable function $f : X \to \Re$ is said to be $(\rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$

$$f(x) - f(x^*) \le 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \le 0.$$

Definition 2.12. A twice differentiable function $f : X \to \Re$ is said to be strictly $(\rho, \eta, \theta)-$quasi-invex at x^* of second if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$

$$f(x) - f(x^*) \le 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0.$$

Definition 2.13. A twice differentiable function $f : X \to \Re$ is said to be prestrictly $(\rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \Re^n \times \Re^n \to \Re^n$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\theta : X \times X \to \Re^n$ and $z \in \Re^n$

$$f(x) - f(x^*) < 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.$$

3. The $\epsilon-$Solvability Conditions

Now we consider the $\epsilon-$solvability conditions for (P) and (Pλ) problems motivated by the publications (see (Kim *et al.*, 2011)), where they have investigated the $\epsilon-$efficiency as well as the weak $\epsilon-$efficiency conditions for multiobjective fractional programming problems under constraint qualifications. Based on these developments in the literature, first we introduce a second order generalization of $(\Phi, \Psi, \rho, \eta, \theta)-$invexities to the existing notion of $(\rho, \eta, \theta)-$invexities, and then using the parametric approach, we develop some parametric sufficient $\epsilon-$efficiency conditions for multiobjective fractional programming problem (P) under this framework. We need to recall some auxiliary results crucial to the problem on hand.

Definition 3.1. A point $x^* \in Q$ is an $\epsilon-$efficient solution to (P) if there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i \,\forall\, i = 1, \cdots, p,$$
$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j,\ some\ j \in \{1, \cdots, p\},$$

where ϵ_i=$(\epsilon_1, \cdots, \epsilon_p)$ is with $\epsilon_i \geq 0$ for $i = 1, \cdots, p$.

For $\epsilon = 0$, Definition 3.1 reduces to the case that $x^* \in Q$ is an efficient solution to (P).

Definition 3.2. A point $x^* \in Q$ is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} \,\forall\, i = 1, \cdots, p.$$

Next to this context, we have the following auxiliary problem:
(P$\bar{\lambda}$)

$$minimize_{x \in Q}(f_1(x) - \bar{\lambda}_1 g_1(x), \cdots, f_p(x) - \bar{\lambda}_p g_p(x)),$$

subject to $x \in Q$,

where $\bar{\lambda}_i$ for $i \in \{1, \cdots, p\}$ are parameters, $\epsilon_i^* = \epsilon_i g_i(x^*)$ and $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$.

Next, we introduce the ϵ^*-solvability conditions for (P$\bar{\lambda}$) problem.

Definition 3.3. A point $x^* \in Q$ is an ϵ^*-efficient solution to (P$\bar{\lambda}$) if there does not exist an $x \in Q$ such that

$$f_i(x) - \bar{\lambda}_i g_i(x) \leq f_i(x^*) - \bar{\lambda}_i g_i(x^*) - \epsilon_i^* \,\forall\, i = 1, \cdots, p,$$
$$f_j(x) - \bar{\lambda}_j g_j(x) < f_j(x^*) - \bar{\lambda}_j g_j(x^*) - \epsilon_j^*,\ some\ j \in \{1, \cdots, p\},$$

where $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$, and $\epsilon_i^* = \epsilon_i g_i(x^*)$ with $\epsilon = (\epsilon_1, \cdots, \epsilon_p)$, $\epsilon_i \geq 0$ for $i = 1, \cdots, p$.

For $\epsilon = 0$, it reduces to the case that x^* is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right) \leq \left(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\right).$$

Lemma 3.1. *(Kim et al., 2011) Let $x^* \in Q$. Suppose that $f_i(x^*) \geq \epsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) x^* is an $\epsilon-$efficient solution to (P).

(ii) x^* is an ϵ^*-efficient solution to $(P\bar{\lambda})$, where

$$\bar{\lambda} = \left(\frac{f_1(x^*)}{g_1(x^*)} - \epsilon_1, \cdots, \frac{f_p(x^*)}{g_p(x^*)} - \epsilon_p\right)$$

and $\epsilon^* = (\epsilon_1 g_1(x^*), \cdots, \epsilon_p g_p(x^*))$.

Lemma 3.2. *(Kim et al., 2011) Let $x^* \in Q$. Suppose that $f_i(x^*) \geq \epsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) x^* is an $\epsilon-$efficient solution to (P).

(ii) There exists $c = (c_1, \cdots, c_p) \in \mathfrak{R}^p_+ \setminus \{0\}$ such that

$$\Sigma_{i=1}^p c_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)] \geq 0 = \Sigma_{i=1}^p c_i[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x^*)] - \Sigma_{i=1}^p c_i\epsilon_i g_i(x^*),$$

for any $x \in Q$.

Lemma 3.3. *Let $x^* \in Q$. Suppose that $f_i(x^*) \geq \epsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) x^* is an ϵ^*-efficient solution to $(P\bar{\lambda})$.

(ii) There exists $c = (c_1, \cdots, c_p) \in \mathfrak{R}^p_+ \setminus \{0\}$ such that

$$\Sigma_{i=1}^p c_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)] \geq 0 = \Sigma_{i=1}^p c_i[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x^*)] - \Sigma_{i=1}^p c_i\epsilon_i g_i(x^*),$$

for any $x \in Q$.

4. Auxiliary results on Parametric sufficiency conditions

This section deals with some auxiliary parametric sufficient $\epsilon-$ efficiency conditions for problem (P) under the generalized frameworks for generalized invexity. We start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)],\ i \in \{1, \cdots, p\},$$

and

$$B_j(.,v) = v_j H_j(x), \ \ j = 1, \cdots, m.$$

Verma (see (Verma, 2013)) recently established the following result based on parametric sufficient weak $\epsilon-$ efficiency conditions for problem (P) under the generalized (ρ, η, θ) frameworks for generalized invexities. These results are significant to developing our main results on hand.

Theorem 4.1. *Let $x^* \in Q$. Let f_i, g_i for $i \in \{1, \cdots, p\}$ with $f_i(x^*) \geq \epsilon_i g_i(x^*)$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that*

$$\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)] + \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*) \rangle \geq 0 \,, \tag{4.1}$$

and

$$v_j^* H_j(x^*) = 0, \ j \in \{1, \cdots, m\}. \tag{4.2}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$) :

(i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are $(\rho, \eta, \theta)-$quasi-invex at x^*.

(ii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly $(\rho, \eta, \theta)-$quasi-invex at x^*.

(iii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly $(\rho, \eta, \theta)-$quasi-invex at x^*, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly $(\rho, \eta, \theta)-$pseudo-invex at x^*.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is $(\rho_1, \eta, \theta)-$invex and $-g_i$ is $(\rho_2, \eta, \theta)-$invex at x^*. $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ is $(\rho_3, \eta, \theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^* (\rho_1 + \phi(x^*)\rho_2)$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$.

Then x^* is a weakly $\epsilon-$efficient solution to (P).

Next, we recall the following result (see (Verma & Zalmai, 2012)) that is crucial to developing the results for the next section based on second Order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities.

Theorem 4.2. *Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, \overline{t})$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\begin{aligned} & \{\Big(\nabla G_j(x^*, t), \langle y, \nabla^2 G_j(x^*, t) y \rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \\ + \ \ & span\{\Big(\nabla H_k(x^*, s), \langle y, \nabla^2 H_k(x^*, s) y \rangle\Big) : s \in S_k, k \in \underline{r}\}, \\ & where \ \hat{T}_j(x^*) \equiv \{t \in T_j \ : \ G_j(x^*, t) = 0\}, \end{aligned}$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^p u_i = 1\}$ *and integers* ν_0^* *and* ν^**, with* $0 \leq \nu_0^* \leq \nu^* \leq n+1$*, such that there exist* ν_0^* *indices* j_m*, with* $1 \leq j_m \leq q$*, together with* ν_0^* *points* $t^m \in \hat{T}_{j_m}(x^*),\ m \in \underline{\nu_0^*},\ \nu^* - \nu_0^*$ *indices* k_m*, with* $1 \leq k_m \leq r$*, together with* $\nu^* - \nu_0^*$ *points* $s^m \in S_{k_m}$ *for* $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$*, and* ν^* *real numbers* v_m^**, with* $v_m^* > 0$ *for* $m \in \underline{\nu_0^*}$*, with the property that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^*(\nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*, s^m) = 0, \tag{4.3}$$

$$\langle y, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla^2 G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla^2 H_k(x^*, s^m)\Big] y\rangle \geq 0, \tag{4.4}$$

where $\hat{T}_{j_m}(x^*) = \{t \in T_{j_m} : G_{j_m}(x^*, t) = 0\}$, $U = \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^p u_i = 1\}$*, and* $\underline{\nu^*} \backslash \underline{\nu_0^*}$ *is the complement of the set* $\underline{\nu_0^*}$ *relative to the set* $\underline{\nu^*}$*.*

5. Second Order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities

This section deals with some parametric sufficient $\epsilon-$ efficiency conditions for problem (P) under the generalized frameworks of $(\Phi, \Psi, \rho, \eta, \theta)-$invexities for generalized invex functions. We start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big) g_i(x)],\ i \in \{1, \cdots, p\}$$

and

$$B_j(., v) = v_j H_j(x),\ \ j = 1, \cdots, m.$$

Theorem 5.1. *Let* $x^* \in Q$*. Let* f_i, g_i *for* $i \in \{1, \cdots, p\}$ *with* $f_i(x^*) \geq \epsilon_i g_i(x^*)$*,* $g_i(x^*) > 0$ *and* H_j *for* $j \in \{1, \cdots, m\}$ *be twice continuously differentiable at* $x^* \in Q$*, and let there exist* $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$*,* $v^* \in \Re_+^m$ *and* $z \in \Re^n$ *such that*

$$\Sigma_{i=1}^p u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \triangledown H_j(x^*) = 0, \tag{5.1}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0, \tag{5.2}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{5.3}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$*):*

(i) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are* $(\Phi, \Psi, \rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are* $(\Phi, \Psi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0$*.*

(ii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ *are prestrictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0.$

(iii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ *are strictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0.$

(iv) For each $i \in \{1, \cdots, p\}$*,* f_i *is* $(\Phi, \Psi, \rho_1, \eta)-$*invex and* $-g_i$ *is* $(\Phi, \Psi, \rho_2, \eta)-$*invex at* x^**.* $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ *is* $(\Phi, \Psi, \rho_3, \eta)-$*quasi-invex at* x^**,* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0$*, and* $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ *for* $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \phi(x^*)\rho_2)$ *and for* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$*.*

Then x^* *is an* $\epsilon-$*efficient solution to (P).*

Proof. If (i) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*)\rangle + \langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle = 0\, \forall\, x \in Q, \quad (5.4)$$

$$\Big\langle z, \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \quad (5.5)$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

and in light of the $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, and assumptions on Ψ, we find

$$\Psi\Big(\Sigma_{j=1}^m v_j^* H_j(x) - \Sigma_{j=1}^m v_j^* H_j(x^*)\Big) \leq 0,$$

which results in

$$\langle \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0. \quad (5.6)$$

It follows from (5.3), (5.4), (5.5) and (5.6) that

$$\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\tfrac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*)\rangle + \tfrac{1}{2}\Big\langle z, \Sigma_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\tfrac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*) z]\Big\rangle$$
$$\geq \rho(x, x^*)\|\theta(x, x^*)\|^2. \quad (5.7)$$

As a result, since $\rho(x, x^*) \geq 0$, applying the $(\Phi, \Psi, \rho, \eta, \theta)-$ pseudo-invexity at x^* to (5.7) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \epsilon_i) g_i(x)] - \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)]\Big) \geq 0,$$

which implies

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)]$$

$$\geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] - \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) = 0.$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{5.8}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \ \textit{some}\ j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

Next, if (ii) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\langle \Sigma_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*)\rangle + \langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle = 0 \ \forall\, x \in Q, \tag{5.9}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{5.10}$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and in light of the strict $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, and assumptions on Ψ, we find

$$\Psi\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0,$$

which results in

$$\langle \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{5.11}$$

It follows from (5.3), (5.9), (5.10) and (5.11) that

$$\begin{aligned}
&\langle \Sigma_{i=1}^{p} u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)], \eta(x, x^*\rangle \\
+ \quad & \frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle \\
> \quad & \rho(x, x^*)\|\theta(x, x^*)\|^2. \qquad (5.12)
\end{aligned}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrict $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invexity at x^* to (5.12) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \epsilon_i)g_i(x)] - \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned}
&\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] \\
\geq \quad & \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] - \Sigma_{i=1}^{p} u_i^*\epsilon_i g_i(x^*) = 0.
\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \qquad (5.13)$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \ some\ j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

The proofs applying (iii) is similar to that of (ii), so we just need to include the proof using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, which implies $\Psi\Big(H_j(x) - H_j(x^*)\Big) \leq 0$.

Then applying the $(\Phi, \Psi, \rho_3, \eta)-$quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\langle \Sigma_{j=1}^{m} v_j^* \triangledown H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\Big\langle z, \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z\Big\rangle \leq -\Sigma_{j=1}^{m} v_j^* \rho_3 \|\theta(x, x^*)\|^2.$$

Since $u^* \geq 0$ and $f_i(x^*) \geq \epsilon_i g_i(x^*)$, it follows from $(\Phi, \Psi, \rho_3, \eta)-$invexity assumptions that

$$\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)]\Big) \\
= \; & \Phi\Big(\Sigma_{i=1}^{p} u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)[g_i(x) - g_i(x^*)] + \epsilon_i g_i(x^*)\}\Big) \\
\geq \; & \Sigma_{i=1}^{p} u_i^*\{\langle \nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*), \eta(x, x^*)\rangle\} \\
+ \; & \frac{1}{2}\langle z, \Sigma_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z\rangle] \\
+ \; & \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) \\
\geq \; & -[\langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\Big\langle z, \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z\Big\rangle] \\
+ \; & \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) \\
\geq \; & (\Sigma_{j=1}^{m} v_j^* \rho_3 + \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2])\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) \\
= \; & (\Sigma_{j=1}^{m} v_j^* \rho_3 + \rho^*)\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) \\
\geq \; & (\Sigma_{j=1}^{m} v_j^* \rho_3 + \rho^*)\|\theta(x, x^*)\|^2,
\end{aligned}$$

where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$ and $\rho^* = \Sigma_{i=1}^{p} u_i^*(\rho_1 + \phi(x^*)\rho_2)$. □

We note that Theorem 5.1 can be specialized to the context of second order $(\rho, \eta, \theta)-$ invexities as follows:

Theorem 5.2. *Let $x^* \in Q$. Let f_i, g_i for $i \in \{1, \cdots, p\}$ with $f_i(x^*) \geq \epsilon_i g_i(x^*)$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$ and $v^* \in \Re_+^m$ such that*

$$\Sigma_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)] + \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*) = 0 \tag{5.14}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0, \tag{5.15}$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \tag{5.16}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are* $(\rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are* $(\rho, \eta, \theta)-$*quasi-invex at* x^**.*

(ii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are prestrictly* $(\rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\rho, \eta, \theta)-$*quasi-invex at* x^**.*

(iii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are strictly* $(\rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\rho, \eta, \theta)-$*quasi-invex at* x^**.*

(iv) For each $i \in \{1, \cdots, p\}$*,* f_i *is* $(\rho_1, \eta, \theta)-$*invex and* $-g_i$ *is* $(\rho_2, \eta, \theta)-$*invex at* x^**.* $H_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *is* $(\rho_3, \eta, \theta)-$*quasi-invex at* x^**, and* $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ *for* $\rho^* = \Sigma_{i=1}^p u_i^* (\rho_1 + \phi(x^*)\rho_2)$ *and for* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$*.*

Then x^* *is an* $\epsilon-$*efficient solution to (P).*

Proof. If (i) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*\rangle + \langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle = 0\ \forall\, x \in Q, \tag{5.17}$$

$$\Big\langle z, \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{5.18}$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

and in light of the $(\rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, we have

$$\langle \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle + \rho(x, x^*) \|\theta(x, x^*)\|^2 \leq 0. \tag{5.19}$$

It follows from (5.19) that

$$\begin{aligned} & \langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*\rangle \\ + \ & \frac{1}{2}\Big\langle z, \sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*) z]\Big\rangle \\ \geq \ & \rho(x, x^*) \|\theta(x, x^*)\|^2. \end{aligned} \tag{5.20}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the $(\rho, \eta, \theta)-$ pseudo-invexity at x^* to (5.20), we have

$$\begin{aligned} & \Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] \\ \geq \ & \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] - \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{5.21}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \ some\ j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

Next, if (ii) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\begin{aligned} &\langle \Sigma_{i=1}^{p} u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)], \eta(x, x^*\rangle \\ + \ &\langle \Sigma_{j=1}^{m} v_j^* \triangledown H_j(x^*), \eta(x, x^*)\rangle = 0 \ \forall\, x \in Q, \end{aligned} \tag{5.22}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0. \tag{5.23}$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and in light of the strict $(\rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, we find

$$\langle \triangledown H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{5.24}$$

It follows from (5.23) and (5.24) that

$$\begin{aligned} &\langle \Sigma_{i=1}^{p} u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)], \eta(x, x^*\rangle \\ + \ &\frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle > \rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{5.25}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrict $(\rho, \eta, \theta)-$pseudo-invexity at x^* to (5.25), we have

$$\begin{aligned} &\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] \\ \geq \ &\Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] - \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq 0. \tag{5.26}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \ some\ j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P). □

6. Concluding Remarks

We observe that the obtained results in this communication can be generalized to the case of multiobjective fractional subset programming with generalized invex functions, for instance based on the work of Mishra et al. (see (Mishra *et al.*, 2010)) and Verma (see (Verma, 2013))) to the case of the $\epsilon-$ efficiency and weak $\epsilon-$efficiency conditions to the context of minimax fractional programming problems involving n-set functions.

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