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Refined Estimates for the Equivalence Between Ditzian-Totik Moduli of Smoothness and K-Functionals

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Abstract

The aim of this note is to study the magnitude of the constants in the equivalence between the first and second order Ditzian-Totik moduli of smoothness and related K-functionals. Applications to some classic approximation operators are given.

Keywords: Ditzian-Totik moduli of smoothness, K-functional, smooth functions, constants in the equivalence.
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1. Introduction and main results

Ditizian-Totik moduli of smoothness have become a standard tool in approximation theory. This is true in particular for second order moduli which play a crucial role in approximation by positive linear operators. For their properties and many applications see (Z. Ditzian, 1987). For $f \in C[0,1]$ the second order Ditzian-Totik modulus is defined by

$$\omega_2^{\varphi}(f,h) = \sup\{|\Delta^2_{\rho\varphi(x)} f(x)|,\ x \pm \rho\varphi(x) \in [0,1],\ 0 < \rho \leq h.\} \tag{1.1}$$

Here $\varphi(x) = \sqrt{x(1-x)}$, and $\Delta^2_{\eta} f(y) = f(y-\eta) - 2f(y) + f(y+\eta)$ if $\eta > 0$ and $y \pm \eta \in [0,1]$ and as 0 otherwise. In the sequel we will use the following notation:

$$AC_{loc}[0,1] := \{h : h \text{ is absolutely continuous in } [a,b] \text{ for every } 0 < a < b < 1\};$$

$$W^{\varphi}_{2,\infty}[0,1] := \{g : g' \in AC_{loc}[0,1] \text{ and } \|\varphi^2 g''\|_{\infty} < \infty\}.$$

The related K-functional $K_2^{\varphi}(f,h^2)$ is given by

$$K_2^{\varphi}(f,h^2) := \inf_g \{\|f-g\|_{\infty} + h^2\|\varphi^2 g''\|_{\infty}\}. \tag{1.2}$$

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Here the infimum is taken over all $f \in W^{\varphi}_{2,\infty}[0,1]$. The definitions of the second order Ditzian-Totik modulus of smoothness and related K-functional can be generalized in a natural way for all $r \geq 1$. The equivalence between these two constructive characteristics is well-known (see Theorem 6.2 in (DeVore & Lorentz, 1993)). We cite it here as

Theorem A *There are constants $c_1, c_2 > 0$, which depend only on r, such that for all $f \in C[0,1]$*

$$c_1\omega^{\varphi}_r(f,h) \leq K^{\varphi}_r(f,h^r) \leq c_2\omega^{\varphi}_r(f,h),\ 0 < h \leq (2r)^{-1}. \tag{1.3}$$

In many problems in approximation theory it is a difficult task to prove direct or inverse estimates directly in terms of the Ditzian-Totik moduli of smoothness. Instead of this, the K-functionals have become a powerful tool to establish such statements. However, the main disadvantage of the latter, is the fact, that practically it is impossible to calculate the value of the K-functional for a given function f. Therefore, the usual way is first to prove direct or inverse estimates in terms of the K-functional, and after this using Theorem A to reformulate the results in terms of the moduli of smoothness. This explains how important is to have a good information about the magnitude of the constants c_1, c_2. To the best of our knowledge the problem to find the best possible values of c_2 and c_1 (in the sense c_2-minimal and c_1-maximal in (1.3)) is still not solved. Hardly anything appears to be known about the explicit description of the size of c_1 and c_2 for $r = 1, 2$ and about their asymptotic dependence on r for $r > 2$. The first attempt in this direction is Theorem 3.5 in (Gonska & Tachev, 2003) which we cite here as

Theorem B *For $m \geq 2$, $h \in \left[\frac{\sqrt{2}}{md(m)}, \frac{\sqrt{2}}{(m-1)d(m-1)}\right]$ the following inequalities hold for any $f \in C[0,1]$:*

$$\frac{1}{16}\omega^{\varphi}_2(f,h) \leq K^{\varphi}_2(f,h^2) \leq c_2(m)\omega^{\varphi}_2(f,h),$$

where

$$c_2(m) := 1 + \left(\frac{m}{m-1}\right)^2 \frac{48}{d^2(m-1)},$$

and the sequence $d(m)$ is defined as

$$d(m) = \frac{\sqrt{m^4+m^2+1}-1}{\sqrt{m^4+m^2+1}+m^2},\ d(m) \to \frac{1}{2},\ m \to \infty.$$

It is clear that $\lim_{m\to\infty} c_2(m) = 193$. If we restrict our attention to values $h \leq 1$, as a corollary from Theorem B we get

$$\frac{1}{16}\omega^{\varphi}_2(f,h) \leq K^{\varphi}_2(f,h^2) \leq 404 \cdot \omega^{\varphi}_2(f,h). \tag{1.4}$$

The difficulties in the proof of Theorem B are connected with the construction of an appropriate auxiliary function g in the definition of K^{φ}_2. Actually we apply a "smoothing" technique to the linear interpolant on certain places near the points of interpolation to obtain an appropriate quadratic C^1-spline based upon the knot sequence. This method was developed in (Gonska & Kovacheva, 1994; Gonska & Tachev, 2003; H. Gonska, 2002) and further refined in (Gavrea, 2002). In this note we essentially improve the value of the constant 404 in (1.4). Our main result states the following:

Theorem 1 *The following inequalities hold for any $f \in C[0,1]$, $h \in (0,1]$:*

$$\frac{1}{16}\omega_2^{\varphi}(f,h) \le K_2^{\varphi}(f,h^2) \le (5+2\sqrt{2})\cdot\omega_2^{\varphi}(f,h). \tag{1.5}$$

In Section 2 we give the proof of Theorem 1. In Section 3 we apply Theorem 1 to obtain quantitative estimates in terms of second order Ditzian-Totik modulus of smoothness for approximation by genuine Bernstein-Durrmeyer operator, considered in (P. E. Parvanov, 1994) and also for pointwise estimates, established in (Felten, 1998). In the last section we consider the case $r = 1$, which is closely related to piecewise linear interpolation at specific knot sequence.

2. Proof of Theorem 1

To obtain as small as possible value of the constant c_2 in (1.3) we need an appropriate auxiliary function g in the definition of the K-functional. We use the construction, developed by Gavrea in (Gavrea, 2002) and based on the ideas from (Gonska & Kovacheva, 1994; Gonska & Tachev, 2003). Let m be fixed natural number, $m \ge 1$. The patition Δ_m of the interval $[0,1]$ is given by

$$\Delta_m: \ 0 = x_0 < x_1 < \cdots < x_{2m+2} = 1,$$

where

$$x_k = \sin^2\frac{k\pi}{4(m+1)},\ k = 0,1,\ldots,2m+2. \tag{2.1}$$

We denote by $S_m(f)$ a piecewise linear interpolant with interpolation knots-the points x_k, $k = 0,1,\ldots,2m+2$. Each point $(x_k, S_m(f,x_k))$, $k = 1,2,\ldots,2m+1$ we associate with two other points $(a_k, S_m(f,a_k)), (b_k, S_m(f,b_k))$ such that

$$a_1 = \frac{x_1}{2},\ b_1 - x_1 = x_1 - a_1,$$

and

$$a_k = \frac{x_k + x_{k-1}}{2},\ b_k - x_k = x_k - a_k,\ k = 1,2,\ldots,2m+1.$$

The function g is defined as follows:

For $x \in [0,a_1] \cup [b_{2m+1},1]$ we set $g(x) = S_m(f,x)$.

For $x \in [a_k,b_k]$, $k = 1,\ldots,2m+1$, $g(x)$ is the 2nd degree Bernstein polynomial over the interval $[a_k,b_k]$, determined by the ordinates $S_m(f,a_k), f(x_k), S_m(f,b_k)$.

For $x \in [b_k,a_{k+1}]$, $k = 1,2,\ldots,2m$ we set $g(x) = S_m(f,x)$. Thus $g(x)$ is uniquely determined by the interpolation conditions and is C^1-continuous. For this function the following two crucial estimates are proved in Theorem 6 in (Gavrea, 2002):

$$\|f-g\|_\infty \le \omega_2^{\varphi}\left(f,\sin\frac{\pi}{2(m+1)}\right), \tag{2.2}$$

$$\|\varphi^2 g''\|_\infty \le \frac{1}{\sin^2 \frac{\pi}{4(m+1)}} \cdot \omega_2^\varphi\left(f, \sin\frac{\pi}{2(m+1)}\right). \tag{2.3}$$

For any positive number $h \in (0,1]$ there exists a natural number $m \ge 1$, such that

$$h \in \left[\sin\frac{\pi}{2(m+1)}, \sin\frac{\pi}{2m}\right].$$

Hence (2.2) and (2.3) imply

$$\|f-g\|_\infty \le \omega_2^\varphi(f,h), \tag{2.4}$$

$$h^2\|\varphi^2 g''\|_\infty \le \frac{\sin^2\frac{\pi}{2m}}{\sin^2\frac{\pi}{4(m+1)}} \cdot \omega_2^\varphi(f,h). \tag{2.5}$$

It is easy to verify that the sequence

$$c(m) := \frac{\sin^2\frac{\pi}{2m}}{\sin^2\frac{\pi}{4(m+1)}}$$

is monotone decreasing, i.e. $c(m) \le c(1) = 4 + 2\sqrt{2}$. Consequently the right-hand side of (1.5) is proved. Lastly we point out that the constant $\frac{1}{16}$ in (1.5) could be derived from Theorem 6.1 in (DeVore & Lorentz, 1993). Thus the proof of Theorem 1 is completed.

3. Applications

1. The genuine Bernstein-Durrmeyer operator. As first application of Theorem 1 let us consider the so-called genuine Bernstein-Durrmeyer operator, introduced by Goodman and Sharma in (Goodman & Sharma, 1991) and given by

$$U_n(f,x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, k = 0,\dots,n$, are the fundamental Bernstein polynomials. Parvanov and Popov proved in (P. E. Parvanov, 1994) in an elementary and very elegant manner a direct and a strong converse inequality of type A, thus completely characterizing the approximation speed of the operators. The main result in (P. E. Parvanov, 1994) states the following:

For any $f \in C[0,1]$ we have

$$\frac{1}{2}\|U_n f - f\|_\infty \le K_2^\varphi(f, \frac{1}{2n}) \le (4+\sqrt{2})\|U_n f - f\|_\infty. \tag{3.1}$$

As a corollary from Theorem 1 and (3.1) we obtain

$$\frac{1}{2(5+2\sqrt{2})}\|U_n f - f\|_\infty \le \omega_2^\varphi(f, \frac{1}{\sqrt{2n}}) \le 16(4+\sqrt{2})\|U_n f - f\|_\infty. \tag{3.2}$$

2. The Bernstein operator The classical Bernstein operator $B_n(f,x)$ for a given function $f \in C[0,1]$ is defined by

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x).$$

Let $\Phi : [0,1] \to R$, $\Phi \neq 0$ be a function such that Φ^2 is concave. Then the pointwise approximation

$$|B_n(f,x) - f(x)| \leq 2K_2^{\varphi}\left(f, n^{-1}\frac{\varphi^2(x)}{\Phi^2(x)}\right),\ x \in [0,1], \tag{3.3}$$

holds true for all $f \in C[0,1], n \in N$. This result was proved by Felten in (Felten, 1998). As a straightforward corollary from Theorem 1 we get

$$|B_n(f,x) - f(x)| \leq 2(5 + 2\sqrt{2})\omega_2^{\varphi}\left(f, n^{-\frac{1}{2}}\frac{\varphi(x)}{\Phi(x)}\right),\ x \in [0,1]. \tag{3.4}$$

4. The case $r = 1$

In this section we consider the interval $[-1,1]$ instead of $[0,1]$. After a linear transformation it is clear that each estimate in one of these two cases can be obtained from the other. The weight function over $[-1,1]$ is now $\varphi(x) = \sqrt{1-x^2}$. Let $\Delta_n : -1 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of the interval $[-1,1]$ such that the inequalities

$$c_3(x_{k+1} - x_k) \leq \frac{\varphi(x)}{n} \leq c_4(x_{k+1} - x_k) \tag{4.1}$$

are satisfied for $k = 1, 2, \ldots, n-2,\ x \in [x_k, x_{k+1}]$, and also

$$c_3(x+1) \leq \frac{\varphi(x)}{n} \leq c_4(x+1),\ x \in [x_0, x_1],$$

$$c_3(1-x) \leq \frac{\varphi(x)}{n} \leq c_4(1-x),\ x \in [x_{n-1}, x_n],$$

where $c_i, i = 3,4$ are absolute positive constants independent of n. The function g in the definition of K_1^{φ} we define as the linear interpolant of $f \in C[-1,1]$ with knots $\{x_k\}$. For $x \in [x_k, x_{k+1}]$, $k = 1, \ldots, n-2$, from the properties of linear interpolation it follows that

$$|g(x)-f(x)| \leq \omega_1(f, x_{k+1}-x_k) = \sup\{|f(x+\frac{h}{2})-f(x-\frac{h}{2})|,\ x, x\pm\frac{h}{2} \in [x_k, x_{k+1}]\} \leq \omega_1^{\varphi}(f, \frac{1}{c_3 n}). \tag{4.2}$$

Let $x \in [-1, x_1]$. The case $x \in [x_{n-1}, 1]$ is analogous. Obviously

$$|g(x) - f(x)| \leq \sup\left\{|f(x+\frac{h}{2}) - f(x-\frac{h}{2})|,\ x, x\pm\frac{h}{2} \in [-1, x_1]\right\}.$$

The inequality $x - \frac{h}{2} \geq -1$ yields $h \leq 2(x+1) \leq \frac{2}{c_3}\frac{\varphi(x)}{n}$, which follows from (4.1). To summarize we proved

$$\|f - g\|_{\infty} \leq \omega_1^{\varphi}(f, \frac{2}{c_3 n}). \tag{4.3}$$

Next we evaluate the second term in the definition of the K-functional.

For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \ldots, n-2$, it is easy to verify that

$$\frac{1}{n}|\varphi(x)g'(x)| = \frac{\varphi(x)}{n(x_{k+1} - x_k)}|f(x_k) - f(x_{k+1})|.$$

Using (4.1) and (4.2) we get

$$\frac{1}{n}\|\varphi g'\|_{L_\infty[x_1, x_{n-1}]} \le c_4 \omega_1^\varphi(f, \frac{1}{c_3 n}).$$

It remains to consider $x \in [-1, x_1]$. In this case we observe that

$$\frac{\varphi(x)}{n} \le c_4(x+1).$$

Therefore for $x \in [-1, x_1]$ we have

$$\frac{1}{n}|\varphi(x)g'(x)| \le c_4 \omega_1^\varphi(f, \frac{2}{c_3 n}).$$

Finally we arrive at

$$\frac{1}{n}\|\varphi g'\| \le c_4 \omega_1^\varphi(f, \frac{2}{c_3 n}). \tag{4.4}$$

For every $0 < t < 1$ there exists $n \ge 2$ such that

$$\frac{2}{c_3 n} < t < \frac{2}{c_3(n-1)}.$$

Combining (4.3) and (4.4) we get

$$K_1^\varphi(f, t) \le \left[1 + \frac{2c_4}{c_3}\left(\frac{n}{n-1}\right)\right]\omega_1^\varphi(f, t). \tag{4.5}$$

It is clear that the condition number $\frac{c_4}{c_3}$ of our system of knots determines the value of the constant in front of the modulus. By the previous considerations we have shown the validity of

Theorem 2. *For $f \in C[-1, 1]$, $n \ge 2$, $t \le \frac{1}{2}$ we have*

$$\frac{1}{8}\omega_1^\varphi(f, t) \le K_1^\varphi(f, t) \le c_2(n)\omega_1^\varphi(f, t), \tag{4.6}$$

where

$$c_2(n) = \left[1 + \frac{2c_4}{c_3}\left(\frac{n}{n-1}\right)\right].$$

Remark 1. The constant $\frac{1}{8}$ in the left side of (4.6) follows easily if we verify the computations made in Theorem 6.1 in Chapter 6 in (DeVore & Lorentz, 1993).

Remark 2. If we strictly follow the construction in the proof of Theorem 2, it is possible to improve the value of $c_2(n)$, i.e. to obtain the value of the latter as small as possible. In order to do this, we would find an optimal set of knots, satisfying (4.1) with a condition number as small as possible. In this case we formulate the following

Open problem. Find the optimal set $\{x_k\}$ satisfying (4.1) in the sense that the condition number $\frac{c_4}{c_3}$ is minimal.

Here we give two examples of knots.

Example A. In this example we choose $\{x_k\}$ to be the well-known zeros of the Chebyshev polynomial of the first kind

$$x_k = \cos\theta_k,\ \theta_k := \frac{(2k-1)\pi}{2n},\ k = 1, \ldots, n,\ x_{n+1} := -1, x_0 := 1.$$

Following (7.7-7.8) in Chapter 8 in [1] we get $c_3 = \frac{1}{3\pi}$, $c_4 = 3\pi$. The condition number is $9\pi^2$.

Example B. Here we choose the extremal points of the Chebyshev polynomial of the first kind

$$x_k = \cos(\frac{k\pi}{n}),\ k = 0, \ldots, n.$$

This is the same set of interpolation knots , considered in Section 2 for the interval $[0, 1]$. In this case we compute $c_3 = \frac{1}{2\pi}$, $c_4 = 2\pi$. The condition number is $4\pi^2$-better as in Example A.

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