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Approach Merotopies and Associated Near Sets

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Abstract

This article introduces associated near sets of a collection of sets. The proposed approach introduces a means of defining as well as describing an ε-approach merotopy in terms of the members of associated sets of collections that are sufficiently near. A characterization for continuous functions is established using associated near sets. This article also introduces p-containment considered in the context of near sets. An application of the proposed approach is given in terms of digital image classification.

Keywords: Approach space, associated set, p-containment, merotopy, near sets.
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1. Introduction

For any real-valued function f of a real variable, the associated sets of f (Agronsky, 1982) are the sets

$$E^{\alpha}(f) = \{x : f(x) < \alpha\} \text{ and } E_{\alpha}(f) = \{x : f(x) > \alpha\},$$

where $\alpha \in \mathbb{R}$ (the set of all real numbers). Many classes of functions can be characterized in terms of their associated sets. The study of associated sets of a function started in 1922 (Coble, 1922) and elaborated in (Zahorski, 1950; Bruckner, 1967; Agronsky, 1982; Petrakiev, 2009). For example, a function is continuous, if and only if, all of its associated sets are open, a function is approximately continuous if, and only if, all of its associated sets F sets with the property that every point of an associated set is a point of Lebesgue density of that set. More generally, A. Bruckner (Bruckner, 1967, p. 228) has shown that if κ is a class of functions characterized in terms of an associated set P and h is a homeomorphism, then the associated sets of the function $h \circ f$ are all members

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of P and $h \circ f \in \kappa$. S. Agronsky (Agronsky, 1982, p. 767) has observed that an associated set for a function in $\mathcal{M}_i$ must be 'more dense' near each of its members than an associated set for a function in $\mathcal{M}_{i-1}$.

In this paper, associated sets defined in terms of ε-approach merotopies are considered. In particular, we consider associated sets containing members that are sufficiently near each other relative to ε-approach merotopies. Carrying forward the idea of defining and characterizing a function in terms of an associated set, it is possible to define and characterize an approach merotopy in terms of an associated set of collections. Using the concept of associated sets, an equivalent condition for continuous functions is obtained.

2. Preliminaries

Let X be a nonempty ordinary set. The power set of X is denoted by $\mathcal{P}(X)$, the family of all collections of subsets of $\mathcal{P}(X)$ is denoted by $\mathcal{P}^2(X)$. We denote by $\aleph_0$ the first infinite cardinal number, by J an arbitrary index set, and $|A|$ is the cardinality of A, where $A \subseteq X$. For $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$, we say $\mathcal{A} \vee \mathcal{B} \equiv \{A \cup B : A \in \mathcal{A},\ B \in \mathcal{B}\}$; $\mathcal{A}$ *corefines* $\mathcal{B}$ (written as $\mathcal{A} \prec \mathcal{B}$), if and only if, for all $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $B \subseteq A$. For $\mathcal{A} \subseteq \mathcal{P}(X)$, $stack(\mathcal{A}) = \{A \subseteq X : B \subseteq A$, for some $B \in \mathcal{A}\}$ and $sec(\mathcal{A}) = \{B \subseteq X : A \cap B \neq \emptyset$, for all $A \in \mathcal{A}\} = \{B \subseteq X : X - B \notin \mathrm{stack}(\mathcal{A})\}$. Observe that $\mathrm{sec}^2(\mathcal{A}) = \mathrm{stack}(\mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$. A *filter* on X is a nonempty subset $\mathcal{F}$ of $\mathcal{P}(X)$ satisfying: $\emptyset \notin \mathcal{F}$; if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$; and if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. A maximal filter on X is called an *ultrafilter* on X. A *grill* on X is a subset $\mathcal{G}$ of $\mathcal{P}(X)$ satisfying: $\emptyset \notin \mathcal{G}$; if $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$; and if $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$. Note that for any $x \in X, \dot{x} = \{A \subseteq X : x \in A\}$ is an ultrafilter on X, which is also a grill on X. There is one-to-one correspondence between the set of all filters and the set of all grills on X by the relation: $\mathcal{F}$ is a filter on X if and only if $\mathrm{sec}(\mathcal{F})$ is a grill on X; and $\mathcal{G}$ is a grill on X if and only if, $\mathrm{sec}(\mathcal{G})$ is a filter on X.

In its most basic form, an approach merotopy is a measure of the nearness of members of a collection. For collections $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$, a function $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) :\longrightarrow [0, \infty]$ satisfying a number of properties is a called an ε-approach merotopy. A pair of collections are near, provided $\nu(\mathcal{A}, \mathcal{B}) = 0$. For $\varepsilon \in (0, \infty]$, the pair $\mathcal{A}, \mathcal{B}$ are *sufficiently near*, provided $\nu(\mathcal{A}, \mathcal{B}) < \varepsilon$.

Let cl be a Kuratowski closure operator on X. Then the topological space (X, cl) is called a *symmetric topological space* if and only if $x \in cl(\{y\}) \Longrightarrow y \in cl(\{x\})$, for all $x, y \in X$.

Definition 2.1. A function $\delta : X \times \mathcal{P}(X) \longrightarrow [0, \infty]$ is called a distance on X (Lowen, 1997; Lowen *et al.*, 2003) if for any $A, B \subseteq X$ and $x \in X$, the following conditions are satisfied:
(**D**.1) $\delta(x, \{x\}) = 0$,
(**D**.2) $\delta(x, \emptyset) = \infty$,
(**D**.3) $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$,
(**D**.4) $\delta(x, A) \leq \delta(x, A^{(\alpha)}) + \alpha$, for all $\alpha \in [0, \infty]$, where $A^{(\alpha)} \doteqdot \{x \in X : \delta(x, A) \leq \alpha\}$.
The pair (X, δ) is called an approach space.

Definition 2.2. A generalized approach space (X, ρ) (Peters & Tiwari, 2011, 2012) is a nonempty set X equipped with a generalized distance function $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0, \infty]$, if and only if, for all nonempty subsets $A, B, C \in \mathcal{P}(X)$, ρ satisfies properties (A.1)-(A.5), *i.e.*,

(**A**.1) $\rho(A, A) = 0$,
(**A**.2) $\rho(A, \emptyset) = \infty$,
(**A**.3) $\rho(A, B \cup C) = \min\{\rho(A, B), \rho(A, C)\}$,
(**A**.4) $\rho(A, B) = \rho(B, A)$,
(**A**.5) $\rho(A, B) \leq \rho(A, B^{(\alpha)}) + \alpha$, for every $\alpha \in [0, \infty]$, where $B^{(\alpha)} = \{x \in X : \rho(\{x\}, B) \leq \alpha\}$.

It has been observed that the notion of distance in an approach space is closely related to the notion of nearness (Khare & Tiwari, 2012, 2010; Tiwari, Jan. 2010). In particular, consider the Čech distance between sets.

Definition 2.3. Čech Distance (Čech, 1966). For nonempty subsets $A, B \in \mathcal{P}(X)$, $\rho(a, b)$ is the standard distance between $a \in A, b \in B$ and the Čech distance $D_\rho : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0, \infty]$ is defined by

$$D_\rho(A, B) \doteqdot \begin{cases} \inf\{\rho(a, b) : a \in A, b \in B\}, & \text{if } A \text{ and } B \text{ are not empty,} \\ \infty, & \text{if } A \text{ or } B \text{ is empty.} \end{cases}$$

Remark. Observe that (X, D_ρ) is a generalized approach space. The distance $D_\rho(A, B)$ is a variation of the distance function introduced by E. Čech in his 1936–1939 seminar on topology (Čech, 1966) (see, also, (Beer *et al.*, 1992; Hausdorff, 1914*a*; Leader, 1959)).

3. Approach merotopic spaces

Definition 3.1. Let $\varepsilon \in (0, \infty]$. Then a function $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ is an ε-approach merotopy on X, if and only if, for any collections $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}^2(X)$, the properties (AN.1)-(AN.5) are satisfied.

(AN.1) $\mathcal{A} \prec \mathcal{B} \Longrightarrow \nu(\mathcal{C}, \mathcal{A}) \leq \nu(\mathcal{C}, \mathcal{B})$,

(AN.2) $\mathcal{A} \neq \emptyset, \mathcal{B} \neq \emptyset$ and $(\bigcap \mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \Longrightarrow \nu(\mathcal{A}, \mathcal{B}) < \varepsilon$,

(AN.3) $\nu(\mathcal{A}, \mathcal{B}) = \nu(\mathcal{B}, \mathcal{A})$ and $\nu(\mathcal{A}, \mathcal{A}) = 0$,

(AN.4) $\mathcal{A} \neq \emptyset \Longrightarrow \nu(\emptyset, \mathcal{A}) = \infty$,

(AN.5) $\nu(\mathcal{C}, \mathcal{A} \vee \mathcal{B}) \geq \nu(\mathcal{C}, \mathcal{A}) \wedge \nu(\mathcal{C}, \mathcal{B})$.

The pair (X, ν) is termed as an *ε-approach merotopic space*.

For an ε-approach merotopic space (X, ν), we define: $cl_\nu(A) \doteqdot \{x \in X : \nu(\{\{x\}\}, \{A\}) < \varepsilon\}$, for all $A \subseteq X$. Then cl_ν is a Čech closure operator on X.

Let $cl_\nu(\mathcal{A}) \doteqdot \{cl_\nu(A) : A \in \mathcal{A}\}$. Then an ε-approach merotopy ν on X is called an *ε-approach nearness* on X, if the following condition is satisfied:

(AN.6) $\nu(cl_\nu(\mathcal{A}), cl_\nu(\mathcal{B})) \geq \nu(\mathcal{A}, \mathcal{B})$.

In this case, cl_ν is a Kuratowski closure operator on X.

Lemma 3.1. *Let $\varepsilon \in (0, \infty]$, and let (X, ν) and (Y, ν') be ε-approach nearness spaces. Then $f : (X, \nu) \longrightarrow (Y, \nu')$ is a contraction if and only if $\nu(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B})) \geq \nu'(\mathcal{A}, \mathcal{B})$, for all $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(Y)$.*

Example 3.1. Let D_ρ be a gap functional. Then the function $\nu_{D_\rho} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ defined as

$$\nu_{D_\rho}(\mathcal{A}, \mathcal{B}) \doteqdot \sup_{A \in \mathcal{A}, B \in \mathcal{B}} D_\rho(A, B); \quad \nu_{D_\rho}(\mathcal{A}, \mathcal{A}) \doteqdot \sup_{A \in \mathcal{A}} D_\rho(A, A) = 0,$$

is an ε-approach merotopy on X. Define $cl_\rho(A) = \{x \in X : \rho(\{x\}, A) < \varepsilon\}$, $A \subseteq X$. Then cl_ρ is a Čech closure operator on X. Further, if $\rho(cl_\rho(A), cl_\rho(B)) \geq \rho(A, B)$, for all $A, B \subseteq X$, then cl_ρ is a Kuratowski closure operator on X, and we call ρ as an *ε-approach function* on X; and (X, ρ) is an *ε-approach space*. In this case, ν_{D_ρ} is an ε-approach nearness on X.

So, there are many instances of ε-approach nearness on X just as there are many instances of ε-approach spaces (Lowen, 1997) and metric spaces on X.

Definition 3.2. Near and Almost Near Collections
For collections $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$, assume that the function $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) :\longrightarrow [0, \infty]$ is an ε-approach merotopy. A pair of collections are *near*, provided $\nu(\mathcal{A}, \mathcal{B}) = 0$. For $\varepsilon \in (0, \infty]$, the pair $\mathcal{A}, \mathcal{B}$ are *ε-near* (*almost near*), provided $\nu(\mathcal{A}, \mathcal{B}) < \varepsilon$ (Peters & Tiwari, 2011). Otherwise, collections $\mathcal{A}, \mathcal{B}$ are far, *i.e.*, *sufficiently apart*, provided $\nu(\mathcal{A}, \mathcal{B}) \geq \varepsilon$.

4. Associated collections

It is possible to characterise ε-approach merotopies in terms of associated collections.

Definition 4.1. Associated Collections of an ε-Approach Merotopy
Let X denote an ordinary nonempty set and let $\mathcal{A} \in \mathcal{P}^2(X)$ denote collections of subsets of X. Suppose that $\varepsilon \in (0, \infty]$ and ν be an ε-approach merotopic space. The upper associated set of $\mathcal{A}$ with respect to ν is defined by

$$E^\varepsilon(\mathcal{A}) \doteqdot \{\mathcal{B} \in \mathcal{P}^2(X) : \ \nu(\mathcal{A}, \mathcal{B}) > \varepsilon\}.$$

and the lower associated set of $\mathcal{A}$ with respect to ν is defined by

$$E_\varepsilon(\mathcal{A}) \doteqdot \{\mathcal{B} \in \mathcal{P}^2(X) : \ \nu(\mathcal{A}, \mathcal{B}) < \varepsilon\}.$$

Example 4.1. Let D_ρ be a gap functional. For $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$, the function $\nu_{D_\rho} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ is defined by

$$\nu_{D_\rho}(\mathcal{A}, \mathcal{B}) \doteqdot \sup_{A \in \mathcal{A}, B \in \mathcal{B}} D_\rho(A, B); \quad \nu_{D_\rho}(\mathcal{A}, \mathcal{A}) \doteqdot \sup_{A \in \mathcal{A}} D_\rho(A, A) = 0.$$

From Def. 4.1, $E_\varepsilon(\mathcal{A})$ is the lower associated set of $\mathcal{A}$ for a given $\varepsilon \in \mathbb{R}$. Similarly, obtain the upper associated set $E^\varepsilon(\mathcal{A})$ of $\mathcal{A}$ as a collection $\mathcal{B} \in \mathcal{P}^2(X)$, provided $\nu_{D_\rho}(\mathcal{A}, \mathcal{B}) > \varepsilon$.

Additional examples of lower and upper associated collections are given next.

Example 4.2. Let (X, ν) be an ε-approach nearness on X, $r < \varepsilon < \infty$ and $\varepsilon' < \varepsilon$. Then

(**ASet**.1) Associated sets $E^{\varepsilon}(\mathcal{A}), E_{\varepsilon}(\mathcal{A})$ of $\mathcal{A}$ with respect to $\nu_1 : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ such that

$$\nu_1(\mathcal{A}, \mathcal{B}) = \begin{cases} \infty, & \text{if } \emptyset \in \mathcal{A} \text{ or } \emptyset \in \mathcal{B}, \\ r, & \text{otherwise,} \end{cases}$$

is defined by:
if $\emptyset \in \mathcal{A}$, $E^{\varepsilon}(\mathcal{A}) = \mathcal{P}^2(X)$ and $E_{\varepsilon}(\mathcal{A}) = \emptyset$,
if $\emptyset \notin \mathcal{A}$, $E^{\varepsilon}(\mathcal{A}) = \{\mathcal{A} \in \mathcal{P}^2(X) : \emptyset \in \mathcal{B}\}$ and $E_{\varepsilon}(\mathcal{A}) = \{\mathcal{A} \in \mathcal{P}^2(X) : \emptyset \notin \mathcal{B}\}$.

(**ASet**.2) Associated sets $E^{\varepsilon}(\mathcal{A}), E_{\varepsilon}(\mathcal{A})$ of $\mathcal{A}$ with respect to $\nu_2 : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ such that

$$\nu_2(\mathcal{A}, \mathcal{B}) = \begin{cases} \infty, & \text{if } \emptyset \in \mathcal{A} \text{ or } \emptyset \in \mathcal{B}, \\ \inf\{\nu(\mathcal{A}, \mathcal{B}), \varepsilon'\}, & \text{otherwise,} \end{cases}$$

is defined by:
if $\emptyset \in \mathcal{A}$, $E^{\varepsilon}(\mathcal{A}) = \mathcal{P}^2(X)$ and $E_{\varepsilon}(\mathcal{A}) = \emptyset$,
if $\emptyset \notin \mathcal{A}$, $E^{\varepsilon}(\mathcal{A}) = \{\mathcal{A} \in \mathcal{P}^2(X) : \emptyset \in \mathcal{B}\}$ and $E_{\varepsilon}(\mathcal{A}) = \{\mathcal{A} \in \mathcal{P}^2(X) : \emptyset \notin \mathcal{B}\}$.

Proposition 1. *A collection in the lower associated set of $\mathcal{A}$ with respect to the ε-approach merotopy ν is sufficiently near $\mathcal{A}$.*

Proof. Assume $\mathcal{B} \in E_{\varepsilon}(\mathcal{A})$, the lower associated set of $\mathcal{A}$ with respect to ν. From Def. 3.2, $\mathcal{A}, \mathcal{B}$ are sufficiently near. □

Proposition 2. *A collection in upper associated set of $\mathcal{A}$ with respect to the ε-approach merotopy ν are sufficiently apart.*

Proof. Immediate from from Def. 4.1 and Def. 3.2. □

We now present a characterization for continouous functions.

Theorem 4.1. *Let ν_X and ν_Y be ε-approach merotopies on X and Y, respectively. A mapping $f : X \longrightarrow Y$ is continuous, if and only if, $\mathcal{A} \in E_{\varepsilon}(x) \implies f(\mathcal{A}) \in E_{\varepsilon}(f(x))$, for all $\mathcal{A} \in \mathcal{P}^2(X)$ and for all $x \in X$.*

Proof. Let $f : X \longrightarrow Y$ be continuous, $x \in X$ and $\mathcal{A} \in \mathcal{P}^2(X)$. Suppose that $\mathcal{A} \in E_{\varepsilon}(x)$. Then $\nu(\mathcal{A}, \{\{x\}\}) < \varepsilon$, which gives $\nu(\{A\}, \{\{x\}\}) < \varepsilon$, for all $A \in \mathcal{A}$. That is, $x \in cl_{\nu_X}(A)$, for all $A \in \mathcal{A}$. Consequently, $f(x) \in f(cl_{\nu_X}(A)) \subseteq cl_{\nu_Y}(f(A))$, for all $A \in \mathcal{A}$. Hence, $f(\mathcal{A}) \in E_{\varepsilon}(f(x))$. The converse is obvious. □

Definition 4.2. Finite Strong Containment Property (Agronsky, 1982).
Let p be a property defined for sets of real numbers with respect to sets containing them. If $A \subset B$, then A is p-contained in B (written $A \underset{p}{\subset} B$), provided A has the property p with respect to B. Put $k \in [0, \infty)$. Then p is a finite strong containment property, provided

(**p**.1) If $A \underset{p}{\subset} B \subset F$ and p is defined for $A \subset F$, then $A \underset{p}{\subset} F$,

(**p**.2) If $A \subset B \underset{p}{\subset} F$, then $A \underset{p}{\subset} F$,

(**p**.3) If, for each $n \in \mathbb{N}, E_n \underset{p}{\subset} F_n$, then $\bigcup_{n=1}^{k} E_n \underset{p}{\subset} \bigcup_{n=1}^{k} F_n$.

Example 4.3. Strong Containment of Sufficiently Near Collections
Put $\varepsilon \in (0, \infty]$. Let (X, ν) be an ε-approach nearness on X and $p \doteqdot$ 'sufficiently near' defined for $\mathcal{A}, \mathcal{B} \in \mathcal{P}(X)$ such that $\nu(\mathcal{A}, \mathcal{B}) < \varepsilon$. From Example 4.2, assume $\mathcal{A}, \mathcal{B} \in E_\varepsilon(\nu_2)$ and $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{A} \underset{p}{\subset} \mathcal{B}$.

Proof.
(**p**.1) Assume $\mathcal{A}, \mathcal{B}, C \in E_\varepsilon(\nu_2)$. By definition, $\mathcal{A} \underset{p}{\subset} \mathcal{B}$. Assume $\mathcal{B} \subset C$, then $\mathcal{A} \underset{p}{\subset} \mathcal{B} \subset C$. Since $\mathcal{B}, C \in E_\varepsilon(\nu_2)$, then $\mathcal{A} \underset{p}{\subset} C$.

(**p**.2) Assume $\mathcal{A}, \mathcal{B}, C \in E_\varepsilon(\nu_2)$ and that $\mathcal{A} \subset \mathcal{B} \subset C$. By definition, $\mathcal{A} \subset \mathcal{B} \underset{p}{\subset} C$ and by assumption $\mathcal{A} \subset C$. Since $\mathcal{A}, C \in E_\varepsilon(\nu_2)$, then $\mathcal{A} \underset{p}{\subset} C$.

(**p**.3) The proof of this strong containment property follows by mathematical induction.

□

5. Description-based neighbourhoods

For N. Bourbaki, a set is a neighbourhood of each of its points if, and only if, the set is open (Bourbaki, 1971, §1.2) (Bourbaki, 1966, §1.2, p. 18). A set A is *open*, if and only if, for each $x \in A$, all points *sufficiently near*[1] x belong to A.

For a Hausdorff neighbourhood (denoted by N_r), *sufficiently near* is explained in terms of the distance between points y and x being less than some radius r (Hausdorff, 1914*b*, §22). In other words, a Hausdorff neighbourhood of a point is an open set such that each of its points is sufficiently close to its centre.

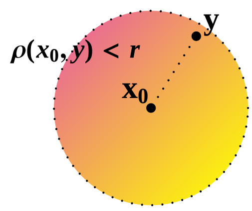

Figure 1: Nbd $N_r(x_0)$

Traditionally, nearness of points is measured in terms of the location of the points. Let $\rho : X \times X :\rightarrow [0, \infty]$ denote the standard distance[2] between points in X. For $r \in (0, \infty]$, a neighbourhood of $x_0 \in X$ is the set of all $y \in X$ such that $\rho(x_0, y) < r$ (see, *e.g.*, Fig. 1, where the distance $\rho(x, y)$ between each pair x_0, y is less than r in the neighbourhood). In that case, a neighbourhood is called an open ball (Engelking, 1989, §4.1) or spherical neighbourhood (Hocking & Young, 1988, §1-4). In the plane, the points in a spherical neighbourhood (nbd) are contained in the interior of a circle.

Next, an alternative to a spherical neighbourhood is called a visual neighbourhood (denoted nbd$_v$), which stems from recent work on descriptively near sets (Naimpally & Peters, 2013; Peters, 2013; Peters & Naimpally, 2012).

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[1]...*tous les points assez voisins d'un point x* (Bourbaki, 1971, p. TG I.3)
[2]*i.e.*, for $x, y \in X \subset, \rho(x, y) = |x - y|$.

Definition 5.1. Visual Neighbourhood
A visual nbd$_v$ of a point x_0 (denoted N_{r_ϕ}) is an open set A such that the visual information values extracted from all of the points in A are sufficiently near the corresponding visual information values at x_0. Let ϕ denote a probe function used to extract visual information from a point in nbd$_v$. Sufficient nearness of points in a visual nbd$_v$ is defined in terms of bound r_ϕ, a real number. That is, points $x_0, x \in A$ are sufficiently near, *i.e.*, provided

$$\rho_\phi(x_0, y) = |\phi(x_0) - \phi(y)| < r_\phi.$$

Example 5.1. Visual Neighbourhood in a Drawing
In its simplest form (see, *e.g.*, Fig. 2), a nbd$_v$ (denoted by N_{r_ϕ}) is defined in terms of a real-valued probe function ϕ used to extract visual information from the pixels in a digital image, reference point x_0 (*not* necessarily the centre of the nbd$_v$) and 'radius' r_ϕ such that

$$X = \{\text{drawing visual pixels}\}, x, y \in X,$$

$$\phi : X \to [0, \infty], (\text{probe function}, \textit{e.g.}, \text{probe } \phi(x) = \text{pixel } x \text{ intensity}),$$

$$\rho_\phi(x_0, y) = |\phi(x_0) - \phi(y)|, (\text{visual distance}),$$

$$x_0 \in X, (\text{nbd}_v \text{ reference point}),$$

$$r_\phi \in (0, \infty], (\text{sufficient nearness bound}),$$

$$N_{r_\phi}(x_0) = \{y \in X : \rho_\phi(x_0, y) < r_\phi\}, (\text{visual nbd}_v).$$

At this point, observe that the appearance of a visual neighbourhood can be quite different from the appearance of a spherical neighbourhood. For this reason, x_0 is called a *reference point* (not a centre) in a *nbd*$_v$. A visual neighbourhood results from a consideration of the features of a point in the neighbourhood and the measurement of the distance between neighbourhood points[3]. For example, $\phi(x_0)$ in Fig. 2 is a description of x_0 (probe ϕ is used to extract a feature value from x in the form of pixel intensity). Usually, a complete description of a point x in a *nbd*$_v$ is in the form of a feature vector containing probe function values extracted from x (see, *e.g.*, (Henry, 2010, §4), for a detailed explanation of the near set approach to perceptual object description). Observe that the members $y \in N_{r_\phi}(x_0)$ in the visual neighbourhood in Fig. 2 have descriptions that are *sufficiently near* the description of the reference point x_0.

For example, each of the points in the green shaded regions in Fig. 2 have intensities that are very close to the intensity of the point x_0. By contrast, many points in the purple shaded region have higher intensities (*i.e.*, more light) than the pixel at x_0, For example, consider the intensities of the points in the visual nbd represented by the green wedge-shaped region and some outlying green circular regions and the point x_4 in the purple region in Fig. 2, where

$$r_\phi = 5 \text{ low intensity difference},$$

[3]It is easy to prove that *a visual neighbourhood is an open set*

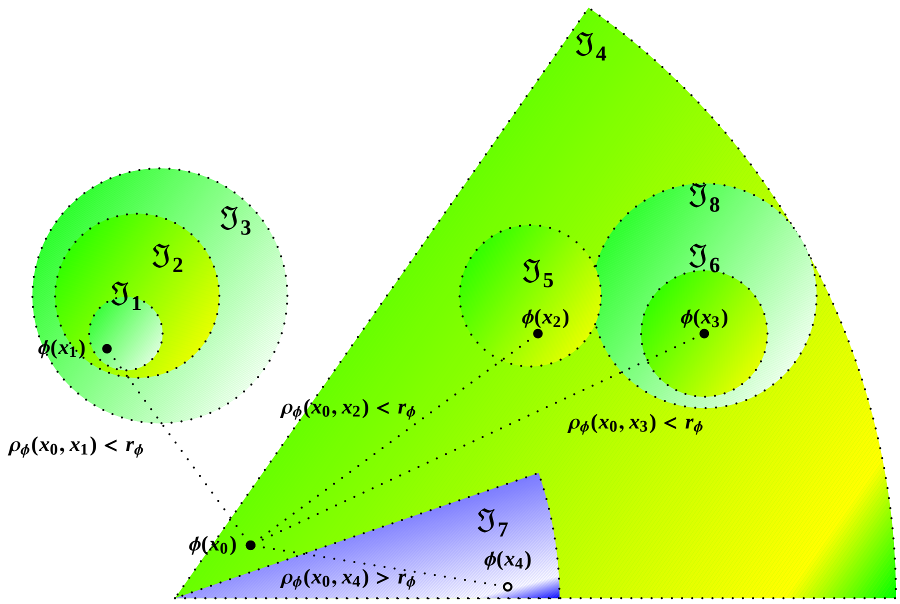


Figure 2: Sample Visual Nbd $N_{r_\phi}(x_0)$ in a Drawing

$$\rho_\phi(x_0, x_1) = |\phi(x_0) - \phi(x_1)| < r_\phi,$$

$$\rho_\phi(x_0, x_2) = |\phi(x_0) - \phi(x_2)| < r_\phi,$$

$$\rho_\phi(x_0, x_3) = |\phi(x_0) - \phi(x_3)| < r_\phi, \text{ but}$$

$$\rho_\phi(x_0, x_4) = |\phi(x_0) - \phi(x_4)| > r_\phi, \text{ where } \phi(x_4) = \text{high intensity (white)}.$$

In the case of the point x_4 in Fig. 2, the intensity is high (close to white), *i.e.*, $\phi(x_4) \sim 255$. By contrast the point x_0 has low intensity (less light), *e.g.*, $\phi(x_0) \sim 20$. Assume $r_\phi = 5$. Hence, $|\phi(x_0) - \phi(x_4)| > r_\phi$. As in the case of C. Monet's paintings[4], the distance between probe function values representing visual information extracted from image pixels can be sufficiently near a centre x_0 (perceptually) but the pixels themselves can be *far apart*, *i.e.*, not sufficiently near, if one considers the locations of the pixels.

Remark. **Filters and Grills**
In Fig. 2, observe that $\mathcal{F}_1 = \Im_1 \subset \Im_2 \subset \Im_3$ is a filter. Again, observe that $\mathcal{F}_2 = \{\Im_4, \Im_6, \Im_8\}$ is a filter. It can be shown that the set $\mathcal{G} = \{\Im_4, \Im_6, \Im_8\}$ is a grill.

Proof. Let $A = \Im_6, B = \Im_4$ in Fig. 2. From $\mathcal{F}_2$, we know that $\Im_6 \subset \Im_4$ and $\Im_4 \subset \mathcal{G}$. Then $B \in \mathcal{G}$. Observe that $\Im_5 \cup \Im_6 \in \mathcal{G}$, then $\Im_5 \in \mathcal{G}$ or $\Im_6 \in \mathcal{G}$.

[4] A comparison between Z. Pawlak's and C. Monet's waterscapes is given in Peters (2011).

In addition, let X denote the set of regions shown in Fig. 2. Obviously, $\mathcal{F} = \{\mathfrak{I}_4, \mathfrak{I}_6, \mathfrak{I}_8\}$ is a filter, if and only if, $sec(\mathcal{F})$ is a grill $\mathcal{G}_2$ on X. Further, $\mathcal{G}_3 = \{\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3\}$ is a grill, if and only if, $sec(\mathcal{G}_3)$ is a filter. $\square$

Example 5.2. Sample Associated Sets
Let D_{ρ_ϕ} be a gap functional such that

$$D_{\rho_\phi}(A,B) \doteqdot \begin{cases} \inf\{\rho_\phi(a,b) : a \in A, b \in B\}, & \text{if } A \text{ and } B \text{ are not empty,} \\ \infty, & \text{if } A \text{ or } B \text{ is empty.} \end{cases}$$

Then the function $\nu_{D_{\rho_\phi}} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0,\infty]$ defined as

$$\nu_{D_{\rho_\phi}}(\mathcal{A},\mathcal{B}) \doteqdot \sup_{A\in\mathcal{A}, B\in\mathcal{B}} D_{\rho_\phi}(A,B); \quad \nu_{D_{\rho_\phi}}(\mathcal{A},\mathcal{A}) \doteqdot \sup_{A\in\mathcal{A}} D_{\rho_\phi}(A,A) = 0,$$

is an ε-approach merotopy on X. In terms of the labelled sets $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4, \mathfrak{I}_5, \mathfrak{I}_6$ in Fig. 2, we can identify the following lower associated set E_ε in (Assoc.1) and upper associated E^ε in (Assoc.2) with respect to $\nu_{D_{\rho_\phi}}$.

(**Assoc**.1) $E_\varepsilon(\mathfrak{I}_1) = \{\mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4, \mathfrak{I}_5, \mathfrak{I}_6\}$, where

$$\nu_{D_{\rho_\phi}}(\mathfrak{I}_1, \mathfrak{I}_i) < \varepsilon \text{ for } i \in \{2,3,5,6\},$$

i.e., for $a \in \mathfrak{I}_1, b \in \mathfrak{I}_i, i \neq 1$, $\rho_\phi(a,b) < \varepsilon$, since the colours of all of the pixels are similar in each set $\mathfrak{I}_i \in E_\varepsilon(\mathfrak{I}_1)$ in Fig. 2. The sets in $E_\varepsilon(\mathfrak{I}_1)$ are sufficiently near $\mathfrak{I}_1$.

(**Assoc**.2) $E^\varepsilon(\mathfrak{I}_4) = \{\mathfrak{I}_7\}$, where

$$\nu_{D_{\rho_\phi}}(\mathfrak{I}_4, \mathfrak{I}_7) > \varepsilon,$$

i.e., for $a \in \mathfrak{I}_4, b \in \mathfrak{I}_7$, $\rho_\phi(a,b) > \varepsilon$, due to the fact that the green colour of each the pixels in $\mathfrak{I}_4$ is dissimilar to the purple or white colour of the pixels in $\mathfrak{I}_7$ in Fig. 2. In effect, the sets in $E^\varepsilon(\mathfrak{I}_4)$ are far apart from $\mathfrak{I}_4$ with respect to $\nu_{D_{\rho_\phi}}$.

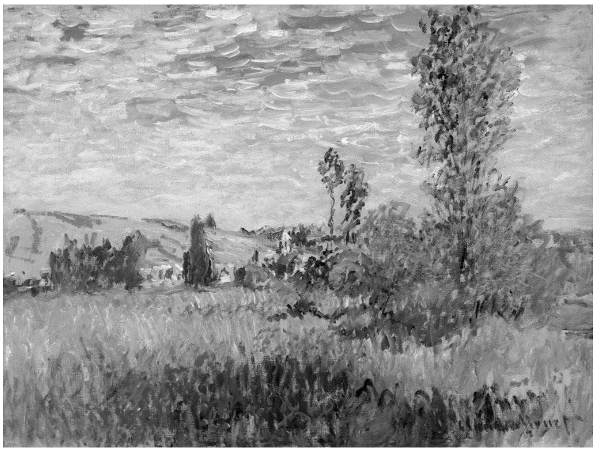
Example 5.3. Sufficiently Near Strong *p*-Containment
After a manner similar to Example 4.3, let (X, ν_{ρ_ϕ}) be an ε-approach nearness on X and $p \doteqdot$ 'sufficiently near' defined for $\{A\},\{B\} \in \mathcal{P}(X)$ such that $\nu_{\rho_\phi}(\{A\},\{B\}) < \varepsilon$. Consider $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3$ in Fig. 2. It is a straightforward task to verify that

(**p**.1) $\mathfrak{I}_1 \underset{p}{\subset} \mathfrak{I}_2 \subset \mathfrak{I}_3$ implies $\mathfrak{I}_1 \underset{p}{\subset} \mathfrak{I}_3$,

(**p**.2) $\mathfrak{I}_1 \subset \mathfrak{I}_2 \underset{p}{\subset} \mathfrak{I}_3$ implies $\mathfrak{I}_1 \underset{p}{\subset} \mathfrak{I}_3$,

(**p**.3) Considering only $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3$,

$$\mathfrak{I}_1 \underset{p}{\subset} \mathfrak{I}_3 \text{ and} \mathfrak{I}_2 \underset{p}{\subset} \mathfrak{I}_3 \text{ implies } \bigcup_{i=1}^{2} \mathfrak{I}_i \underset{p}{\subset} \mathfrak{I}_3.$$



3.1: Monet meadow

3.2: Nbd $N_{r_{\phi_{grey}}}$

Figure 3: Sample Monet Meadow nbd $N_{r_{\phi_{grey}}}$, with $r_{\phi_{grey}}$=10

Example 5.4. Visual Neighbourhood in a Digital Image
Consider visual neighbourhoods in digital images, where each point is an image pixel (picture element). A pixel is described in terms of its feature values. Pixel features include grey level intensity and primary colours red, green, and blue with wavelengths 700 nm, 546.1 nm and 435.8 nm, respectively)[5], texture, and shape information. Visual information (feature values) is extracted from each pixel with a variety of probe functions.

For example, consider a xixth century, St. Martin, Vetheuil landscape by C. Monet rendered as a greyscale image in Fig. 3.1. Let $\phi_{grey}(x)$ denote a probe that extracts the greylevel intensity from a pixel x and let $r_{\phi_{grey}} = 10$. This will lead to the single visual neighbourhood represented by the green-shaded regions shown in Fig. 3.2. To obtain the visual nbd in Fig. 3.2, replace the greylevel intensity of each point sufficiently near the intensity $\phi_{grey}(x_0)$ with a green colour. The result is green-coloured visual nbd $N_{r_{\phi_{grey}}}$ in Fig. 3.2. This set of intensities in the visual nbd shown in $N_{r_{\phi_{grey}}}$ is an example of an open set contain numbers representing intensities that are sufficiently near x_0. To verify this, notice that the pixel intensities for large regions of the sky, hills and meadow in Fig. 3.1 are quite similar. This is the case with the sample pixels (points of light) x_0, x_1, x_2 in Fig. 3.2, where the in $\left|\phi_{grey}(x_0) - \phi_{grey}(x_1)\right| < r_{\phi_{grey}}$ and $\left|\phi_{grey}(x_0) - \phi_{grey}(x_2)\right| < r_{\phi_{grey}}$.

In summary, the lower associated set $E_\varepsilon(\{\{x_i\}\})$ is the set of all visual neighbourhoods of the pixel x_i in Fig. 3.2 that are descriptively ε-near each other. In addition, one can also observe

[5]The amounts of red, green and blue that form a particular colour are called *tristimulus* values. Let R, G, B denote red, green, blue tristimulus values, respectively, with green almost in the middle of the wavelengths of the visual spectrum, which is at 568 nm. Then define the following probe functions to extract the colour components of a pixel.

$$r = \frac{R}{R+G+B}, \quad g = \frac{G}{R+G+B}, \quad b = 1 - r - g.$$

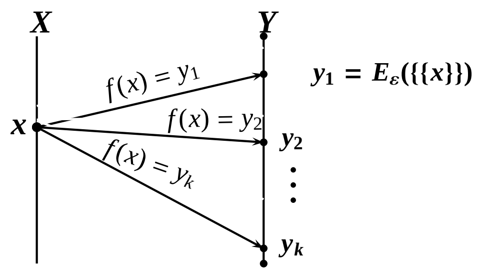


Figure 4: $f(x) = |E_\varepsilon(\{\{x\}\})| > 0$

that the upper associated set $E_\varepsilon(\{\{x_i\}\})$ contains all visual neighbourhoods that are descriptively dissimilar to x_i.

Example 5.5. Bipartite Graph for Associated Sets
Although this example continues the discussion of paintings, the proposed bipartite graph representation of associated sets is easily extended to members of any pair of nonempty sets. For example, consider classifying paintings by a particular artist by collecting together nonempty associated lower sets of sufficiently near neighbourhoods extracted from pairs of pictures. To see this, let X denote a set of query images and let Y denote a set of test images (*i.e.*, X contains pictures showing paintings, where each painting in X is compared with the paintings in the set of sample paintings Y).

The goal is to collect together those pictures in Y containing neighbourhoods of points in $y \in Y$ that are sufficiently similar to neighbourhoods of points in a picture $x \in X$. Let $N_a \in X, N_b \in Y$ denote neighbourhoods that are sufficiently near. Then construct the lower associated set $E_\varepsilon(N_a) = \{N_b, \ldots\}$. A query image is similar to a test image if, and only if, $E_\varepsilon(N_a) > 0$.

Given approach spaces $(X, \nu_{D_{\rho_{\phi grey}}}), (Y, \nu_{D_{\rho_{\phi grey}}})$, consider a function $f : X \to Y$ defined by $f(x) = |E_\varepsilon(x)|$, where $x \in X$. Then the relation between a particular painting and one or more associated lower sets can be represented by a bipartite graph (see Fig. 4). The image set

$$\mathcal{O} = \{f(x_i) : i \in \text{ and } |f(x_i)| > 0\}$$

can be extracted from Fig. 4. The set $\mathcal{O}$ has interest, since two of its members reveal the least similar and most similar paintings in relation to a particular query image. That is, $\inf\{\mathcal{O}\}, \sup\{\mathcal{O}\}$ function values correspond to the least similar and most similar of the paintings that are sufficiently near the query image $x \in X$.

Similarly, one can determine the collection of those paintings dissimilar to a given query picture with a nonempty associated upper set containing visual neighbourhoods taken from the query image and a test image.

References

Agronsky, S.J. (1982). A generalization of a theorem of maximoff and applications. *Trans. Amer. Math. Soc.* **273**(2), 767–779.

Beer, G., A. Lechnicki, S. Levi and S.A. Naimpally (1992). Distance functionals and suprema of hyperspace topologies. *Annali di Matematica pura ed applicata* **CLXII**(IV), 367–381.

Bourbaki, N. (1966). *Elements of Mathematics. General Topology, Part 1*. Hermann & Addison-Wesley. Paris & Reading, MA, U.S.A. i-vii, 437 pp.

Bourbaki, N. (1971). *Topologie générale, 1-4*. Hermann. Paris. Springer-Verlag published a new edition in 2007.

Bruckner, A.M. (1967). On characterizing classes of functions in terms of associated sets. *Canad. Math. Bull.* **10**(2), 227–231.

Čech, E. (1966). *Topological Spaces, revised Ed. by Z. Frolik and M. Katătov*. John Wiley & Sons. NY.

Coble, A.B. (1922). Associated sets of points. *Trans. Amer. Math. Soc.* **24**(1), 1–20.

Engelking, R. (1989). *General Topology, Revised & completed edition*. Heldermann Verlag. Berlin.

Hausdorff, F. (1914*a*). *Grundzüge der Mengenlehre*. Veit and Company. Leipzig. viii + 476 pp.

Hausdorff, F. (1914*b*). *Set Theory*. AMS Chelsea Publishing. Providence, RI. 352 pp.

Henry, C.J. (2010). Near Sets: Theory and Applications, Ph.D. dissertation, supervisor: J.F. Peters. PhD thesis. Department of Electrical & Computer Engineering.

Hocking, J.G. and G.S. Young (1988). *Topology*. Dover. NY.

Khare, M. and S. Tiwari (2010). Grill determined L-approach merotopological spaces. *Fund. Inform.* **48**, 1–12.

Khare, M. and S. Tiwari (2012). L-approach merotopies and their categorical perspective. *Demonstratio Math.* **45**(3), 699–716.

Leader, S. (1959). On clusters in proximity spaces. *Fundamenta Mathematicae* **47**, 205–213.

Lowen, R. (1997). *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*. Oxford Mathematical Monographs, Oxford University Press. Oxford, UK. viii + 253pp.

Lowen, R., D. Vaughan and M. Sioen (2003). Completing quasi metric spaces: an alternative approach. *Houstan J. Math.* **29**(1), 113–136.

Naimpally, S.A. and J.F. Peters (2013). *Topology with Applications. Topological Spaces Via Near and Far*. World Scientific. Singapore.

Peters, J. F. and S. Tiwari (2012). Completing extended metric spaces: an alternative approach. *Appl. Math. Letters* **25**, 1544–1547.

Peters, J.F. (2011). How near are Zdzisław Pawlak's paintings? Merotopic distance between regions of interest. pp. 1–19. , *in* A. Skowron, S. Suraj, Eds., Intelligent Systems Reference Library volume dedicated to Prof. Zdzisław Pawlak, Springer, Berlin.

Peters, J.F. (2013). Local near sets: Pattern discovery in proximity spaces. *Math. Comp. Sci.* **7**(1), *to appear*.

Peters, J.F. and S. Tiwari (2011). Approach merotopies and near filters. *Gen. Math. Notes* **3**(1), 32–45.

Peters, J.F. and S.A. Naimpally (2012). Applications of near sets. *Amer. Math. Soc. Notices* **59**(4), 536–542.

Petrakiev, I. (2009). On self-associated sets of points in small projective spaces. *Comm. in Algebra* **37**(2), 397–405.

Tiwari, S. (Jan. 2010). Some Aspects of General Topology and Applications. Approach Merotopic Structures and Applications, supervisor: M. Khare. PhD thesis. Department of Mathematics, Allahabad (U.P.), India.

Zahorski, Z. (1950). Sur la première dérivée. *Trans. Amer. Math. Soc.* **69**, 1–54.