Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 3 (1) (2013) 32–37

On an Unified Class of Functions of Complex Order

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Abstract

In this paper, we obtain a necessary and sufficient condition for functions in an unified class of functions of complex order. Some of our results generalize previously known results.

Keywords: Analytic function, univalent, starlike, convex, subordination.
2012 MSC: 30C45.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that f and g are analytic in $\mathbb{U}$. We say that the function f is subordinate to g in $\mathbb{U}$, or g superordinate to$f$ in $\mathbb{U}$, and we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function ω in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. If g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true, see (Miller & Mocanu, 1981) and (Miller & Mocanu, 2000):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on ϕ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $\mathcal{S}^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \tag{1.2}$$

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and $C(\phi)$ class of functions in $f \in \mathcal{S}$ for which

$$1+\frac{zf''(z)}{f'(z)} \prec \phi(z). \tag{1.3}$$

These classes were introduced and studied by (Ma & Minda, 1992). (Ravichandran *et al.*, 2005) defined classes $\mathcal{S}_b^*(\phi)$ and $C_b(\phi)$ of complex order defined as follows :

$$\mathcal{S}^*(\phi;b)=\left\{f\in\mathcal{A}:1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\phi(z)\quad(b\in\mathbb{C}^*=\mathbb{C}\backslash\{0\})\right\} \tag{1.4}$$

and

$$C(\phi;b)=\left\{f\in\mathcal{A}:1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\prec\phi(z)\quad(b\in\mathbb{C}^*)\right\}. \tag{1.5}$$

From (1.4) and (1.5), we have

$$f\in C(\phi;b)\Longleftrightarrow zf'\in\mathcal{S}^*(\phi;b).$$

Now, we introduce a more general class of complex order $\mathcal{T}(\phi;\lambda,b)$ as follows:

Definition 1.1. Let $\phi(z)$ be an analytic function with positive real part on ϕ with $\phi(0)=1$, $\phi'(0)>0$ which maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $\mathcal{T}(\phi;\lambda,b)$ consists of all analytic functions $f\in\mathcal{A}$ satisfying:

$$1+\frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\prec\phi(z)\ (b\in\mathbb{C}^*;\lambda\geq 0). \tag{1.6}$$

We note that

(*i*) $\mathcal{T}(\phi;0,b)=\mathcal{S}^*(\phi;b)$ and $\mathcal{T}(\phi;1,b)=C(\phi;b)$ (Ravichandran *et al.*, 2005);

(*ii*) $\mathcal{T}(\phi;0,1)=\mathcal{S}^*(\phi)$ and $\mathcal{T}(\phi;1,1)=C(\phi)$ (Ma & Minda, 1992);

(*iii*) $\mathcal{T}\left(\frac{1+(1-2\alpha)z}{1-z};0,b\right)=\mathcal{S}_\alpha^*(b)$ and $\mathcal{T}\left(\frac{1+(1-2\alpha)z}{1-z};1,b\right)=C_\alpha(b)\,(0\leq\alpha<1;b\in\mathbb{C}^*)$ (Frasin, 2006);

(*iv*) $\mathcal{T}\left(\frac{1+z}{1-z};0,b\right)=\mathcal{T}\left(\frac{1+(2b-1)z}{1-z};0,1\right)=\mathcal{S}^*(b)\,(b\in\mathbb{C}^*)$ (Nasr & Aouf, 1985) and (Wiatrowski, 1970);

(*v*) $\mathcal{T}\left(\frac{1+z}{1-z};1,b\right)=\mathcal{T}\left(\frac{1+(2b-1)z}{1-z};1,1\right)=C(b)\,(b\in\mathbb{C}^*)$ (Nasr & Aouf, 1982) and (Wiatrowski, 1970);

(vi) $\mathcal{T}\left(\frac{1+z}{1-z};0,1-\alpha\right)=\mathcal{T}\left(\frac{1+(1-2\alpha)z}{1-z};0,1\right)=\mathcal{S}^{*}(\alpha)$

and $\mathcal{T}\left(\frac{1+z}{1-z};1,1-\alpha\right)=\mathcal{T}\left(\frac{1+(1-2\alpha)z}{1-z};1,1\right)=\mathcal{C}(\alpha)\,(0\leq\alpha<1)$ (Robertson, 1936);

(vii) $\mathcal{T}\left(\frac{1+z}{1-z};0,be^{-i\gamma}\cos\gamma\right)=\mathcal{S}^{\gamma}(b)$ and $\mathcal{T}\left(\frac{1+z}{1-z};1,be^{-i\gamma}\cos\gamma\right)=\mathcal{C}^{\gamma}(b)\left(|\gamma|<\frac{\pi}{2},b\in\mathbb{C}^{*}\right)$ (Al-Oboudi & Haidan, 2000) and (Aouf *et al.*, 2005).

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $\mathcal{T}(\phi;\lambda,b)$ which we have defined. The motivation of this paper is to generalize the results obtained by (Ravichandran *et al.*, 2005), (Aouf *et al.*, 2005), (Srivastava & Lashin, 2005) and also (Obradovic *et al.*, 1989).

2. Main Results

Unless otherwise mentioned, we assume throughout the sequel that $b\in\mathbb{C}^{*}$, $\lambda\geq0$ and all powers are understood as principle values. To prove our main result, we need the following lemmas.

Lemma 2.1. *(Ruscheweyh, 1982) Let ϕ be a convex function defined on $\mathbb{U}$, $\phi(0)=1$. Define $F(z)$ by*

$$F(z)=z\,\exp\left(\int_{0}^{z}\frac{\phi(t)-1}{t}dt\right). \tag{2.1}$$

Let $p(z)=1+p_1z+p_2z^2+\ldots$be analytic in $\mathbb{U}$. Then

$$1+\frac{zq'(z)}{q(z)}\prec\phi(z) \tag{2.2}$$

if and only if for all $|s|\leq1$ and $|t|\leq1$, we have

$$\frac{p(tz)}{p(sz)}\prec\frac{sF(tz)}{tF(sz)}. \tag{2.3}$$

Lemma 2.2. *(Miller & Mocanu, 2000) Let $q(z)$ be univalent in $\mathbb{U}$ and let $\varphi(z)$ be analytic in a domain containing $q(\mathbb{U})$. If $\frac{zq'(z)}{q(z)}$ is starlike, then*

$$zp'(z)\varphi(p(z))\prec zq'(z)\varphi(q(z)),$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

Theorem 2.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f\in\mathcal{T}(\phi;\lambda,b)$ if and only if for all $|s|\leq1$ and $|t|\leq1$, we have*

$$\left[\left(\frac{sf(tz)}{tf(sz)}\right)^{1-\lambda}\left(\frac{f'(tz)}{f'(sz)}\right)^{\lambda}\right]^{\frac{1}{b}}\prec\frac{sF(tz)}{tF(sz)}. \tag{2.4}$$

Proof. Define the function $p(z)$ by

$$p(z) = \left[\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\right]^{\frac{1}{b}} \quad (z \in \mathbb{U}). \tag{2.5}$$

Taking logarithmic derivative of (2.5), we get

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right].$$

Since $f \in \mathcal{T}(\phi; \lambda, b)$, then we have

$$1 + \frac{zp'(z)}{p(z)} \prec \phi(z)$$

and the result now follows from Lemma 2.1. □

Putting $\lambda = 0$ in Theorem 2.1, we obtain the following result of (Shanmugam *et al.*, 2009).

Corollary 2.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in \mathcal{S}^*(\phi; b)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{sf(tz)}{tf(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.6}$$

For $\lambda = 1$ in Theorem 2.1, we obtain the following result of (Shanmugam *et al.*, 2009).

Corollary 2.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in C(\phi; b)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{f'(tz)}{f'(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.7}$$

Theorem 2.2. *Let $\phi(z)$ be starlike with respect to 1 and $F(z)$ given by (2.1) be starlike. If $f \in \mathcal{T}(\phi; \lambda, b)$, then we have*

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec \left(\frac{F(z)}{z}\right)^{b}. \tag{2.8}$$

Proof. Let $p(z)$ be given by (2.5) and $q(z)$ be given by

$$q(z) = \frac{F(z)}{z} \quad (z \in \mathbb{U}). \tag{2.9}$$

After a simple computation we obtain

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{zf'(z)}\right) - 1\right].$$

and

$$\frac{zq'(z)}{q(z)} = \frac{zF'(z)}{F(z)} - 1 = \phi(z) - 1.$$

Since $f \in \mathcal{T}(\phi; \lambda, b)$, we have

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}.$$

The result now follows by an application of Lemma 2.2. □

Putting $\lambda = 0$ in Theorem 2.2, we obtain the following results of (Shanmugam *et al.*, 2009).

Corollary 2.3. *Let $\phi(z)$ be starlike with respect to 1 and $F(z)$ given by* (2.1) *be starlike. If $f \in \mathcal{S}^*(\phi; b)$, then we have*

$$\frac{f(z)}{z} \prec \left(\frac{F(z)}{z}\right)^b.$$

Taking $\phi(z) = \dfrac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.2, we get the following corollary.

Corollary 2.4. *If $f \in \mathcal{T}\left(\dfrac{1 + Az}{1 + Bz}; \lambda, b\right)$ $(-1 \leq B < A \leq 1)$, then we have*

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^\lambda \prec (1 + Bz)^{\frac{(A-B)b}{B}} \quad (B \neq 0).$$

For $\phi(z) = \dfrac{1+z}{1-z}$ and $\lambda = 0$ in Theorem 2.2, we get the following result of (Obradovic *et al.*, 1989), and (Srivastava & Lashin, 2005).

Corollary 2.5. *If $f \in \mathcal{S}^*(b)$, then we have*

$$\frac{f(z)}{z} \prec (1 - z)^{-2b}.$$

Putting $\phi(z) = \dfrac{1+z}{1-z}$ and $\lambda = 1$ in Theorem 2.2, we get the following result of (Obradovic *et al.*, 1989), and (Srivastava & Lashin, 2005).

Corollary 2.6. *If $f \in C(b)$, then we have*

$$f'(z) \prec (1 - z)^{-2b}.$$

For $\phi(z) = \dfrac{1+z}{1-z}, \lambda = 0$ and replacing b by $be^{-i\gamma}\cos\gamma$ $\left(|\gamma| < \dfrac{\pi}{2}, b \in \mathbb{C}^*\right)$ in Theorem 2.2, we get the following result of (Aouf *et al.*, 2005).

Corollary 2.7. *If $f \in \mathcal{S}^{\gamma}(b)$, then we have*

$$\frac{f(z)}{z} \prec (1-z)^{-2be^{-i\gamma}\cos\gamma}.$$

Taking $\phi(z) = \dfrac{1+z}{1-z}$, $\lambda = 1$ and replacing b by $be^{-i\gamma}\cos\gamma \left(|\gamma| < \dfrac{\pi}{2}, b \in \mathbb{C}^{*}\right)$ in Theorem 2.2, we get the following result of (Aouf *et al.*, 2005).

Corollary 2.8. *If $f \in C^{\gamma}(b)$, then we have*

$$f'(z) \prec (1-z)^{-2be^{-i\gamma}\cos\gamma}.$$

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