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Third Order Boundary Value Problem with Integral Condition at Resonance

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Abstract

This paper deals with a class of third order boundary value problem with integral condition at resonance. Some existence results are obtained by using the coincidence degree theory of Mawhin.

Keywords: Fixed point theorem, coincidence degree theory of Mawhin, third order boundary value problem, integral condition, Fredholm operators, resonance.
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1. Introduction

Let us consider the following third-order differential equation:

$$x''' (t) = f (t, x (t) , x' (t)) , 0 < t < 1, \tag{1.1}$$

subject to the following nonlocal conditions

$$x (0) = x'' (0) = 0, x (1) = \frac{2}{\eta^2} \int_0^{\eta} x (t) dt, \eta \in (0, 1) , \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, and $\eta \in (0, 1)$. We say that the boundary value problem (1.1), (1.2) is a resonance problem if the linear equation $Lx = x'''$, with the boundary value conditions (1.2) has a non-trivial solution i.e., $\dim \ker L \geq 1$.

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The research of ordinary differential equations with nonlocal conditions plays a very important role in both theory and applications. It is widely used in describing a large number of physical, biological and chemical phenomena. Moreover, the theory of boundary-value problems with integral boundary conditions arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. In recent years, the multi-point boundary value problems at resonance for second order, third order ordinary differential equations have been extensively studied and many excellent results have been obtained, for instance, see (Feng & Webb, 1997*a*), (Feng & Webb, 1997*b*), (Gupta, 1995), (Gupta & Tsamatos, 1994), (Liu & Yu, 2002), (Liu, 2003), (Liu & Zhao, 2007), (Kosmatov, 2006), (Du, 2008), (Du & Ge, 2005), (Ma, 2005), (Nagle & Pothoven, 1995), (Xue & Ge, 2004). (see, also, (Y. Liu, 2005), (X. Lin, 2009), (H Zhang & Chen, 2009)). However, to our knowledge, the corresponding results for third-order with integral boundary conditions, are rarely seen (see, for example, (X. Lin & Meng, 2011), (Karakostas & Tsamatos, 2002), (Yang, 2006); (A. Yang, 2011) and references therein). (Meng & Du, 2010) studied the following second-order multi-point boundary value problem at resonance:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), t \in (0,1), \\ x(0) = \sum\limits_{i=1}^{m} \alpha_i x(\xi_i), \ x'(1) = \sum\limits_{j=1}^{n} \beta_j x\left(\eta_j\right), \end{cases}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, $e \in L^1[0,1], 0 < \xi_1 < .... < \xi_m < 1$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, ...m, m \geq 2$ and $0 < \eta_1 < ..... < \eta_n < 1, \beta_j \in \mathbb{R}$, $j = 1, ...n, n \geq 1$. By using coincidence degree of Mawhin the authors obtain many excellent results about the existence of solutions for the above problem under the resonance conditions $\sum\limits_{i=1}^{m} \alpha_i = \sum\limits_{j=1}^{n} \beta_j = 1$ and $\sum\limits_{i=1}^{m} \alpha_i \xi_i = 0$.

By using coincidence degree of Mawhin (Lin & Meng, 2011), established the existence of solutions for the following third-order multi-point boundary value problem at resonance

$$\begin{cases} x'''(t) = f(t, x(t), x'(t), x''(t)), 0 < t < 1 \\ x''(0) = \sum\limits_{i=1}^{m} \alpha_i x''(\xi_i), \quad x'(0) = 0, x(1) = \sum\limits_{j=1}^{n} \beta_j x\left(\eta_j\right), \end{cases}$$

where $0 < \xi_1 < .... < \xi_m < 1$, $\alpha_i \in \mathbb{R}$, $i = 1, ...m, m \geq 1$ and $0 < \eta_1 < ..... < \eta_n < 1, \beta_j \in \mathbb{R}$, $j = 1, ...n, n \geq 2$, and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function.

More recently, (X. Zhang & Ge, 2009) studied the following nonlocal boundary value problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), t \in (0,1) \\ x'(0) = \int_0^1 h(t) x'(t) dt, x'(1) = \int_0^1 g(t) x'(t) dt, \end{cases}$$

where $f, g \in C([0,1], [0,\infty))$. Especially by using the coincidence degree of Mawhin, and under the resonance conditions $\int_0^1 h(t) dt = 1$, and $\int_0^1 g(t) dt = 1$, the authors proved at least one solution.of the boundary value problem.

The purpose of this paper is to study the existence of solutions for nonlocal boundary value problem $(1,1), (1,2)$ at resonance and establish an existence theorem . Our method is based upon the coincidence degree theory of (Mawhin, 1979).

2. Main results

We first recall some notation and an abstract existence result (Mawhin, 1979).

Let X, Y be two real Banach spaces and let $L : domL \subset X \to Y$ be a linear operator which is Fredholm map of index zero and $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $ImP = KerL$, $KerQ = ImL$ and $X = KerL \oplus KerP$, $Y = ImL \oplus ImQ$. It follows that $L|_{domL\cap KerP}$:$domL \cap KerP \to ImL$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of X such that $domL \cap \Omega \neq \emptyset$, the map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if the map $QN\left(\overline{\Omega}\right)$ is bounded and $K_P(I - QN) : \overline{\Omega} \to X$ is compact. To obtain our existence results we use the following fixed point theorem of (Mawhin, 1979).

Theorem 2.1. *Let be L a Fredholm operator of index zero and N be L− compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied:*

i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(domL \backslash KerL) \cap \partial\Omega] \times (0, 1)$.

ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in KerL \cap \partial\Omega$.

iii) $\deg(QN|_{KerL}, \Omega \cap KerL, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection as above with* $\operatorname{Im} L = KerQ$.

Then the equation $Lx = Nx$ *has at least one solution in* $domL \cap \overline{\Omega}$.

In the following, we shall use the classical spaces $C[0,1]$, $C^1[0,1]$, $C^2[0,1]$ and $L^1[0,1]$. For $x \in C^2[0,1]$, we use the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$ where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and denote the norm in $L^1[0,1]$ by$\|\cdot\|_1$. We will use the Sobolev space $W^{3,1}(0,1)$ which is defined by $W^{3,1}(0,1) = \{x : [0,1] \to \mathbb{R} : x, x', x'' \text{ are absolutely continuous on } [0,1] \text{ with } x''' \in L^1[0,1]\}$.

Let $X = C^2[0,1]$, $Y = L^1[0,1]$, L is the linear operator from $domL \subset X$ to Y with $domL = \left\{x \in W^{3,1}(0,1) : x(0) = x''(0) = 0,\ x(1) = \frac{2}{\eta^2}\int_0^\eta x(t)\,dt\right\}$ and $Lx = x'''$, $x \in \operatorname{dom} L$. We define $N : X \to Y$ by setting

$$Nx = f(t, x(t), x'(t)),\ t \in (0,1).$$

Then the BVP (1.1) and (1.2) can be written as $Lx = Nx$.

Theorem 2.2. *Assume that the following conditions are satisfied:*

1) There exists functions $\alpha, \beta, \gamma \in L^1[0,1]$, *such that for all* $(x, y) \in \mathbb{R}^2$, $t \in [0,1]$ *then*

$$|f(t,x,y)| \leq \alpha(t)|x| + \beta(t)|y| + \gamma(t). \tag{2.1}$$

2) There exists a constant $M > 0$, *such that for* $x \in domL$, *if* $|x'(t)| > M$ *for all* $t \in [0,1]$, *then*

$$\int_0^1 (1-s)^2 f(s, x(s), x'(s))\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta - s)^3 f(s, x(s), x'(s))\,ds \neq 0. \tag{2.2}$$

3) There exists a constant $M^* > 0$, *such that for any* $x(t) = bt \in KerL$ *with* $|b| > M^*$, *either*

$$b\left[\int_0^1 (1-s)^2 f(s, b(s), b)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta - s)^3 f(s, b(s), b)\,ds\right] < 0, \tag{2.3}$$

or else

$$b\left[\int_0^1 (1-s)^2 f(s, b(s), b)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta - s)^3 f(s, b(s), b)\,ds\right] > 0. \tag{2.4}$$

then BVP $(1,1)$ *and* $(1,2)$ *has at least one solution in* $C^2[0,1]$, *provided*

$$\|\alpha\| + \|\beta\| < \frac{1}{2}. \tag{2.5}$$

2.1. Proof of Theorem 2.2

For the proof of Theorem 2.2 we shall apply Theorem 2.1 and the following lemmas.

Lemma 2.1. *The operator $L : domL \subset X \to Y$ is a Fredholm operator of index zero. Furthermore, the linear projector operator $Q : Y \to Y$ can be definied by*

$$Qy(t) = k\left[\int_0^1 (1-s)^2 y(s)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds\right]t,$$

where $k = 60/5 - 2\eta^3$ and the linear operator $K_P : \operatorname{Im} L \to dom\, L \cap Ker\, P$ can be written by

$$K_p y(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds, \forall y \in \operatorname{Im} L.$$

Furthermore

$$\left\|K_p y\right\| \le \|y\|_1, \forall y \in \operatorname{Im} L.$$

Proof. It is clear that

$$kerL = \{x \in \operatorname{dom} L : x = bt,\ b \in \mathbb{R},\ t \in [0,1]\} \simeq \mathbb{R}.$$

Now we show that

$$\operatorname{Im} L = \left\{y \in Y : \int_0^1 (1-s)^2 y(s)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0\right\}. \tag{2.6}$$

The problem

$$x''' = y \tag{2.7}$$

has a solution $x(t)$ that satisfies the conditions $x(0) = x''(0) = 0,\ x(1) = \frac{2}{\eta^2}\int_0^\eta x(t)\,dt$, if and only if

$$\int_0^1 (1-s)^2 y(s)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0. \tag{2.8}$$

In fact from (2.7) we have

$$x(t) = x''(0)\frac{t^2}{2} + x'(0)t + x(0) + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds = x'(0)t + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

According to $x(1) = \frac{2}{\eta^2}\int_0^\eta x(t)\,dt$,we obtain

$$\int_0^1 (1-s)^2 y(s)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds = 0.$$

On the other hand, if (2.8) holds, setting

$$x(t) = bt + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds,$$

where b is an arbitrary constant, then $x(t)$ is a solution of (2.7). Hence (2.6) holds.
Setting

$$Ry = \int_0^1 (1-s)^2 y(s)\,ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 y(s)\,ds,$$

define $Qy(t) = k.(Ry).t$, it is clear that $\dim \operatorname{Im} Q = 1$. We have

$$Q^2y = Q(Qy) = k(k.Ry)\left(\int_0^1 (1-s)^2\, s ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3\, s ds\right)t = (kRy)\,t = Qy,$$

which implies that the operator Q is a projector. Furthermore, $\operatorname{Im} L = kerQ$.

Let $y = (y - Qy) + Qy$, where $y - Qy \in KerQ = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$. It follows from $KerQ = \operatorname{Im} L$ and $Q^2y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then, we have $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Since $\dim KerL = 1 = \dim \operatorname{Im} Q = co\dim \operatorname{Im} L = 1$, L is a Fredholm map of index zero.

Now we define a projector P from X to X by setting

$$Px(t) = x'(0)\,t.$$

Then the generalized inverse $K_P : \operatorname{Im} L \to domL \cap KerP$ of L can be written by

$$K_p = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

Obviously, $\operatorname{Im} P = KerL$ and $P^2x = Px$. It follows from $x = (x - Px) + Px$ that $X = KerP + KerL$. By simple calculation, we can get that $KerL \cap KerP = \{0\}$. Then $X = KerL \oplus KerP$. From the definitions of P and K_P it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \operatorname{Im} L$, we have

$$\left(LK_p\right) y(t) = \left[\left(K_p y\right) t\right]''' = y(t),$$

and for $x \in domL \cap kerP$, we know

$$\left(K_p L\right) x(t) = \left(K_p\right) x'''(t) = \frac{1}{2}\int_0^t (t-s)^2 x'''(s)\,ds = x(t) - x(0) - x'(0)\,t - \frac{1}{2}x''(0)\,t^2,$$

in view of $x \in domL \cap kerP$, $x(0) = x''(0) = 0$ and $Px = 0$, thus

$$\left(K_p L\right) x(t) = x(t).$$

This shows that $K_p = \left(L|_{\mathrm{dom}L\cap \ker P}\right)^{-1}$. Also we have

$$\left\|K_p y\right\|_\infty \le \int_0^1 (1-s)^2 |y(s)|\,ds \le \int_0^1 |y(s)|\,ds = \|y\|_1,$$

and from $\left(K_p y\right)'(t) = \int_0^1 (1-s)\, y(s)\, ds$, we obtain

$$\left\|\left(K_p y\right)'\right\|_\infty \le \int_0^1 (1-s)\, |y(s)|\, ds \le \int_0^1 |y(s)|\, ds = \|y\|_1$$

then $\left\|K_p y\right\| \le \|y\|_1$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let* $\Omega_1 = \{x \in domL \backslash KerL : Lx = \lambda Nx, for\ some\ \lambda \in [0,1]\}$. *Then* Ω_1 *is bounded.*

Proof. Suppose that $x \in \Omega_1$, and $Lx = \lambda Nx$. Thus $\lambda \neq 0$ and $QNx = 0$, so it yields

$$\int_0^1 (1-s)^2 f(s, x(s), x'(s))\, ds - \frac{2}{3\eta^2} \int_0^\eta (\eta - s)^3 f(s, x(s), x'(s))\, ds = 0.$$

Thus, by condition (2), there exists $t_0 \in [0,1]$, such that $|x'(t)| \le M$. In view of

$$x'(0) = x'(t_0) - \int_0^{t_0} x''(t)\, dt, x''(t) = x''(0) + \int_0^t x'''(s)\, ds,$$

then, we have

$$|x'(0)| \le M + \int_0^1 \left(\int_0^1 |x'''(s)\, ds|\right) dt = M + \|x'''\|_1 = M + \|Lx\|_1 \le M + \|Nx\|_1 . \tag{2.9}$$

Again for $x \in \Omega_1$, $x \in domL \backslash KerL$, then $(I-P)x \in domL \cap KerP$ and $LPx = 0$, thus from Lemma 3, we know

$$\|(I-P)x\| = \left\|K_p L(I - Px)\right\| \le \|L(I-Px)\|_1 = \|Lx\|_1 \le \|Nx\|_1 . \tag{2.10}$$

From (2.9) and (2.10), we have

$$\|x\| \le \|Px\| + \|(I-P)x\| = |x'(0)| + \|(I-P)x\| \le M + 2\|Nx\|_1 . \tag{2.11}$$

From (2.1) and (2.11), we obtain

$$\|x\| \le 2\left[\|\alpha\|_1 \|x\|_\infty + \|\beta\|_1 \|x'\|_\infty + \|\gamma\|_1 + \frac{M}{2}\right]. \tag{2.12}$$

Thus, from $\|x\|_\infty \le \|x\|$ and (2.12) we have

$$\|x\|_\infty \le \frac{2}{1 - 2\|\alpha\|_1}\left[\|\beta\|_1 \|x'\|_\infty + \|\gamma\|_1 + \frac{M}{2}\right]. \tag{2.13}$$

From $\|x'\|_\infty \le \|x\|$, and (2.12) and (2.13), one has

$$\|x'\|_\infty \left[1 - \frac{2\|\beta\|_1}{1 - 2\|\alpha\|_1}\right] \le \frac{2}{1 - 2\|\alpha\|_1}\left[\|\gamma\|_1 + \frac{M}{2}\right].$$

Therefore,

$$\|x'\|_\infty \left[\frac{1-2\|\alpha\|_1 - 2\|\beta\|_1}{1-2\|\alpha\|_1}\right] \le \frac{1}{1-2\|\alpha\|_1}\left[2\|\gamma\|_1 + M\right].$$

i.e.,

$$\|x'\|_\infty \le \frac{2\left[\|\gamma\|_1 + \frac{M}{2}\right]}{1-2\|\alpha\|_1 - 2\|\beta\|_1} = M_1. \tag{2.14}$$

From (2.14), there exists $M_1 > 0$, such that

$$\|x'\|_\infty \le M_1, \tag{2.15}$$

thus from (2.15) and (2.13), there exists $M_2 > 0$, such that

$$\|x\|_\infty \le M_2. \tag{2.16}$$

Hence

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \le \max\{M_1, M_2\}.$$

Again from (2.1), (2.15) and (2.16), we have

$$\|x'''\|_1 = \|Lx\|_1 \le \|Nx\|_1 \le \|\alpha\|_1 M_2 + \|\beta\|_1 M_1 + \|\gamma\|_1.$$

So Ω_1 is bounded. □

Lemma 2.3. *The set* $\Omega_2 = \{x \in KerL : Nx \in \operatorname{Im} L\}$ *is bounded.*

Proof. Let $x \in \Omega_2$, then $x \in KerL = \{x \in domL : x = bt,\ b \in \mathbb{R},\ t \in [0,1]\}$, and $QNx = 0$, therefore

$$\int_0^1 (1-s)^2 f(s, bs, b)\, ds - \frac{2}{3\eta^2}\int_0^\eta (\eta - s)^3 f(s, bs, b)\, ds = 0.$$

From condition (2) of Theorem 2.2, $\|x\|_\infty = |b| \le M$, so $\|x\| = |b| \le M$, thus Ω_2 is bounded. □

Lemma 2.4. *If the first part of condition (3) of Theorem 2.2 holds, then*

$$b\frac{60}{5-2\eta^3}\left[\int_0^1 (1-s)^2 f(s, b(s), b)\, ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s, b(s), b)\, ds\right] < 0, \tag{2.17}$$

for all $|b| > M^*$. *Let* $\Omega_3 = \{x \in KerL : -\lambda Jx + (1-\lambda) QNx = 0,\ \ \lambda \in [0,1]\ \}$ *where* $J : KerL \to \operatorname{Im} Q$ *is the linear isomorphism given by* $J(bt) = bt$, $\forall b \in \mathbb{R}$, $t \in [0,1]$. *Then* Ω_3 *is bounded.*

Proof. Suppose that $x = b_0 t \in \Omega_3$, then we obtain

$$\lambda b_0 = (1-\lambda)\frac{60}{5-2\eta^3} \times \left(\int_0^1 (1-s)^2 f(s, b(s), b)\, ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s, b(s), b)\, ds\right).$$

If $\lambda = 1$, then $b_0 = 0$. Otherwise, if $|b_0| > M^*$, then in view of (2.17) one has $\lambda b_0^2 = b_0(1-\lambda)\frac{60}{5-2\eta^3} \times \left(\int_0^1 (1-s)^2 f(s, b(s), b)\, ds - \frac{2}{3\eta^2}\int_0^\eta (\eta-s)^3 f(s, b(s), b)\, ds\right) < 0$, which contradicts the fact that $\lambda b_0^2 \geq 0$. Then $|x| = |b_0 t| \leq |b_0| \leq M^*$, we obtain $\|x\| \leq M^*$, therefore $\Omega_3 \subset \{x \in KerL : \|x\| \leq M^*\}$ is bounded.

If $\lambda = 0$, it yields

$$\int_0^1 (1-s)^2 f(s, b(s), b)\, ds - \frac{2}{3\eta^2} \int_0^\eta (\eta - s)^3 f(s, b(s), b)\, ds = 0,$$

taking condition (2) of Theorem 2.2 into account, we obtain $\|x\| = |b| \leq M^*$. □

Lemma 2.5. *If the second part of condition (3) of Theorem 2.2 holds, then*

$$b\frac{60}{5-2\eta^3}\left[\int_0^1 (1-s)^2 f(s, b(s), b)\, ds - \frac{2}{3\eta^2} \int_0^\eta (\eta - s)^3 f(s, b(s), b)\, ds\right] > 0, \tag{2.18}$$

for all $|b| > M^*$. *Let* $\Omega_3 = \{x \in KerL \ : \ \lambda Jx + (1-\lambda) QNx = 0, \ \ \lambda \in [0,1] \ \}$, *here J is defined as in Lemma 2.4. Similar to the above argument, we can verify that* Ω_3 *is bounded.*

Now the proof of Theorem 2.2 is a consequence of Theorem 2.1 and the above lemmas.

Proof. **of Theorem 2.2**. Let Ω to be an open bounded subset of X such that $\cup_{i=1}^3 \overline{\Omega_i} \subset \Omega$. By using the Arzela- Ascoli theorem, we can prove that $K_P(I - QN) : \overline{\Omega} \to X$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by Lemmas 2.2 and 2.3, we have

i) $Lx \neq \lambda Nx$ pour tout $(x, \lambda) \in [(domL \backslash KerL) \cap \partial\Omega] \times (0, 1)$.

ii) $Nx \notin \operatorname{Im} L$ pour tout $x \in KerL \cap \partial\Omega$.

iii) Let $H(x, \lambda) = \pm\lambda Jx + (1-\lambda) QNx = 0$.

According to Lemmas 2.4 and 2.5, we know that $H(x, \lambda) \neq 0$ for every $x \in KerL \cap \partial\Omega$. Thus, by the homotopy property of degree, $\deg(QN|_{KerL}, \Omega \cap KerL, 0) = \deg(H(\cdot, 0), \Omega \cap KerL, 0)$ $= \deg(H(\cdot, 1), \Omega \cap KerL, 0) = \deg(\pm J, \Omega \cap KerL, 0) \neq 0$. Then by Theorem 2.1, $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$, so the BVP (1.1), (1, 2) has at least one solution in $C^2[0, 1]$. The proof is complete. □

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