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On Univalent Functions with Logarithmic Coefficients by Using Convolution

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Abstract

The purpose of this present paper is to derive some inclusion results and coefficient estimates for certain analytic functions with logarithmic coefficients by using Hadamard product. Relevant connections of the results with various known properties are also investigated.

Keywords: Convolution, univalent function, logarithmic coefficient, coefficient bounds, extreme point, convex set.
2010 MSC: 30C45, 30C50.

1. Introduction and Motivation

let A denote the class of normalized functions $f(z)$ of the from

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \tag{1.1}$$

which are holomorphic in the open unit disk $\Delta = \{z : |z| < 1\}$. Let N denote the subclass of A consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{+\infty} a_n z^n. \quad (a_n \geq 0). \tag{1.2}$$

Associated with each f in A is a well defined logarithmic function

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{+\infty} \gamma_n z^n. \ z \in \Delta. \tag{1.3}$$

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The numbers γ_n are called the logarithmic coefficients of $f(z)$. See (Girela, 2000). For log $\frac{f(z)}{z}$ given by (1.3) and $G(z) \in N$ given by

$$G(z) = z - \sum_{n=2}^{+\infty} b_n z^n, \tag{1.4}$$

the convolution (or Hadamard product) of

$$F(z) = -(\log \frac{f(z)}{z}) + (1 + 2\gamma_1)z, \tag{1.5}$$

and $G(z)$ denoted by $F * G$, is defined by

$$H(z) = F * G := z - \sum_{n=2}^{+\infty} 2\gamma_n b_n z^n. \tag{1.6}$$

We denote by $\Pi(\eta,\beta)$ and $Q(\eta,\beta)$ consisting of the functions $H(z) = F * G$ in N which satisfy

$$Re\{\frac{\frac{zH'(z)}{H(z)}}{\eta\frac{zH'(z)}{H(z)} + (1-\eta)}\} > \beta \tag{1.7}$$

and

$$Re\{\frac{1 + \frac{zH''(z)}{H'(z)}}{1 + \eta\frac{zH''(z)}{H'(z)}}\} > \beta\,, \quad 0 \le \beta < 1, 0 \le \eta < 1, \tag{1.8}$$

respectively. Also the functions $H(z)$ in N are said to be in the class $\Lambda(\eta,\beta,\psi)$, if there exists a function $\psi(z) \in N$ such that

$$Re\{\frac{\frac{zH'(z)}{\psi(z)}}{\eta\frac{zH'(z)}{\psi(z)} + (1-\eta)}\} > \beta. \tag{1.9}$$

For these subclassess we prove some interesting theorems include coefficient bounds, inclusion results, extreme points and property of convex sets.

Several other interesting subclasses of univalent functions were investigated recently, for example, by Ghanim and Darus (Ghanim & Darus, 2008), Prajapat and Goyal (Prajapat & Goyal, 2009), Acu and Owa (Acu & Owa, 2000) and etc. See also (Najafzadeh & Kulkarni, 2006) and (Najafzadeh & Ebadian, 2009).

2. Main result

Theorem 2.1. *If $H(z) \in \Lambda(\eta,\beta,\psi)$, then*

$$\sum_{n=2}^{+\infty} [2\gamma_n b_n(1 - \eta\beta) - \beta(1-\eta)c_n] \le 1 - \beta. \tag{2.1}$$

Proof. Since $H(z) \in \Lambda(\eta,\beta,\psi)$, then there exists a function $\psi(z) = z - \sum_{n=2}^{+\infty} c_n z^n \in N$ such that (1.9) holds true. By putting (1.6) and $H'(z) = (F * g)' = 1 - \sum_{n=2}^{+\infty} 2\gamma_n b_n z^n$ in (1.9) we get $Re\{\frac{1-\sum_{n=2}^{+\infty} 2\gamma_n b_n z^{n-1}}{1-\sum_{n=2}^{+\infty}(2\eta\gamma_n b_n+(1-\eta)c_n)z^{n-1}}\} > \beta$. By choosing the values of z on the real axis so that $\frac{z(F*G)'}{\psi(z)}$ is real and letting $r \to 1^-$ through real values, we have $\frac{1-\sum_{n=2}^{+\infty} 2\gamma_n b_n}{1-\sum_{n=2}^{+\infty}(2\eta\gamma_n b_n+(1-\eta)c_n)} \geq \beta$, or equivalently $\sum_{n=2}^{+\infty}[2\gamma_n b_n(1-\eta\beta) - \beta(1-\eta)c_n] \leq 1-\beta$. Now the proof is complete. □

Theorem 2.2. *If* $H(z) \in Q(\eta,\beta)$ *, then* $\sum_{n=2}^{+\infty} 2\gamma_n b_n(1+\eta(n-1)+\beta n^2) \leq 1-\beta$.

Proof. Since $H(z) \in Q(\eta,\beta)$, then by (1.6) and (1.8) we get $Re\{\frac{1-\sum_{n=2}^{+\infty} 2n^2\gamma_n b_n z^{n-1}}{1-\sum_{n=2}^{+\infty} 2n\gamma_n(1+\eta(n-1))z^{n-1}}\} > \beta$. By choosing the values of z on the real axis so that $\frac{z(F*G)''}{(F*G)'}$ is real and letting $r \to 1^-$ though real values we have $\frac{1-\sum_{n=2}^{+\infty} 2n^2\gamma_n b_n}{1-\sum_{n=2}^{+\infty} 2n\gamma_n b_n(1+\eta(n-1))} > \beta$. The above inequality gives the required result. □

Definition 2.1. A function $H(z) \in N$ is said to be in $W(\eta,\beta)$, if there exists a function $\psi(z) = z - \sum_{n=2}^{+\infty} c_n z^n$ such that

(a) The condition (2.1) holds true;

(b) For every n, $2\gamma_n b_n - c_n \geq 0$.

In the next theorem we prove an inclusion property.

Theorem 2.3. $W(\eta,\beta) \subseteq \Lambda(\eta,\beta,\psi)$.

Proof. Let $H(z) \in W(\eta,\beta)$, we must show that $H(z) \in \Lambda(\eta,\beta,\psi)$ or equivalently the condition (1.9) holds. But

$$\left|\frac{\frac{z(F*G)'}{\psi(z)}}{\eta\frac{z(F*G)'}{\psi(z)} + (1-\eta)} - 1\right| = \left|\frac{1-\sum_{n=2}^{+\infty} 2\gamma_n b_n z^{n-1}}{1-\sum_{n=2}^{+\infty}(2\eta\gamma_n b_n+(1-\eta)c_n)z^{n-1}} - 1\right| = \left|\frac{(\eta-1)\sum_{n=2}^{+\infty}(2\gamma_n b_n - c_n)z^{n-1}}{1-\sum_{n=2}^{+\infty}(2\eta\gamma_n b_n+(1-\eta)c_n)z^{n-1}}\right|$$

$$\leq \frac{(1-\eta)\sum_{n=2}^{+\infty}(2\gamma_n b_n - c_n)}{1-\sum_{n=2}^{+\infty}(2\eta\gamma_n b_n+(1-\eta)c_n)}.$$

If (a) holds , above fraction is bounded above by $1-\alpha$ and hence (1.9) is satisfied. So $H(z) \in \Lambda(\eta,\beta,\psi)$. □

Remark. By putting $\psi(z) = G(z)$, in the last Theorem we obtain $\Pi(\eta,\beta) \subseteq W(\eta,\beta)$, and also by putting $\psi(z) = G(z)$ in (2.1) we have $\sum_{n=2}^{+\infty}[2\gamma_n(1-\eta\beta) - \beta(1-\eta)]b_n \leq 1-\beta$. This is the necessary and sufficient condition for functions $H(z) \in N$ to be in the class $\Pi(\eta,\beta)$.

3. Coefficient estimates and Distortion bounds for functions in $W(\eta, \beta)$

In this section we find coefficient bounds and verify distortion Theorem for the class $W(\eta,\beta)$.

Remark. If $H(z)$ be in the class $W(\eta,\beta)$, then

$$\sum_{n=2}^{+\infty} \gamma_n b_n \leq \frac{n(1-\beta)+\beta(1-\eta)}{2(1-\eta\beta)}. \tag{3.1}$$

Proof. From definition of $W(\eta,\beta)$ and taking $\psi(z) = z - \sum_{n=2}^{+\infty} c_n z^n$, we have $\sum_{n=2}^{+\infty}(1-\eta\beta)(2\gamma_n b_n) \leq 1-\beta+\beta(1-\eta)c_n$. If $c_n \leq \frac{1}{n}$ $(\forall n)$, thus we have $\sum_{n=2}^{+\infty} \gamma_n b_n \leq \frac{n(1-\beta)+\beta(1-\eta)}{2(1-\eta\beta)}$. □

Remark. The function $H_n(z) = z - \frac{n(1-\beta)+\beta(1-\eta)}{2(1-\eta\beta)} z^n$ is an extremal function for the class $W(\eta,\beta)$.

Theorem 3.1. *Let $H(z) = F * G$ be in the class $W(\eta,\beta)$, then for $|z| \leq r < 1$*

$$r - \frac{2-\beta-\beta\eta}{4(1-\eta\beta)} r^2 \leq |F * G| \leq r + \frac{2-\beta-\beta\eta}{4(1-\eta\beta)} r^2 \tag{3.2}$$

Proof. Since

$$H(z) = F * G = z - \sum_{n=2}^{+\infty} 2\gamma_n b_n z^n, \tag{3.3}$$

so by (2.1) we get $\sum_{n=2}^{+\infty} 2\gamma_n b_n (1-\eta\beta) - \beta(1-\eta)c_n \leq 1-\beta$. Since $c_n \leq \frac{1}{n} \leq \frac{1}{2}$ we have $\sum_{n=2}^{+\infty} 2n\gamma_n b_n(1-\eta\beta) \leq \frac{\beta(1-\eta)}{2} + 1 - \beta$, or $2\sum_{n=2}^{+\infty} 2n\gamma_n b_n(1-\eta\beta) \leq 2-\beta-\beta\eta$, or $2\sum_{n=2}^{+\infty} 2\gamma_n b_n \leq 2\sum_{n=2}^{+\infty} n\gamma_n b_n \leq \frac{2-\beta-\beta\eta}{2(1-\eta\beta)}$, or $\sum_{n=2}^{+\infty} 2\gamma_n b_n \leq \frac{2-\beta-\beta\eta}{4(1-\eta\beta)}$. From this inequality and (3.3) we have $|F * G| \leq |z| + \sum_{n=2}^{+\infty} 2\gamma_n b_n |z|^n \leq r + \frac{2-\beta-\beta\eta}{4(1-\eta\beta)} r^2$, and $|F * G| \geq r - \frac{2-\beta-\beta\eta}{4(1-\eta\beta)} r^2$. □

Theorem 3.2. *The class $W(\eta,\beta)$ is convex.*

Proof. Let $H_1(z)$ and $H_2(z)$ be in the class $W(\eta,\beta)$ with respect to functions $\psi_1(z) = z - \sum_{n=2}^{+\infty} c_n z^n$ and $\psi_2(z) = z - \sum_{n=2}^{+\infty} c'_n z^n$. For $0 \leq j \leq 1$ we must show that $H(z) = jH_1(z) + (1-j)H_2(z)$ belongs to $W(\eta,\beta)$ with respect to $\psi(z) = j\psi_1(z) + (1-j)\psi_2(z)$. But $H_1(z) = z - \sum_{n=2}^{+\infty} 2\gamma_n b_n z^n$, $H_2(z) = z - \sum_{n=2}^{+\infty} 2\gamma_n b'_n z^n$, and $H(z) = z - \sum_{n=2}^{+\infty} s_n(j) z^n$, where $s_n(j) = 2\gamma_n(jb_n + (1-j)b'_n)$. Also $\psi(z) = z - \sum_{n=2}^{+\infty} r_n(j) z^n$ where $r_n(j) = jc_n + (1-j)c'_n$.

The function $H(z)$ will belong to $W(\eta,\beta)$ if

(i) $\sum_{n=2}^{+\infty} [s_n(j)(1-\eta\beta) - \beta(1-\eta)r_n(j)] \leq 1-\beta$,

(ii) $s_n(j) - r_n(j) \geq 0$ for every n.

Since H_1 and H_2 are in $W(\eta,\beta)$ then $2\gamma_n b_n - c_n \geq 0$ and $2\gamma_n b'_n - c'_n \geq 0$, for all n. With direct calculation since $0 \leq j \leq 1$ we have, $s_n(j) - r_n(j) = 2\gamma_n(jb_n + (1-j)b'_n) - (jc_n + (1-j)c'_n) = j(2\gamma_n b_n - c_n) + (1-j)(2\gamma_n b'_n - c'_n) \geq 0$. Also $\sum_{n=2}^{+\infty}[s_n(j)(1-\eta\beta) - \beta(1-\eta)r_n(j)] = j\sum_{n=2}^{+\infty} 2\gamma_n b_n(1-\eta\beta) - \beta(1-\eta)c_n + (1-j)\sum_{n=2}^{+\infty} 2\gamma_n b'_n(1-\eta\beta) - \beta(1-\eta)c'_n \leq j(1-\beta) + (1-j)(1-\beta) = 1-\beta$. Now the proof is complete. □

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