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Satellite Constellation Reconfiguration Using the Approximating Sequence Riccati Equations

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Abstract

In this work we study the reconfiguration of a constellation of satellite. In this work we consider the non-linear feedback optimal control of the motion of a spacecraft under the influence of the gravitational attraction of a central body, the Earth in our case, and we would like to transfer the spacecraft from lower circular orbit to a higher one. Both orbits around the Earth are assumed to be circular and coplanar. We use both radial and tangential thrust control. The nonlinear dynamics of the system will be factorized in such a way that the new factorized system is accessible. The problem is tackled using the Approximating Sequence Riccati Equations (ASRE) method. The technique is based on Linear Quadratic Regulator (LQR) with fixed terminal state, which guarantees closed loop solution. The method is tested through GNSS circular constellation.

Keywords: Nonlinear feedback, linear quadratic regulator, approximation sequence Riccati equation, GNSS satellite.
2010 MSC: 49.

1. Introduction

In some instances, it is desirable to deploy a constellation in stages to gradually expand its capacity. This requires launching additional satellites and reconfiguring the existing on-orbit satellites (de Weck *et al.*, 2008). Also, a constellation might be re-structured and reconfigured after it is initially set for operational reasons.

The most common way of raising or lowering the orbit of a spacecraft is the low thrust orbit rendezvous approach, which is a nonlinear optimal control problem. Historically, there are several method to solve the nonlinear optimal control problem in both open and closed loop contexts. In the open loop context the problem can be solved via indirect and then direct method. The indirect method was developed through Pontryagin Maximum Principle (PMP) (Bryson & Ho, 1975),

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(Pontryagin *et al.*, 1952). The direct method was developed using the Karush-Kuhn-Tucker(KKT) algebric equation (Enright & Conway, 1992).

One of the most common methods for solving the nonlinear optimal control problem in the closed loop context is the State Dependent Riccati Equations (SDRE) (Cimen, 2006), (Owis, 2013). The Approximating Sequence of Riccati Equations (ASRE) (Cimen, 2004) technique is an iterative approach to solve the nonlinear optimal control problem. The ASRE is developed (Topputo & Bernelli-Zazzera, 2012) using the state transition matrix. The guidance designed with these methods is obtained in an open-loop context. In other words, the optimal path, even if minimizing the prescribed performance index, is not able to respond to any perturbation that could alter the state of the spacecraft. Furthermore, if the initial conditions are slightly varied (e.g. the launch date changes), the optimal solution needs to be recomputed again. The outcome of the classical problem is in fact a guidance law expressed as a function of the time, the initial and final time, and u the control vector, respectively. We develop a closed loop approach. With this approach the solutions that minimize the performance index are also functions of the generic initial state x_0; the outcome is in fact a guidance law written as $u = u(x_0, t_0, t)$, $t \in [t_0, t_f]$. This represents a closed-loop solution: given the initial conditions (t_0, x_0) it is possible to extract the optimal control law that solves the optimal control problem. Moreover, if for any reason the state is perturbed and assumes the new value $(t_0^{'}, x_0^{'}) = (x_0 + \delta x, t_0 + \delta t)$, we are able to compute the new optimal solution by simply evaluating so avoiding the solution of another optimal control problem. This property holds by virtue of the closed loop characteristics of the control law that can be viewed as a one-parameter family of solutions. Due to such property, a trajectory designed in this way has the property to respond to perturbations acting during the transfer that continuously alter the state of the spacecraft. The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation: this is a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers (Bryson & Ho, 1975). Recently, the nonlinear feedback control of circular coplanar low-thrust orbital transfers has been faced using continuous orbital elements feedback and Lyapunov functions (Chang & Marsden, 2002) and proved optimal by (Alizadah & Villac, 2011). Later on the problem has been solved using the primer vector approximation method (Haung, 2012).

The analytical low-thrust optimal feedback control problem is solved, with modulated inverse-square-distance, in the frame of a nonlinear vector field, the two-body dynamics, supported by a nonlinear objective function by applying a globally diffeomorphic linearizing transformation that rearranges the original problem into a linear system of ordinary differential equations and a quadratic objective function written in a new set of variables with radial thrust (Topputo *et al.*, 2008). In this work we consider the nonlinear feedback optimal control of the motion of a spacecraft under the influence of the gravitational attraction of a central body, the Earth in our case, and we would like to transfer the spacecraft from lower to higher orbit. Both lower and higher orbits around the Earth are assumed to be circular and coplanar. We use both radial and tangential thrust control. The nonlinear dynamics of the system will be factorized in such a way that the new factorized system is accessible. The problem is tackled using the Approximating Sequence Riccati Equation (ASRE) method. The technique is based on Linear Quadratic Regulator (LQR) with fixed terminal state. The method is applied to GNSS circular constellation Figure 1.

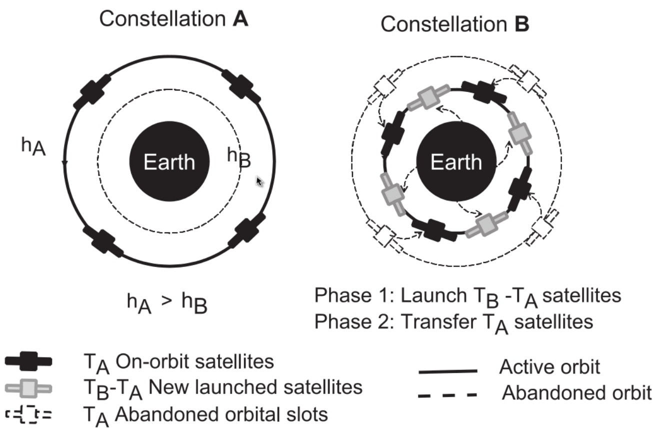

Figure 1. Constellation reconfiguration.

Linear Quadratic Regulator(LQR) with Fixed Terminal State.

Consider the following system with linear dynamics and quadratic performance index as follows:

$$\dot{X} = AX + BU, \qquad X(t_0) = X_0 \in \mathbb{R}^n, \tag{1.1}$$

the following performance index

$$J = X_f^T Q_f X_f + \frac{1}{2}\int_{t_0}^{t_f} [X^T QX + U^T RU]dt, \tag{1.2}$$

where A , B , Q , and R are constant coefficients matrices of the suitable dimensions. we have to find the m-dimensional control functions $U(t)$, $t \in [t_0\ t_f]$ which minimizes the J, which is an open loop (with t_0 fixed) optimal control. We optimize the performance index J, by adjoining the dynamics and the performance index (integrand) to form the Hamiltonian:

$$H(X, \lambda, U, t) = \frac{1}{2}(X^T QX + U^T RU) + \lambda^T (A(t)X + B(t)U),$$

where the Lagrange multiplier λ is called the adjoint variable or the costate. The necessary conditions for optimality are:

1. $\dot{X} = H_\lambda = A(t)X + B(t)U, \qquad X(t_0) = X_0,$
2. $\dot{\lambda} = -H_x = -QX - A^T\lambda, \qquad \lambda(t_f) = Q_f X_f,$
3. $H_u = 0 \Longrightarrow RU + B^T\lambda = 0 \Longrightarrow U^\star = -R^{-1}B^T\lambda.$

To find the minimum solution we have to check for $H_{uu} = \dfrac{\partial^2 H}{\partial \lambda^2} > 0$ or equivalently $R > 0$. Now we have that $\dot{X} = AX + BU^\star = AX - BR^{-1}B^T\lambda$, which can be combined to the the equation of the costate as follows

$$\begin{bmatrix} \dot{X} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ \lambda \end{bmatrix}, \tag{1.3}$$

which is called the Hamiltonian matrix, it represents a $2n$ boundary value problem with $X(t_0) = X_0$ and, $\lambda(t_f) = Q_f X_f$.

We can solve this $2n$ boundary value problem using the transition matrix method as follows. Let's define a transition matrix

$$\phi_(t_1, t_0) = \begin{bmatrix} \phi_{11}(t_1, t_0) & \phi_{12}(t_1, t_0) \\ \phi_{21}(t_1, t_0) & \phi_{22}(t_1, t_0) \end{bmatrix},$$

we use this matrix to relate the current values of X and λ to the final values X_f and λ_f as follows

$$\begin{bmatrix} X \\ \lambda \end{bmatrix} = \begin{bmatrix} \phi_{11}(t, t_f) & \phi_{12}(t, t_f) \\ \phi_{21}(t, t_f) & \phi_{22}(t, t_f) \end{bmatrix} \begin{bmatrix} X(t_f) \\ \lambda(t_f) \end{bmatrix},$$

so we have $X = \phi_{11}(t, t_f)X(t_f) + \phi_{12}(t, t_f)\lambda(t_f) = [\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f]X(t_f)$, we can eliminate $X(t_f)$ to get $X = [\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f][\phi_{11}(t_0, t_f) + \phi_{12}(t_0, t_f)Q_f]^{-1}X(t_0) = X(t, X_0, t_0)$, now we can find $\lambda(t)$ in terms of $X(t_f)$ as $\lambda(t) = [\phi_{21}(t, t_f) + \phi_{22}(t, t_f)Q_f]X(t_f)$, then we can eliminate $X(t_f)$ to get $\lambda(t) = [\phi_{21}(t, t_f) + \phi_{22}(t, t_f)Q_f][\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f]^{-1}X(t) = \phi_{\lambda x}X(t)$. Now we search a solution for $\phi_{\lambda x}$. By differentiating $\lambda(t)$ we get $\dot{\lambda}(t) = \dot{\phi}_{\lambda x}X(t) + \phi_{\lambda x}\dot{X}(t)$. Comparing the last equation with the Hamiltonian matrix we get $-QX(t) - A^T\lambda(t) = \dot{\phi}_{\lambda x}X(t) + \phi_{\lambda x}\dot{X}(t)$, then we have

$$\begin{aligned} -\dot{\phi_{\lambda x}}(t)X(t) &= QX(t) + A^T\lambda(t) + \phi_{\lambda x}\dot{X}(t) \\ &= QX(t) + A^T\lambda(t) + \phi_{\lambda x}(AX - BR^{-1}B^T\lambda(t)) \\ &= (Q + \phi_{\lambda x}A)X(t) + (A^T - \phi_{\lambda x}BR^{-1}B^T)\lambda(t) \\ &= (Q + \phi_{\lambda x}A)X(t) + (A^T - \phi_{\lambda x}BR^{-1}B^T)\phi_{\lambda x}X(t) \\ &= [Q + \phi_{\lambda x}A + A^T\phi_{\lambda x} - \phi_{\lambda x}BR^{-1}B^T\phi_{\lambda x}]X(t). \end{aligned}$$

Since this is true for arbitrary $X(t)$, $\phi_{\lambda x}$ must satisfy

$$-\dot{\phi}_{\lambda x}(t) = Q + \phi_{\lambda x}A + A^T\phi_{\lambda x} - \phi_{\lambda x}BR^{-1}B^T\phi_{\lambda x}, \tag{1.4}$$

which is the matrix differential Riccati Equation . We can solve for $\phi_{\lambda x}$ by solving Riccati Equation backwards in time from t_f with $\phi_{\lambda x}(t_f) = Q_f$. The optimal control is then given by

$$U^\star = -R^{-1}B^T\lambda(t) = -R^{-1}B^T\phi_{\lambda x}X = -K(t)X(t, X_0, t_0). \tag{1.5}$$

From 1.5 we notice that the optimal control is a linear full-state feedback control, therefore the linear quadratic terminal controller is feedback by default.

2. The Approximating Sequence of Riccati Equations(ASRE)

Assume that we have the following nonlinear system

$$\dot{X} = f(X, U, t) \tag{2.1}$$

$$X(t_0) = X_0, \quad X(t_f) = X_f \in R^n \tag{2.2}$$

with performance index

$$J = \phi(X_f, t_f) + \int_{t_0}^{t_f} L(X, U, t)dt. \tag{2.3}$$

This system can be rewritten in the state dependent quasi-linear system as follows

$$\dot{X}^i = A(X^{i-1})X^i + B(X^{i-1})U^i \tag{2.4}$$

$$X(t_0) = X_0^0, \quad X(t_f) = X_f^n \in R^n \tag{2.5}$$

$$J = X_f^{i\,T} Q(X_f^{i-1})X_f^i + \frac{1}{2}\int_{t_0}^{t_f} [X^{iT} Q(X^{i-1})X^i + U^{iT} R(X^{i-1})U^i]dt, \tag{2.6}$$

where i represents the iteration step over the time interval $[t_i - 1, t_i]$ Figure 2, the technique is based of the previously introduced Linear Quadratic Regulator with fixed terminal state, which is a full state feedback and therefore the obtained solution will be a closed loop one, i.e. able to respond to the unexpected change in the inputs. The technique works as follows: the initial state is used to compute A_0, and B_0 and we solve for the first LQR iteration and compute X^1 and then used to compute new value of A_1, and B_1 for the second iteration until the final state error reaches a value below a set threshold.

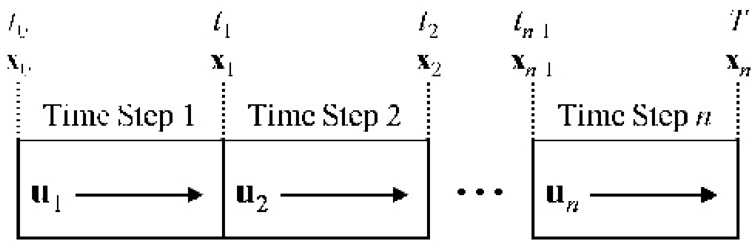

Figure 2. Time Interval Disecretization.

3. Optimal Orbit Transfer

The equations of motion are written in polar coordinates (r, θ), in the inertial Earth-Centered frame. In order to transfer the spacecraft between two circular coplanar orbits two components of the thrust control are used. The tangential component T_θ, and the radial component T_r.

The equations of motion are:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= T_r - \frac{\mu}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= T_\theta \end{aligned} \tag{3.1}$$

where μ is the gravitational constant of the Earth ($3.986005 \times 10^{14} m^3/s^2$) In this system of units the gravitational constant μ is unity, and equations (3.1) are rewritten as:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= T_r - \frac{1}{r^2} \\ \ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} &= \frac{T_\theta}{r} \end{aligned} \tag{3.2}$$

Equations of motion in state variable form, equations (3.2), are then written in state variable form. The state vector $\mathbf{x}$ is chosen to be:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} \tag{3.3}$$

and the control vector is :

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} T_r \\ T_\theta \end{bmatrix}. \tag{3.4}$$

Then equation (3.2) can be written in the form :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}. \tag{3.5}$$

Choosing a suitable factorization equation (3.5) is rewritten in the factored state variable form :

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}, \tag{3.6}$$

where :

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_4^2 & -\frac{1}{x_1^2 x_2} & 0 & 0 \\ -\frac{2x_4}{x_1^2} & 0 & 0 & 0 \end{bmatrix}, \tag{3.7}$$

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{x_1} \end{bmatrix}. \tag{3.8}$$

4. Factored Controllability

For the factored system (3.6) the controllability is established by verifying that the controllability matrix $\mathbf{M}_{cl} = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \mathbf{A}^3\mathbf{B}]$ has a rank equals to $n = 4\ \forall x$ in the domain.

Since $\mathbf{A}$ and $\mathbf{B}$ have nonvanishing rows the controllability matrix $\mathbf{M}_{cl}$ for the System (3.6) is of rank 4.

Nondimensionalization of the problem in order to simplify the calculation we dimensionalize the system by removing the units from the equations of motion via multiplying or dividing some constants. The two constant we divide by are the radial distance of the initial orbit and the gravitational constant μ in this case the radius of the initial orbit is unity and velocity is divided by the circular velocity of the initial orbit $\sqrt{\frac{\mu}{r_0^2}}$ and the time is multiplied by $\sqrt{\frac{\mu}{r_0^3}}$. In the first two example we would like to make an optimal orbit transfer (i.e. from ($r = 1$) to ($r = 1.2$) in time $t_f = 4.469, 5.2231$ (time unit) Figure 3 with an optimal control function of both radial and tangential components Figure 4. The initial angle is ($\theta_0 = \frac{\pi}{2}$) and the final angle is ($\theta_f = \frac{3\pi}{2}$). $\dot{r}_0 = 0$ and $\dot{r}_f = 0$ for the initial and final orbits. $\dot{\theta}_0 = \sqrt{\frac{1}{r_0^3}} = 1$ and $\dot{\theta}_f = \sqrt{\frac{1}{r_f^3}} = 0.54433105395$. In the second $\theta_f = \frac{5\pi}{2}$ with $t_f = 6.866$. In both examples the matrices **Q** and **R** are the identity matrices:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

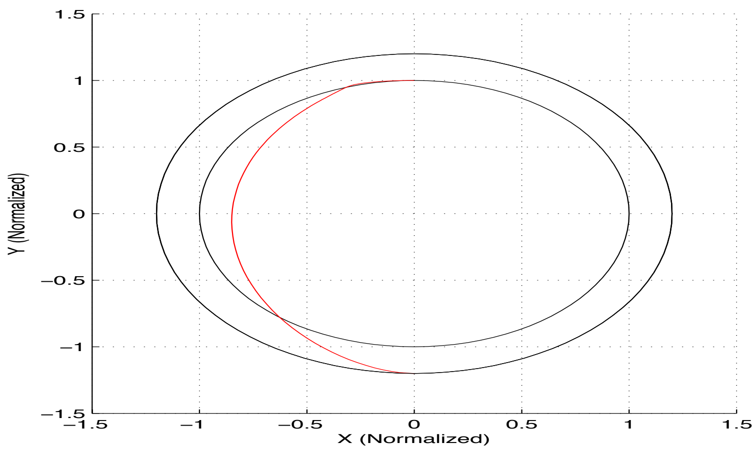

Figure 3. Trajectory of orbit transfer in polar coordinates, from [$r_0 = 1$, $\theta_0 = \pi/2$, $\dot{r}_0 = 0$, $\dot{\theta}_0 = 1$] to [$r_f = 1.2$, $\theta_f = 3\pi/2$, $\dot{r}_f = 0$, $\dot{\theta}_f = 0.72213$]

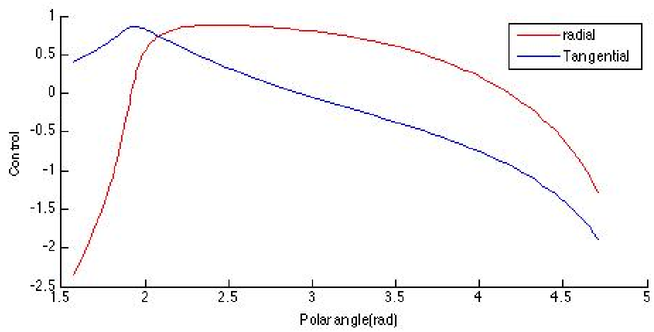

Figure 4. Control function in polar coordinates, from [$r_0 = 1$, $\theta_0 = \pi/2$, $\dot{r}_0 = 0$, $\dot{\theta}_0 = 1$] to [$r_f = 1.2$, $\theta_f = 3\pi/2$, $\dot{r}_f = 0$, $\dot{\theta}_f = 0.54433$].

5. Conclusion

The nonlinear feedback optimal control can be solved by factorizing the original nonlinear dynamics into accessible (weakly controllable) linear dynamics of state dependent factors. The factorized problem has been solved using the the Approximating Sequence Riccati Equations (ASRE) method. The technique is based on Linear Quadratic Regulator (LQR) with fixed terminal state, which guarantees closed loop solution. The method is tested through reconfiguration of a GNSS circular constellation. The result is valid for any circular orbit transfer.

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