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Semitopological Vector Spaces and Hyperseminorms

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Abstract

In this paper, we introduce and study semitopological vector spaces. The goal is to provide an efficient base for developing the theory of extrafunction spaces in an abstract setting of algebraic systems and topological spaces. Semitopological vector spaces are more general than conventional topological vector spaces, which proved to be very useful for solving many problems in functional analysis. To study semitopological vector spaces, hypermetrics and hyperpseudometrics are introduced and it is demonstrated that hyperseminorms, studied in previous works of the author, induce hyperpseudometrics, while hypernorms induce hypermetrics. Sufficient and necessary conditions for a hyperpseudometric (hypermetric) to be induced by a hyperseminorm (hypernorm) are found. We also show that semitopological vector spaces are closely related to systems of hyperseminorms. Then defining boundedness and continuity relative to associated systems of hyperseminorms, we study relations between relative boundedness and relative continuity for mappings of vector spaces with systems of hyperseminorms and systems of hypernorms.

Keywords: Functional analysis, topological vector space, norm, seminorm, hyperseminorm, boundedness, continuity, extrafunction.
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1. Introduction

The concept of a real or complex extrafunction essentially extends the concept of a real or complex function, encompassing, in particular, the concept of a distribution, i.e., distributions are a kind of extrafunctions (Burgin, 2012). Extrafunctions have many advantages in comparison with functions and distributions. For instance, integration of extrafunctions is more powerful than integration of functions allowing integration of a much larger range of functions as it is demonstrated in (Burgin, 2012).

At the same time, spaces of extrafunctions have a more sophisticated structure in comparison with spaces of functions, which are topological vector spaces and have a highly advanced theory (cf., for example, (Bourbaki, 1953-1955); (Robertson & Robertson, 1964); (Riez & Sz.-Nagy,

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1955); (Rudin, 1991); (Grothendieck, 1992); (Kolmogorov & Fomin, 1999)). In particular, it has been demonstrated that topological vector spaces provide an efficient context for the development of integration and are very useful for solving many problems in functional analysis in general (Choquet, 1969); (Edwards & Wayment, 1970); (Shuchat, 1972); (Kurzweil, 2000). In addition, locally convex topological vector spaces offer a convenient structure for studies of summation, which is integration of functions on natural numbers (Pietsch, 1965).

In this paper we introduce and study semitopological vector spaces, operators in these spaces and their mappings. It provides a base for the theory of extrafunction spaces in an abstract setting of algebraic systems and topological spaces. Semitopological vector spaces are more general than conventional topological vector spaces. To study semitopological vector spaces, hypermetrics and hyperpseudometrics are introduced and it is demonstrated that hyperseminorms induce hyperpseudometrics, while hypernorms induce hypermetrics. Norms are special cases of hypernorms, while seminorms are special cases of hyperseminorms. Sufficient and necessary conditions for a hyperpseudometric (hypermetric) to be induced by a hyperseminorm (hypernorm) are found. We also show that semitopological vector spaces are closely related to systems of hyperseminorms.

An essential property of operators in mathematics is continuity (cf. (Dunford & Schwartz, 1958); (Rudin, 1991); (Kolmogorov & Fomin, 1999)). One of the central results of functional analysis is the theorem that establishes equivalence between continuity and boundedness for linear operators. Here we extend the concepts of boundedness and continuity for operators and mappings of semitopological vector spaces with systems of hyperseminorms and seminorms, differentiating between different types of boundedness and continuity and making these concepts relative to systems of hyperseminorms and seminorms. Then we study these concepts, proving a series of theorems, which establish equivalence between a type of relative continuity and the corresponding type of relative boundedness for linear operators in semitopological vector spaces with systems of hyperseminorms or seminorms. Classical results describing continuous operators in convex spaces become direct corollaries of theorems proved in this paper. In conclusion, several problems for further research are formulated.

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2. Semitopological vector spaces

The concept of a semitopological vector space is an extension of the concept of a topological vector space.

Definition 2.1. *A semitopological vector space* L over a field $\mathbf{F}$ is a vector space over $\mathbf{F}$ with a topology in which addition is continuous, while scalar multiplication by elements from $\mathbf{F}$ is continuous with respect to L, i.e., the scalar multiplication mapping $m : \mathbf{F} \times L \to L$ is continuous in the second coordinate.

When the multiplication mapping $m : \mathbf{F} \times L \to L$ is continuous, then L is a topological vector space over the field $\mathbf{F}$. Some authors (cf., for example, (Rudin, 1991)) additionally demand that the point $\mathbf{0}$ in a topological vector space is closed. This condition results in the Hausdorff topology in topological vector spaces.

In what follows, **F** stands either for the field $\mathbb{R}$ of all real numbers or for the field $\mathbb{C}$ of all complex numbers or for a subfield of $\mathbb{C}$ that contains $\mathbb{R}$, while **0** denotes the zero element of any vector space.

Semitopological vector spaces are closely related to hypernorms and hyperseminorms.

Let $\mathbb{R}_\omega$ be the set of all real hypernumbers and $\mathbb{R}^+_\omega$ be the set of all non-negative real hypernumbers (Burgin, 2012).

Definition 2.2. a) A mapping $q : L \to \mathbb{R}^+_\omega$ is called a *hypernorm* if it satisfies the following conditions:

N1 . For any x from L, $q(x) = 0$ if and only if $x = \mathbf{0}$.

N2 . $q(ax) = |a| \cdot q(x)$ for any x from L and any number a from **F**.

N3 . (the triangle inequality or subadditivity).

$$q(x + y) \leq q(x) + q(y) \quad \text{for any } x \text{ and } y \text{ from } L$$

b) A vector space L with a norm is called a *hypernormed vector space* or simply, a *hypernormed space*.

c) The real hypernumber $q(x)$ is called the *hypernorm* of an element x from the hypernormed space L.

Note that *norms* in vector spaces coincide with hypernorms that take values only in the set of real numbers.

Example 2.1. As it is proved in (Burgin, 2012), the set of all real hypernumbers $\mathbb{R}_\omega$ is a hypernormed space where the hypernorm $|| \cdot ||$ is defined by the following formula:

If α is a real hypernumber, i.e.,$\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ with $a_i \in \mathbb{R}$ for all $i \in \omega$, then $||\alpha|| = \mathrm{Hn}(|a_i|)_{i\in\omega}$.

Note that this hypernorm coincides with the conventional norm on real numbers but it is impossible get the same topology by means of a conventional finite norm.

Example 2.2. As it is proved in (Burgin, 2002), the set of all complex hypernumbers $\mathbb{C}_\omega$ of all complex hypernumbers is a hypernormed space where the hypernorm $|| \cdot ||$ is defined by the following formula:

If α is a complex hypernumber,i.e., $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ with $a_i \in \mathbb{C}$ for all $i \in \omega$, then $||\alpha|| = \mathrm{Hn}(|a_i|)_{i\in\omega}$.

Note that this hypernorm coincides with the conventional norm on complex numbers but it is impossible get the same topology by means of a conventional finite norm.

There are hypernormed spaces that are not normed spaces.

Example 2.3. The set $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions is a hypernormed space where the hypernorm $|| \cdot ||$ is defined by the following formula:

If $f : \mathbb{R} \to \mathbb{R}$, then $||f|| = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = max\{|f(x)|; a_i = [-i, i]\}$.

At the same time, it is known that $C(\mathbb{R}, \mathbb{R})$ is not a normed space (Robertson & Robertson, 1964).

There are natural relations between hypernorms and semitopological vector spaces.

Theorem 2.1. *Any hypernormed space is a Hausdorff semitopological vector space.*

Proof. Let us consider a vector space L with a hypernorm q. Taking an element x from L and a positive real number k, we define the neighborhood $O_k x$ of x by the following formula

$$O_k x = \{y \in L;\ q(x - y) < k\}.$$

At first, we show that the system of so defined neighborhoods determines a topology in L. To do this, it is necessary to check the following neighborhood axioms (Kuratowski, 1966):

NB1. Any neighborhood of a point $x \in X$ contains this point.

NB2. For any two neighborhoods $O_1 x$ and $O_2 x$ of a point $x \in X$, there is a neighborhood Ox of x that is a subset of the intersection $O_1 x \cap O_2 x$.

NB3. For any neighborhood Ox of a point $x \in X$ and a point $y \in Ox$, there is a neighborhood Oy of y that is a subset of Ox.

Let us consider a point x from X.

NB1: The point x belongs to $O_k x$ because $q(x - x) = q(\mathbf{0}) = 0 < k$ for any positive real number k.

NB2: Taking two positive real numbers k and h, we see that the intersection $O_k x \cap O_h x = O_l x$ also is a neighborhood of x where $l = \min\{k, h\}$.

NB3: Let $y \in O_k x$. Then $q(x - y) < k$ and by properties of real numbers, there is a positive real number t such that $q(x - y) < k - t$. Then $O_t x \subseteq O_k x$. Indeed, if $z \in O_t x$, then $q(y - z) < t$. Consequently,

$$q(x - z) = q((x - y) + (y - z)) \leq q(x - y) + q(y - z) < (k - t) + t = k.$$

It means that $z \in O_k x$.

Thus, we have a topology in L, and this topology is Hausdorff because any hypernorm separates points, i.e., if $x \neq y$, then $q(x - y) \neq 0$.

Now we show that addition is continuous and scalar multiplication is continuous in the second coordinate with respect to this topology.

Let us consider a sequence $\{x_i;\ i = 1, 2, 3, \ldots\}$ that converges to x, a sequence $\{y_i;\ i = 1, 2, 3, \ldots\}$ that converges to y, and the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \ldots\}$. Convergence of these two sequences means that for any $k > 0$, there are a natural number n such that $q(x_i - x) < k$ for any $i > n$ and a natural number m such that $q(y_i - y) < k$ for any $i > m$. Then by properties of a hypernorm, we have

$$q(z_i - (x + y)) = q((x_i + y_i) - (x + y)) = q((x_i - x) + (y_i - y)) \leq q(x_i - x) + q(y_i - y) < k + k = 2k$$

when $i > \max\{n, m\}$. As k is an arbitrary positive real number, this means that the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \ldots\}$. converges to $x + y$. Consequently, addition is continuous in L.

In addition, for any number a from $\mathbf{F}$, we have

$$q(u_i - ax) = q(ax_i - ax) = q(a(x_i - x)) \leq |a|q(x_i - x) < |a|k$$

where $u_i = ax_i$. As k is an arbitrary positive real number and $|a|$ is a constant, this means that the sequence $\{u_i = ax_i;\ i = 1, 2, 3, \ldots\}$ converges to ax. Consequently, scalar multiplication is continuous in the second coordinate. Theorem is proved. □

Hypernormed spaces are also hypermetric spaces.

Definition 2.3. a) A mapping $\mathbf{d} : X \times X \to \mathbb{R}^+_\omega$ is called a *hypermetric* (or a *hyperdistance function*) in a set X if it satisfies the following axioms:

M1. For any x and y from X, $\mathbf{d}(x, y) = 0$ if and only if $x = y$.

M2. (Symmetry). $\mathbf{d}(x, y) = \mathbf{d}(y, x)$ for all $x, y \in X$.

M3. (the triangle inequality or subadditivity).

$$\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y) \text{ for all } x, y, z \in X.$$

b) A set X with a hypermetric $\mathbf{d}$ is called a *hypermetric space*.

c) The real hypernumber $\mathbf{d}(x, y)$ is called the *distance* between x and y in the hypermetric space X.

Note that the distance between two elements in a hypermetric space can be a real number, finite hypernumber or infinite hypernumber. When the distance between two elements of X is always a real number, $\mathbf{d}$ is a metric.

Lemma 2.1. *a) A hypernorm q in a vector space L induces a hypermetric $\mathbf{d}_q$ in this space.*

b) If q is a norm in L, then $\mathbf{d}_q$ is a metric.

Indeed, if $q : X \to R^+_\omega$ is a hypernorm in L and x and y are elements from L, then we can define $\mathbf{d}_q(x, y) = q(x - y)$. Properties of a hypernorm imply that $\mathbf{d}_q$ satisfies all axioms M1- M3. The statement (b) directly follows from definitions.

Theorem 2.1 and Lemma 2.1 imply the following result.

Corollary 2.1. *$\mathbb{R}_\omega$ and $\mathbb{C}_\omega$ are hypermetric spaces.*

It is interesting to find what hypermetrics in vector spaces are induced by hypernorms and what metrics in vector spaces are induced by norms. To do this, let us consider additional properties of hypermetrics and metrics.

Definition 2.4. A hypermetric (metric) in a vector space L is called *linear* if it satisfies the following axioms:

LM1. $\mathbf{d}(x+z, y+z) = \mathbf{d}(x, y)$ for any $x, y, z \in L$.

LM2. $\mathbf{d}(ax, ay) = |a| \cdot \mathbf{d}(x, y)$ for all $x, y \in L$ and $a \in \mathbf{F}$.

Example 2.4. Let us take the space of all real numbers $\mathbb{R}$ as the space L. The natural metric in this space is defined as $\mathbf{d}(x, y) = |x - y|$. This metric is linear. Indeed,

$$\mathbf{d}(x+z, y+z) = |(x+z) - (y+z)| = |x - y| = \mathbf{d}(x, y)$$

and

$$\mathbf{d}(ax, ay) = |ax - ay| = |a(x - y)| = |a| \cdot |x - y| = |a| \cdot \mathbf{d}(x, y).$$

Example 2.5. Let us take the two-dimensional real vector space $\mathbb{R}^2$ as the space L. The natural metric in this space is defined by the conventional formula

$$\text{If} \quad x = (x_1, x_2) \text{ and } y = (y_1, y_2)\text{, then } \mathbf{d}(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This metric is also linear. Indeed,

$$\mathbf{d}(x+z, y+z) = \sqrt{((x_1 + z_1) - (y_1 + z_1))^2 + ((x_2 + z_2) - (y_2 + z_2))^2} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \mathbf{d}(x, y)$$

and

$$\mathbf{d}(ax, ay) = \sqrt{(ax_1 - ay_1)^2 + (ax_2 - ay_2)^2} = \sqrt{a^2(x_1 - y_1)^2 + a^2(x_2 - y_2)^2} =$$
$$= |a|\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = |a| \cdot \mathbf{d}(x, y).$$

Example 2.6. Let us take the two-dimensional real vector space $\mathbb{R}^2$ as the space L. The natural metric in this space is defined by the conventional formula

$$\text{If } x = (x_1, x_2) \text{ and } y = (y_1, y_2)\text{, then } \mathbf{d}(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

This metric is not linear. Indeed, let us take $x = (3, 3)$, $y = (1, 1)$, and $a = 2$. Then $\mathbf{d}(x, y) = 8$, while $\mathbf{d}(2x, 2y) = 32$.

These examples show that there are linear metrics (hypermetrics) in vector spaces and there are metrics (hypermetrics) in vector spaces that are not linear. The majority of popular metrics are induced by norms and thus, they are linear as the following result demonstrates.

Theorem 2.2. *A hypermetric* $\mathbf{d}$ *is induced by a hypernorm if and only if* $\mathbf{d}$ *is linear.*

Proof. Necessity. Let us consider a vector space L with a hypernorm q. By Lemma 2.1, it induces the hypermetric $\mathbf{d}_q(x, y) = q(x-y)$. Then $\mathbf{d}_q(x+z, y+z) = q((x+z)(y+z)) = q(x-y) = \mathbf{d}_q(x, y)$, i.e., Axiom LM1 is true. In addition, $\mathbf{d}_q(ax, ay) = q(ax-ay) = q(a(x-y)) = |a| \cdot q(x-y) = |a| \cdot \mathbf{d}_q(x, y)$, i.e., Axiom LM2 is also true.

Necessity. Let us consider a vector space L with a linear hypermetric $\mathbf{d}$. We define the hypernorm $q_{\mathbf{d}}$ by the following formula

$$q_{\mathbf{d}}(x) = \mathbf{d}(\mathbf{0}, x).$$

We show that $q_{\mathbf{d}}$ is a hypernorm. Indeed, $q_{\mathbf{d}}(\mathbf{0}) = \mathbf{d}(\mathbf{0}, \mathbf{0}) = 0$. Besides, if $q_d(x) = \mathbf{d}(\mathbf{0}, x) = 0$, then $x = 0$ by Axiom **M1**. This gives us Axiom **N1** for $q_{\mathbf{d}}$.

In addition,

$$q_{\mathbf{d}}(ax) = \mathbf{d}(\mathbf{0}, ax) = \mathbf{d}(a\mathbf{0}, ax) = \mathbf{d}(a(\mathbf{0}, x)) = |a| \cdot \mathbf{d}(\mathbf{0}, x) = |a| \cdot q_{\mathbf{d}}(x)$$

by Axiom LM2. This gives us Axiom N2 for $q_{\mathbf{d}}$.

Likewise, by Axioms M3 and LM1, we have

$$q_{\mathbf{d}}(x + y) = \mathbf{d}(\mathbf{0}, x + y) \leq \mathbf{d}(\mathbf{0}, x) + \mathbf{d}(x, x + y) = \mathbf{d}(\mathbf{0}, x) + \mathbf{d}(\mathbf{0}, y) = q_{\mathbf{d}}(x) + q_{\mathbf{d}}(y).$$

This gives us the triangle inequality (Axiom N3) for $q_{\mathbf{d}}$.

Theorem is proved. □

Corollary 2.2. *A metric* **d** *is induced by a norm if and only if* **d** *is linear.*

Taking only a part of the hypernorm properties, we come to the concept of a hyperseminorm.

Definition 2.5. a) A mapping $q : L \to \mathbb{R}^+_\omega$ is called a *hyperseminorm* if it satisfies the following conditions:

N2. $q(ax) = |a| \cdot q(x)$ for any x from L and any number a from $\mathbb{R}$.

N3. (the triangle inequality or subadditivity).

$$q(x + y) \leq q(x) + q(y) \text{ for any } x \text{ and } y \text{ from } L.$$

b) A vector space L with a norm is called a *hyperseminormed vector space* or simply, a *hyperseminormed space*.
c) The real hypernumber $q(x)$ is called the *hyperseminorm* of an element x from the hyperseminormed space L.
d) A set $X \subseteq L$ is called q - bounded if there is a positive real number h such that for any element a from X, the inequality $q(a) < h$ is true.
e) A set $X \subseteq L$ is called weakly q - *bounded* if there is a positive real hypernumber α such that for any element a from X, the inequality $q(a) < \alpha$ is true.

Note that any seminorm is a hyperseminorm that takes values only in the set of real numbers.

Proposition 2.1. *If* $q : L \to \mathbb{R}$ *is a hyperseminorm, then it has the following properties:*

(1) $q(x) \geq 0$ *for any* $x \in L$.
(2) $q(x - y) = q(y - x)$ *for any* $x, y \in L$.
(3) $q(\mathbf{0}) = 0$.
(4) $|q(x)q(y)| = q(x - y)$ *for any* $x, y \in L$.
(5) $q(x) - q(y) \leq q(x + y)$ *for any* $x, y \in L$.

Proof. (1) By Axiom N3, we have

$$q(x) + q(-x) \geq q(x + (-x)) = q(\mathbf{0}).$$

At the same time, by N2, we have $q(\mathbf{0}) = 0 \cdot q(\mathbf{0}) = 0$ and $q(-x) = q(x)$. This gives us

$$q(x) + q(-x) = q(x) + q(x) = 2q(x) \geq q(x + (-x)) = q(\mathbf{0}) = 0$$

and thus, $q(x) \geq 0$.

(2) By Axiom N2, we have

$$q(x - y) = q(-(y - x)) = |-1| \cdot q(y - x) = q(y - x).$$

(3) By Axiom N2, we have

$$q(\mathbf{0}) = q(0 \cdot \mathbf{0}) = |0| \cdot q(\mathbf{0}) = 0.$$

(4) By Axiom N3, we have

$$q(x) = q(x - y + y) \leq q(x - y) + q(y).$$

Thus,

$$q(x) - q(y) \leq q(x - y).$$

As q is symmetric (property (2)), we have

$$q(y) - q(x) \leq q(x - y).$$

Consequently,

$$|q(x) - q(y)| = q(x - y).$$

Property (5) is a consequence of property (4).

Proposition is proved. □

There are intrinsic relations between hyperseminorms and semitopological vector spaces.

Theorem 2.3. *Any hyperseminormed space is a semitopological vector space, which is Hausdorff if and only if it is a hypernormed space.*

Proof. Let us consider a vector space L with a hyperseminorm q. Taking an element x from L and a positive real number k, we define the neighborhood $O_k x$ of x by the following formula

$$O_k x = \{y \in L; q(x - y) < k\}.$$

To show that the system of so defined neighborhoods determines a topology in L, we check the neighborhood axioms (Kuratowski, 1966).

NB1: The point x belongs to $O_k x$ because by Proposition 1, $q(x - x) = q(\mathbf{0}) = 0 < k$ for any positive real number k.

NB2: Taking two positive real numbers k and h, we see that the intersection $O_k x \cap O_h x = O_l x$ is also a neighborhood of x where $l = \min\{k, h\}$.

NB3: Let $y \in O_k x$. Then $q(x-y) < k$ and by properties of real numbers, there is a positive real number t such that $q(x-y) < k-t$. Then $O_t x \subseteq O_k x$. Indeed, if $z \in O_t x$, then $q(y-z) < t$. Consequently,

$$q(x-z) = q((x-y)+(y-z)) \leq q(x-y) + q(y-z) < (k-t) + t = k.$$

It means that $z \in O_k x$.

Now we show that addition is continuous and scalar multiplication is continuous in the second coordinate with respect to this topology.

Let us consider a sequence $\{x_i;\ i = 1, 2, 3, \dots\}$ that converges to x, a sequence $\{y_i;\ i = 1, 2, 3, \dots\}$ that converges to y, and the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \dots\}$. Convergence of these two sequences means that for any $k > 0$, there are a natural number n such that $q(x_i - x) < k$ for any $i > n$ and a natural number m such that $q(y_i - y) < k$ for any $i > m$. Then by properties of a hyperseminorm, we have

$$q(z_i - (x+y)) = q((x_i + y_i) - (x+y)) = q((x_i - x) + (y_i - y)) \leq q(x_i - x) + q(y_i - y) < k + k = 2k,$$

when $i > \max\{n, m\}$. As k is an arbitrary positive real number, this means that the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \dots\}$ converges to $x + y$. Consequently, addition is continuous in L.

In addition, for any number a from $\mathbf{F}$, we have

$$q(u_i - ax) = q(ax_i - ax) = q(a(x_i - x)) = |a|q(x_i - x) < |a|k,$$

where $u_i = ax_i$. As k is an arbitrary positive real number and $|a|$ is a constant, this means that the sequence $\{u_i = ax_i;\ i = 1, 2, 3, \dots\}$ converges to ax. Consequently, scalar multiplication is continuous in the second coordinate.

By Theorem 2.2, if q is a hypernorm, then the space L is Hausdorff. At the same time, if q is not a hypernorm, then there are x and y from L such that $x \neq y$ but $q(x-y) = 0$. According to definition, these points x and y cannot be separated in the topology defined above. Thus, the space L is not Hausdorff.

Theorem is proved. □

Hyperseminormed spaces are also hyperpseudometric spaces.

Definition 2.6. A *hyperpseudometric* in a set X is a mapping $\mathbf{d} : X \times X \to \mathbb{R}^+_\omega$ that satisfies the following axioms:

P1. $\mathbf{d}(x, y) = 0$ if $x = y$,
i.e., the distance between an element and itself is equal to zero.

M2. (Symmetry). $d(x, y) = d(y, x)$ for all $x, y \in X$,
i.e., the distance between x and y is equal to the distance between y and x.

M3. (the triangle inequality or subadditivity).

$$\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y) \text{ for all } x, y, z \in X.$$

When the distance between two elements of X is always a real number, $\mathbf{d}$ is a *pseudometric* (Kuratowski, 1966).

Note that although it would look natural, we do not use terms semimetric and hypersemimetric because according to the mathematical convention, semimetric is defined by a distance that satisfies only axioms M1 and M2.

Lemma 2.2. *a) A hyperseminorm in a vector space L induces a hyperpseudometric in this space.*

b) If q is a seminorm in L, then $\mathbf{d}_q$ is a pseudometric.

Indeed, if $q : X \times \mathbb{R}^+_\omega$ is a hyperseminorm in L and x and y are elements from L, then we can define $\mathbf{d}_q(x, y) = q(x - y)$. Properties of a hyperseminorm imply that $\mathbf{d}_q$ satisfies all axioms P1, M2 and M3. In addition, if q takes values only in $\mathbb{R}$, then the same is true for $\mathbf{d}_q$, i.e., $\mathbf{d}_q$ is a pseudometric.

It is interesting to find what hyperpseudometrics in vector spaces are induced by hyperseminorms and what pseudometrics in vector spaces are induced by seminorms. To do this, let us consider additional properties of hypermetrics and metrics.

Definition 2.7. A hyperpseudometric (metric) in a vector space L is called *linear* if it satisfies the Axioms LM1 and LM2.

Examples 2.4 - 2.6 show that there are linear pseudometrics (hyperpseudometrics) in vector spaces and there are pseudometrics (hyperpseudometrics) in vector spaces that are not linear. The majority of popular pseudometrics are induced by seminorms and thus, they are linear as the following result demonstrates.

Theorem 2.4. *A hyperpseudometric* $\mathbf{d}$ *is induced by a hyperseminorm if and only if* $\mathbf{d}$ *is linear. Proof is similar to the proof of Theorem 2.2.*

Corollary 2.3. *A pseudometric* $\mathbf{d}$ *is induced by a seminorm if and only if* $\mathbf{d}$ *is linear. We define the kernel* Ker q *of a hyperseminorm q in L as*

$$\operatorname{Ker} q = \{x \in L;\ q(x) = 0\}.$$

Theorem 2.5. *The kernel* Ker q *of a hyperseminorm q in L is a vector subspace of L.*

Indeed, if $q(x) = 0$ and $a \in \mathbf{F}$, then by Axiom N2,

$$q(ax) = |a| \cdot q(x) = |a| \cdot 0 = 0$$

i.e., $ax \in \operatorname{Ker} q$. In addition, $q(x) = 0$ and $q(y) = 0$, then by Axiom N3,

$$q(x + y) \leq q(x) + q(y) = 0 + 0 = 0$$

and $q(x + y) = 0$ because by Proposition 2.1, $q(x + y) \geq 0$.

Theorem 2.5 allows factorization of the hyperseminormed space L by its subspace $\operatorname{Ker} q$, obtaining the quotient space L_q . The hyperseminorm q induces the hypernorm p_q in the space L_q . This gives us the natural projection $\tau : L \to L_q$, which preserves the hyperseminorm q.

Example 2.7. Let us consider the set $C^\infty(\mathbb{R}, \mathbb{R})$ of all smooth real functions. The following seminorms are considered in is the set $C^\infty(\mathbb{R}, \mathbb{R})$. For each point $a \in \mathbb{R}$, and $f \in C^\infty(\mathbb{R}, \mathbb{R})$, we define

$$q_k(f) = (f(a))^2 + (f'(a))^2 + (f''(a))^2 + \ldots + (f^{(k)}(a))^2.$$

The factorization of the space by its subspace $\operatorname{Ker} q$ is called the k - th order jet space $J^k_a(\mathbb{R}, \mathbb{R})$ of $C^\infty(\mathbb{R}, \mathbb{R})$ at the point a. Jet spaces were introduced by Ehresmann (Ehresmann, 1952, 1953) and have various applications in the theory of differential equations and differential relations, as well as in the theory of manifolds (Gromov, 1986), (Krasilshchik *et al.*, 1986).

It is possible to get the same quotient space using the following seminorm

$$m_k(f) = \max\{|f(a)|, |f'(a)|, |f''(a)|, \ldots, |f^{(k)}(a)|\}.$$

Let us consider a Hausdorff space X that is a quotient space of L with the projection $\eta : L \to X$, preserves the hyperseminorm q. Then it is possible to define a projection $\nu : L_q \to X$ preserves the hyperseminorm q and for which $\eta = \nu\tau$, i.e., the following diagram is commutative:

$$\begin{tikzcd} L \arrow[r, "\tau"] \arrow[d, "\eta"'] & L_q \arrow[dl, dashed, "\nu"] \\ X & \end{tikzcd}$$

This gives us the following result.

Theorem 2.6. *a) L_q is the largest Hausdorff quotient space of the topological space L that preserves the hyperseminorm q.*

b) L_q is the largest quotient space of the topological space L in which the hyperseminorm q induces the hypernorm p_q.

It is possible to define two basic operators in a vector space L.

1. If z is an element from **L**, then the translation operator T_z is defined by the formula:

$$T_z(x) = x + z \text{ where } x, z \in L.$$

2. If $a \neq 0$ is an element from **F**, then the multiplication operator M_a is defined by the formula:

$$M_a(x) = ax \text{ where } x \in L.$$

Proposition 2.2. *Operators T_z and M_a are homeomorphisms of the semitopological vector space L.*

Proof. The axioms of a vector space imply that T_z and M_a are one-to-one mappings and their inverses are T_{-z} and M_{-a} , respectively. As addition is continuous in L, the operator T_z is also continuous. As scalar multiplication is continuous with respect to L, the operator M_a is also continuous. Proposition is proved. □

Corollary 2.4. *The topology of a semitopological vector space L is translation-invariant, or simply invariant, i.e., a subset A from L is open if and only if any its translation $A + a$ is open.*

As a result, such a topology is completely determined by any local base and thus, by any local base at **0**.

Let us consider two subsets K and C of a semitopological vector space L.

Theorem 2.7. *If C is compact, K is closed and $K \cap C = \varnothing$, then* **0** *has a neighborhood V such that*

$$(K + V) \cap (C + V) = \varnothing.$$

Proof Proof is similar to the proof of Theorem 1.10 from (Rudin, 1991) because it uses only the first property of semitopological vector spaces.

As topological vector spaces are special cases of semitopological vector spaces, Theorem 1.10 from (Rudin, 1991) is a corollary of Theorem 2.7.

In a topological space X, the weakest separation axiom is $\mathbf{T}_0$ (Kelly, 1955) where:

$$\mathbf{T}_0 \text{ (the } \textit{Kolmogorov Axiom}\text{)}. \; \forall x, y \in X \, (\exists Ox(y \notin Ox) \vee \nexists Oy(x \notin Oy)).$$

Lemma 2.3. *In a topological space X, all points are closed if and only if X satisfies the axiom* $\mathbf{T}_0$.

Proof. Sufficiency. If X satisfies the axiom $\mathbf{T}_0$ and x is a point from X, then each point from the complement Cx of x has a neighborhood that does not contain x. Thus, all these neighborhoods are subsets of Cx. By definition, Cx is an open set (Kuratowski, 1966) and consequently, its complement x is a closed set.

Necessity. If $x, y \in X$ and the point x is closed, then y belongs to the complement Cx of x, which is open as the complement of a closed set (Kuratowski, 1966). Thus, y has a neighborhood Oy that is a subset of Cx. Consequently, Oy does not contain x. As points x and y are arbitrary, X satisfies the axiom $\mathbf{T}_0$.

Lemma is proved. □

We remind (Alexandroff, 1961) that T_3 - *spaces*, or *regular spaces*, are topological spaces in which satisfy Axiom $\mathbf{T}_3$:

$\mathbf{T}_3$ *For every point a and closed set B, there exist disjoint open sets which separately contain a and B.*

It means that points and closed sets are separated.

Note that there are semitopological vector spaces in which not all points are closed. The space $\mathbb{R}^\omega$ of all sequences of real numbers is an example of such a semitopological vector space. Moreover, in $\mathbb{R}^\omega$, there are no closed points.

As a point is a compact space, Theorem 2.5 implies the following result.

Corollary 2.5. *Every semitopological vector space L in which all points are closed is a regular space.*

Lemma 2.3 and Corollary 2.5 imply the following result.

Corollary 2.6. *In semitopological vector spaces, L axiom* $\mathbf{T}_0$ *implies axiom* $\mathbf{T}_3$.

As any regular space is a Hausdorff space (Alexandroff, 1961), we have the following result.

Corollary 2.7. *Every semitopological vector space L in which all points are closed is a Hausdorff space.*

Lemma 2.3 and Corollary 2.7 imply the following result.

Corollary 2.8. *In semitopological vector spaces, L axiom* $\mathbf{T}_0$ *implies axiom* $\mathbf{T}_2$.

As both sets $K + V$ and $C + V$ in Theorem 2.7 are open, the closure of $K + V$ does not intersect $C + V$, while the closure of $C + V$ does not intersect $K + V$. As any point a from L is a compact space, we can take $K = \{a\}$. Applying Theorem 2.7 to this situation, we obtain the result, which has a considerable interest according to (Rudin, 1991).

Corollary 2.9. *Any neighborhood O_a of any point a in a semitopological vector space L contains the closure of some neighborhood V_a of the same point a.*

As topological vector spaces are special cases of semitopological vector spaces, Theorem 1.11 from (Rudin, 1991) is a corollary of Corollary 2.9.

3. Mappings of hyperseminormed vector spaces

Let us consider a hyperseminormed vector space L, i.e., a vector space L with a system of hyperseminorms Q, a hyperseminormed vector space M with a system of hyperseminorms P, a hyperseminorm q from Q, a hyperseminorm p from P, and a subset V of the space L.

Vector spaces with systems of hyperseminorms (of hypernorms) will be called *polyhyperseminormed spaces (polyhypernormed spaces)* because vector spaces over $\mathbb{R}$ with systems of norms or seminorms are called *polynormed spaces* (see (Helemski, 1989); (Dosi, 2011)).

Definition 3.1. a) An operator (mapping) $A : L \to M$ is called (q, p) - *bounded at a point a* from L if for any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

b) An operator (mapping) $A : L \to M$ is called (q, p) - *bounded* if it is (q, p) - bounded at all points of L.

c) An operator (mapping) $A : L \to M$ is called V - *uniformly* (q, p) - *bounded* if for any positive real number k, there is a positive real number h such that for any element a from V and any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

d) An operator (mapping) $A : L \to M$ is called *uniformly* (q, p) - bounded in V if for any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

Note that when the set V contains only one point (say a), then V - uniform (q, p) - boundedness coincides with (q, p) - boundedness at the point a.

Definitions imply the following result.

Lemma 3.1. *Any uniformly (q, p) - bounded in L operator is L - uniformly (q, p) - bounded and any L - uniformly (q, p) - bounded operator is (q, p) - bounded.*

At the same time, as the following example demonstrates, there are (q, p) - bounded operators that are not L - uniformly (q, p) - bounded.

Example 3.1. Let us take $L = M = \mathbb{R}$ and assume that q and p are both equal to the absolute value, while $A(x) = x^2$. This mapping (operator) is (q, p) - bounded but not L - uniformly (q, p) - bounded.

However, for linear operators, the inverse of Lemma 3.1 is also true.

Proposition 3.1. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (q, p) - bounded.

(2) A is uniformly (q, p) - bounded in L.

(3) For some point a, A is uniformly (q, p) - bounded at the point a.

(4) A is L - uniformly (q, p) - bounded.

Proof. Implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) directly follow from definitions. So, we need to prove only (3) $\Rightarrow$ (2), namely, if $A : L \to M$ is (q, p) - bounded at a point a from L, then it is uniformly (q, p) - bounded.

Let us consider another point b from L and assume that $q(b - c) < k$ for some c from L. Then taking $d = c - (b - a)$, we have

$$q(a - d) = q(a - (c - (b - a)) = q(b - c) < k.$$

As A is (q, p) - bounded at a, there is a positive real number h such that $p(A(a) - A(d)) < h$. As A is linear operator, we have

$$p(A(b) - A(c)) = p(A(b - c)) = p(A(a - (c - (b - a)) = p(A(a - d)) = p(A(a) - A(d)) < h.$$

This shows that A is (q, p) - bounded at the point b because c is an arbitrary point for which $q(b - c) < k$. Thus, A is uniformly (q, p) - bounded in L because for a fixed number k, we have the same number h for all points in L.

In addition, we see that by definition, properties (2) and (4) always coincide.

Proposition is proved. □

Corollary 3.1. *A linear operator (mapping) A is* (q, p) *- bounded if and only if it is* (q, p) *- bounded at* $\mathbf{0}$.

The above proof of Proposition 3.1 gives us the following result.

Corollary 3.2. *Any* (q, p) *- bounded linear operator (mapping)* $A : L \to M$ *is* L *- uniformly* (q, p) *- bounded.*

These results show that for linear operators, the concepts of a (q, p) - bounded at a point operator and of a (q, p) - bounded operator coincide.

For operators that are not linear, these results are true as the following examples demonstrate.

Example 3.2. Let us assume that $L = M = \mathbb{R}_\omega$ is the space of all real hypernumbers (cf. Example 2.1), while both hyperseminorms q and p are both equal to the absolute value $\|\cdot\|$ of real hypernumbers. Actually the absolute value $\|\cdot\|$ is a norm in the space $\mathbb{R}_\omega$ (Burgin, 2012).

For the operator A, we define $A(x) = x$ for all real hypernumbers x but the hypernumber $\nu = \mathrm{Hn}(i)_{i\in\omega}$ and put $A(\nu) = 1$. Then $\|\nu - (\nu + 1)\| = 1$ but $\|A(\nu) - A(\nu + 1)\| = \|1 - (\nu + 1)\| = \|\nu\| = \nu$ and this hypernumber is larger than any positive real number (Burgin, 2012). Thus, operator A is (q, p) - bounded at any real number but it is not (q, p) - bounded at the hypernumbers ν.

This shows that an operator can be (q, p) - bounded at one point and not (q, p) - bounded at another point of L.

Example 3.3. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions is a hypernormed space (cf. Example 2.1) where the hypernorm $\|\cdot\|$ is defined by the following formula:

$$\text{If } f : \mathbb{R} \to \mathbb{R}, \text{ then } \|f\| = \mathrm{Hn}(a_i)_{i\in\omega} \text{ where } a_i = max\{|f(x)|;\ a_i \in [-i, i]\}.$$

We define $A(f) = f$ for all real functions f but the function $\nu(x) = x^2$ and put $A(x^2) = e(x)$ where $e(x) = 1$ for all $x \in \mathbb{R}$. This operator A is (q, p) - bounded at any constant function from L but it is not (q, p) - bounded at ν. At the same time, taking $u(x) = x^2 + 1$, we have $\|\nu - u\| = 1$, while $\|A(\nu) - A(u)\| = \|e - u\| = \mathrm{Hn}(i)_{i\in\omega}$ and this hypernumber is larger than any positive real number (Burgin, 2011).

This also shows that an operator can be (q, p) - bounded at one point and not (q, p) - bounded at another point of L.

However, for norms and seminorms, we do not need additional conditions to establish the result of Proposition 3.1.

Proposition 3.2. *If q is a seminorm, then an operator (mapping) $A : L \to M$ is (q, p) - bounded if and only if it is (q, p) - bounded, at least, at one point.*

Proof. Let us consider two points a and c from L and assume that an operator $A : L \to M$ is (q, p) - bounded at the point a. Then taking a point b such that $q(c - b) < u$ where u is a positive real number.

As q is a seminorm, $q(a - c)$ is equal to some positive real number w. Thus, by properties of seminorms, we have

$$q(a - b) = q(a - c + c - b) \leq q(a - c) + q(c - b) < w + u.$$

As the operator A is (q, p) - bounded at the point a and $q(a - c) < w + 1$, we have a positive real number h such that $p(A(a) - A(b)) < h$ and a positive real number k such that $p(A(a) - A(c)) < k$. Consequently,

$$p(A(c) - A(b)) \leq p(A(a) - A(c)) + p(A(a) - A(b)) < k + h.$$

As b is an arbitrary point from L, A is (q, p) - bounded at the point c.

As c is an arbitrary point from L, the operator A is (q, p) - bounded.

Proposition is proved. □

Proposition 3.2 implies the following results.

Corollary 3.3. *The concepts of a (q, p) - bounded at a point operator and of a (q, p) - bounded operator coincide when q is a seminorm.*

Note that Examples 3.2 and 3.3 show this is not true for the general case of hyperseminorms.

Corollary 3.4. *When q is a seminorm, an operator (mapping) A is (q, p) - bounded if and only if it is (q, p) - bounded at* $\mathbf{0}$.

The above proof of Proposition 3.2 gives us the following result.

Corollary 3.5. *If q is a seminorm, then any (q, p) - bounded operator (mapping) $A : L \to M$ is L - uniformly (q, p) - bounded.*

Proposition 3.3. *If q is a seminorm and there is a (q, p) - bounded operator (mapping) A of the linear space L onto the linear space M, then p is a finite hyperseminorm.*

Proof. Let us take a point u from M. As A is a projection (surjection), there are points a and b such that $A(a) = \mathbf{0}$ and $A(b) = u$. As q is a seminorm, $q(b-a)$ is less than some positive real number w. As the operator A is (q, p) - bounded, there is a positive real number h such that $p(A(a)-A(b)) < h$

$$p(u) = p(u - \mathbf{0}) = p(A(b) - A(a)) < h.$$

As u is an arbitrary point from M, the hyperseminorm p is finite.

Proposition is proved. □

Note that a finite hyperseminorm is not always a seminorm and a finite hypernorm is not always a norm.

Definition 3.2. (Burgin, 2012). A real hypernumber is called *monotone* is it has a monotone representative.

For instance, all real numbers are monotone hypernumbers (Burgin, 2012). At the same time, all finite monotone real hypernumbers are real numbers (Burgin, 2012). Thus, Proposition 3.3 implies the following result.

Corollary 3.6. *If q is a seminorm, there is a (q, p) - bounded operator (mapping)A of the linear space L onto the linear space M and all values of p are monotone hypernumbers, then p is a seminorm.*

Definitions imply the following results.

Lemma 3.2. *If $W \subseteq V \subseteq L$, then any V - uniformly (q, p) - bounded operator is W - uniformly (q, p) - bounded and any uniformly (q, p) - bounded in V operator is uniformly (q, p) - bounded in W.*

Lemma 3.3. *Any V - uniformly (q, p) - bounded operator is (q, p) - bounded in V.*

Let us consider a binary relation u between the system of hyperseminorms Q, the system of hyperseminorms P and a subset V of the space L.

Definition 3.3. a) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *bounded at a point a* from L if for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - bounded at the point a.

b) An operator (mapping) $A : L \to M$ is called V - *uniformly* (Q, u, P) - *bounded* if for any hyperseminorms q and p with $(q, p) \in u$ and any positive real number k, there is a positive real number h such that for any element a from V and any element b from L, the inequality $q(a-b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

c) An operator (mapping) $A : L \to M$ is called *uniformly* (Q, u, P) - *bounded* in V if for any hyperseminorms q and p with $(q, p) \in u$ and any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a-b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

d) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *bounded* if it is (Q, u, P) - bounded at all points of L.

It means that an operator (mapping) A is (Q, u, P) - bounded if for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - bounded.

Note that when the set V contains only one point (say a), then V - uniform (Q, u, P) - boundedness coincides with (Q, u, P) - boundedness at the point a.

Lemma 3.1 implies the following result.

Lemma 3.4. *Any uniformly (Q, u, P) - bounded operator in L is L - uniformly (Q, u, P) - bounded, while any L - uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded.*

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Example 3.1 demonstrates that there are (Q, u, P) - bounded operators that are not L - uniformly (Q, u, P) - bounded.

However, for linear operators, the inverse of Lemma 3.4 is also true because Proposition 3.1 implies the following result.

Proposition 3.4. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (Q, u, P) - bounded.

(2) A is uniformly (Q, u, P) - bounded in L.

(3) For some point a, A is uniformly (Q, u, P) - bounded at the point a.

(4) A is L - uniformly (Q, u, P) - bounded.

Corollary 3.7. *A linear operator (mapping) A is (Q, u, P) - bounded if and only if it is (Q, u, P) - bounded at* **0**.

Corollary 3.2 implies the following result.

Corollary 3.8. *Any (Q, u, P) - bounded linear operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - bounded.*

These results show that for linear operators, the concepts of a (Q, u, P) - bounded at a point operator and a (Q, u, P) - bounded operator coincide.

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Examples 3.2 and 3.3 demonstrate that there are operators that are (Q, u, P) - bounded at one point and not (Q, u, P) - bounded at another point.

However, for norms and seminorms, we do not need additional conditions to establish the result of Proposition 3.4. We remind that the definability domain of the relation u is defined as

$$Du = \{q;\ \text{there is a pair } (q, p) \text{ that belongs to } u\}.$$

Then Proposition 3.2 implies the following result.

Proposition 3.5. *If all q from the definability domain Du of u are seminorms, then an operator (mapping)* $A : L \to M$ *is* (Q, u, P) *- bounded if and only if it is* (Q, u, P) *- bounded, at least, at one point.*

Proposition 3.5 implies the following result.

Corollary 3.9. *The concepts of* (Q, u, P) *- bounded at a point operators and* (Q, u, P) *- bounded operator coincide when all q from the definability domain Du of u are seminorms.*

Note that Examples 3.2 and 3.3 show this is not true for the general case of hyperseminorms.

Corollary 3.10. *When all q from the definability domain Du of u are seminorms, an operator (mapping) A is* (Q, u, P) *- bounded if and only if it is* (Q, u, P) *- bounded at* $\mathbf{0}$.

The above proof of Proposition 3.2 gives us the following result.

Corollary 3.11. *If all q from the definability domain Du of u are seminorms, then any* (Q, u, P) *- bounded operator (mapping)* $A : L \to M$ *is L - uniformly* (Q, u, P) *- bounded.*

Proposition 3.3 implies the following result.

Proposition 3.6. *If all q from the definability domain Du of u are seminorms and there is a* (Q, u, P) *- bounded operator (mapping) A of the linear space L onto the linear space M, then all p from the range Rg u of u are finite hyperseminorms.*

Corollary 3.12. *If all q from the definability domain Du of u are seminorms and there is a* (Q, u, P) *- bounded operator (mapping) A of the linear space L onto the linear space M, and all values of all p from the range Rg u are monotone hypernumbers, then all such p are seminorms.*

Definitions imply the following results.

Lemma 3.5. *If* $W \subseteq V \subseteq L$*, then any V - uniformly* (Q, u, P) *- bounded operator is W - uniformly* (q, p) *- bounded and any uniformly* (Q, u, P) *- bounded in V operator is uniformly* (q, p) *- bounded in W.*

Lemma 3.6. *Any V - uniformly* (Q, u, P) *- bounded operator is* (Q, u, P) *- bounded in V.*

Let us take a subset V of the space L.

Definition 3.4. a) An operator (mapping) $A : L \to M$ is called *uniformly* (Q, u, P) *- bounded at a point a* from L if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, and any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

b) An operator (mapping) $A : L \to M$ is called *u - uniformly* (Q, u, P) *- bounded* if it is uniformly (Q, u, P) - bounded at all points of L.

c) An operator (mapping) $A : L \to M$ is called u - *uniformly* (Q, u, P) - *bounded* in V if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, and any elements a and b from V, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

d) An operator (mapping) $A : L \to M$ is called uV - *uniformly* (Q, u, P) - *bounded* in V if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, and any elements a from V and b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

Asking whether any (Q, u, P) - bounded at a point operator (mapping) is uniformly (Q, u, P) - bounded at the same point, we find that the answer is negative.

Example 3.4. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions. It is possible (Burgin, 2012) for all real numbers x, to define seminorms $q_{ptx} = p_{ptx}$ by the following formula

$$q_{ptx}(f) = p_{ptx}(f) = |f(x)|.$$

We define $A(f) = xf(x)$ for all real functions f and $u = \{(q_{ptx}, p_{ptx}); x \in \mathbb{R}\}$. Taking the function $f(x) = x$ as the point a from L, we see that $A(f) = x^2$. Thus, taking some positive real number k, e.g., $k = 1$, the corresponding h from Definition 3.2 always exists but it grows with the growth of x. For instance, when $k = 1$, we have

$$q_{pt1}(f - g) < 1 \text{ implies } p_{pt1}(A(f) - A(g)) = p_{pt1}(xf - xg) < 1.$$

At the same time, $q_{pt10}(f - g) < 1$ does not imply $p_{pt10}(A(f) - A(g)) < 1$. It only implies $p_{pt10}(A(f) - A(g)) = p_{pt10}(xf - xg) < 10$. This means that for any pair (q_{ptx}, p_{ptx}) of seminorms and a number k, we need to find a specific number h to satisfy Definition 3.3 a. Consequently, the operator A is (Q, u, P) - bounded at f but it is not uniformly (Q, u, P) - bounded at f.

The same example shows that there are (Q, u, P) - bounded operators that are not uniformly (Q, u, P) - bounded.

It is also possible to ask whether Propositions 3.4 and 3.5 remain true for uniformly (Q, u, P) - bounded operators. In this case, the answer is positive.

Proposition 3.7. *If all q from the definability domain Du of the relation u are seminorms, then an operator (mapping)* $A : L \to M$ *is uniformly* (Q, u, P) *- bounded if and only if it is uniformly* (Q, u, P) *- bounded, at least, at one point.*

Indeed, Proposition 3.7 is a direct corollary of Proposition 3.5 because any uniformly (Q, u, P) - bounded at a point operator is (Q, u, P) - bounded at the same point and any uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded.

Proposition 3.7 implies the following result.

Corollary 3.13. *The concepts of uniformly* (Q, u, P) *- bounded at a point operators and uniformly* (Q, u, P) *- bounded operators coincide when all q from the definability domain Du of u are seminorms.*

Note that Examples 3.2 and 3.3 show this is not true for the general case of hyperseminorms.

Proposition 3.8. *If all q from the definability domain Du of u are seminorms and there is a uniformly* (Q, u, P) *- bounded operator (mapping) A of the linear space L onto the linear space M, then all p from the range Rg u of u are finite hyperseminorms.*

Indeed, Proposition 3.8 is a direct corollary of Proposition 3.6 because any uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded.

Corollary 3.14. *If all q from the definability domain Du of u are seminorms and there is a uniformly* (Q, u, P) *- bounded operator (mapping) A of the linear space L onto the linear space M, and all values of all p from the range Rg u are monotone hypernumbers, then all such p are seminorms.*

Definitions imply the following results.

Lemma 3.7. *a) Any uniformly* (Q, u, P) *- bounded at a point a operator A is* (Q, u, P) *- bounded at the point* a.

b) Any u-uniformly (Q, u, P) *- bounded operator A is* ((Q, u, P) *- bounded.*

Lemma 3.8. *Any u-uniformly* (Q, u, P) *- bounded in L operator is u-uniformly* (Q, u, P) *- bounded.*

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that hyperseminorms q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Example 3.1 demonstrates that there are u-uniformly (Q, u, P) - bounded operators that are not uniformly (Q, u, P) - bounded because if Q has only one hyperseminorm q, P also has only one hyperseminorm p and u is a complete relation, then any (Q, u, P) - bounded operator is u-uniformly (Q, u, P) - bounded.

However, for linear operators, this is impossible as Proposition 3.1 allows us to prove the following result.

Proposition 3.9. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is u-uniformly (Q, u, P) *- bounded.*

(2) A is u-uniformly (Q, u, P) *- bounded in L.*

(3) For some point a, A is uniformly (Q, u, P) *- bounded at the point a.*

Proof. Implications (2) ⇒ (1) ⇒ (3) directly follow from definitions. So, we need to prove only (3) ⇒ (2), namely, if $A : L \to M$ is uniformly (Q, u, P) - bounded at a point a from L, then it is uniformly (Q, u, P) - bounded in L.

Let us consider another point b from L, take two hyperseminorms q and p with $(q, p) \in u$, and assume that $q(b - c) < k$ for some c from L. Then taking $d = c - (b - a)$, we have

$$q(a - d) = q(a - (c - (b - a))) = q(b - c) < k.$$

As A is uniformly (Q, u, P) - bounded at a, it is also (q, p) - bounded at a. Thus, there is a positive real number h such that $p(A(a) - A(d)) < h$. As A is linear operator, we have

$$p(A(b) - A(c)) = p(A(b - c)) = p(A(a - (c - (b - a)))) = p(A(a - d)) = p(A(a) - A(d)) < h.$$

This shows that A is (q, p) - bounded at the point b because c is an arbitrary point for which $q(b - c) < k$ and thus, A is u-uniformly (Q, u, P) - bounded because q and p are arbitrary hyper-seminorms with $(q, p) \in u$. In addition, A is uniformly (q, p) - bounded in L because for a fixed number k, we have the same number h for all points in L.

Proposition is proved. □

Corollary 3.15. *A linear operator (mapping) $A : L \to M$ is u-uniformly (Q, u, P) - bounded if and only if it is uniformly (Q, u, P) - bounded at* $\mathbf{0}$.

Corollary 3.2 implies the following result.

Corollary 3.16. *Any u-uniformly (Q, u, P) - bounded linear operator (mapping) A is u-uniformly (Q, u, P) - bounded in L.*

These results show that for linear operators, different types of uniformly bounded operators coincide.

Proposition 3.10. *If the relation u is finite, then an operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - bounded (at a point a) if and only if it is (Q, u, P) - bounded (at the point a).*

Proof. As any uniformly (Q, u, P) - bounded (at a point a) operator is (Q, u, P) - bounded (at the same point), we need only to show that when the relation u is finite, a (Q, u, P) - bounded (at a point a) operator $A : L \to M$ is uniformly (Q, u, P) - bounded (at the point a). At first, we consider local boundedness.

Indeed, by Definition 3.3, for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p)-bounded at the point a, that is, by Definition 3.1, the following condition is true:

Condition 1. For any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

This number h can be different for different pairs (q, p), but because u is finite, there is only a finite number of these pairs. So, we can take

$$l = max\{h; h \text{ satisfies Condition 1 for a pair } (q, p) \in u\}$$

and this number l will satisfy the condition from Definition 3.4. Thus, the operator A is uniformly (Q, u, P) - bounded at the point a.

The global case is proved in a similar way.

Proposition is proved. □

Corollary 3.17. *If systems of hyperseminorms Q and P are finite, then an operator (mapping) A is uniformly* (Q, u, P) *- bounded (at a point a) if and only if it is* (Q, u, P) *- bounded (at the point a).*

Now let us study different types of continuity in polyhyperseminormed vector spaces.

Definition 3.5. a) An operator (mapping) $A : L \to M$ is called (q, p) - *continuous at a point a* from L if for any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

b) An operator (mapping) $A : L \to M$ is called (q, p) - *continuous* if it is (q, p) - continuous at all points of L.

c) An operator (mapping) $A : L \to M$ is called *uniformly* (q, p) - *continuous in* $V \subseteq L$ if for any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

d) An operator (mapping) $A : L \to M$ is called V - *uniformly* (q, p) - *continuous* if for any positive real number k, there is a positive real number h such that for any element a from $V \subseteq L$ and any element b from L, the inequality $q(b - a) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Note that when the set V contains only one point (say a), then V - *uniform* (q, p) - *continuity* coincides with (q, p) - continuity at the point a. Besides, to be L - *uniformly* (q, p) - *continuous* or to be *uniformly* (Q, u, P) - *continuous* in L means the same for all operators.

Definitions imply the following results.

Lemma 3.9. *For any* $V \subseteq L$*, any V - uniformly* (q, p) *- continuous operator is* (q, p) *- continuous in V.*

Lemma 3.10. *Any L - uniformly* (q, p) *- continuous operator is* (q, p) *- continuous.*

At the same time, as the following example demonstrates, there are (q, p) - continuous operators that are not L - uniformly (q, p) - continuous.

Example 3.5. Let us take $L = M = \mathbb{R}$ and assume that q and p are both equal to the absolute value, while $A(x) = x^2$. This mapping (operator) is (q, p) - continuous but not L - uniformly (q, p) - continuous.

However, for linear operators, the inverse of Lemma 3.9 is also true.

Proposition 3.11. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (q, p) *- continuous.*

(2) A is uniformly (q, p) *- continuous in L.*

(3) For some point a, A is uniformly (q, p) *- continuous at the point a.*

(4) A is L - uniformly (q, p) - continuous.

Proof. Implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) directly follow from definitions. So, we need to prove only (3) $\Rightarrow$ (2), namely, if $A : L \to M$ is (q, p) - continuous at a point a from L, then it is uniformly (q, p) - continuous in L.

Let us consider a positive real number k. Then because A is (q, p) - continuous at the point a, there is a positive real number h, such that the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Let us take another point b from L and assume that $q(b-c) < h$ for some c from L. Then taking $d = c - (b - a)$, we have

$$q(a - d) = q(a - (c - (b - a))) = q(b - c) < h.$$

As A is (q, p) - continuous at a, we have $p(A(a) - A(d)) < k$. As A is linear operator, we have

$$p(A(b) - A(c)) = p(A(b - c)) = p(A(a - (c - (b - a)))) = p(A(a - d)) = p(A(a) - A(d)) < k.$$

This shows that A is (q, p) - continuous at the point b because c is an arbitrary point for which $q(b - c) < h$. Thus, A is uniformly (q, p) - continuous in L because for a fixed number k, we have the same number h for all points in L.

In addition, we see that by definition, properties (2) and (4) always coincide.

Proposition is proved. □

Corollary 3.18. *A linear operator (mapping) A is (q, p) - continuous if and only if it is (q, p) - continuous at* $\mathbf{0}$.

The above proof of Proposition 3.4 gives us the following result.

Corollary 3.19. *Any (q, p) - continuous linear operator (mapping) $A : L \to M$ is L - uniformly (q, p) - continuous.*

These results show that for linear operators, the concepts of (q, p) - continuous at a point operators and (q, p) - continuous operators coincide.

For operators that are not linear, these results are not true as the following examples demonstrate.

Example 3.6. Let us take $L = M = \mathbb{R}_\omega$ (cf. Example 2.1) and assume that q and p are both equal to the absolute value $\| \cdot \|$ of real hypernumbers. We define $A(x) = x$ for all real hypernumbers x but the hypernumber $\nu = Hn(i)_{i \in \omega}$ and put $A(\nu) = 1$. Then $\|\nu - (\nu + 1)\| = 1$ but $\|A(\nu) - A(\nu + 1)\| = \|1 - (\nu + 1)\| = \|\nu\| = \nu$ and this hypernumber is larger than any positive real number (Burgin, 2012). Thus, operator A is (q, p) - continuous at any real number but it is not (q, p) - continuous at ν.

This shows that an operator can be (q, p) - continuous at one point and not (q, p) - continuous at another point of L and thus not (q, p) - continuous in L, as well as not L - uniformly (q, p) - continuous.

Example 3.7. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions is a hypernormed space (cf. Example 2.1) where the hypernorm $\| \cdot \|$ is defined by the following formula:

$$\text{If } f : \mathbb{R} \to \mathbb{R}, \text{ then } \|f\| = Hn(a_i)_{i\in\omega} \text{ where } a_i = max\{|f(x)|; a_i \in [-i, i]\}.$$

We define $A(f) = f$ for all real functions f but the function $v(x) = x^2$ and put $A(x^2) = e(x)$ where $e(x) = 1$ for all $x \in \mathbb{R}$. This operator A is (q, p) - continuous at any constant function from L, but it is not (q, p) - continuous at v. At the same time, taking $u(x) = x^2 + 1$, we have $\|v - u\| = 1$, while $\|A(v) - A(u)\| = \|e - u\| = Hn(i)_{i\in\omega}$ and this hypernumber is larger than any positive real number (Burgin, 2012).

This also shows that an operator can be (q, p) - continuous at one point and not (q, p) - continuous at another point of L and thus not (q, p) - continuous in L, as well as not L - uniformly (q, p) - continuous.

Definitions imply the following result.

Lemma 3.11. *If $W \subseteq V \subseteq L$, then any V - uniformly (q, p) - continuous operator is W - uniformly (q, p) - continuous.*

Now let us consider continuity with respect to a binary relation u between systems of hyperseminorms.

Definition 3.6. a) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *continuous at a point* a from L if for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - continuous at the point a.

b) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *continuous* if it is (Q, u, P) - continuous at all points of L.

c) An operator (mapping) $A : L \to M$ is called *uniformly* (Q, u, P) - *continuous in* $V \subseteq L$ if for any hyperseminorms q and p such that $(q, p) \in u$ and any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

d) An operator (mapping) $A : L \to M$ is called V - *uniformly* (Q, u, P) - *continuous* if for any hyperseminorms q and p such that $(q, p) \in u$ and for any positive real number k, there is a positive real number h such that for any element a from $V \subseteq L$ and any element b from L, the inequality $q(b - a) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Note that to be L - *uniformly* (Q, u, P) - *continuous* or to be *uniformly* (Q, u, P) - *continuous in* L means the same for all operators.

Lemma 3.10 implies the following result.

Lemma 3.12. *Any uniformly (Q, u, P) - continuous in L operator is (Q, u, P) - continuous.*

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Example 3.5 demonstrates that there are (Q, u, P) - continuous operators that are not L - uniformly (Q, u, P) - continuous.

However, for linear operators, the inverse of Lemma 3.12 is also true as Proposition 3.11 implies the following result.

Proposition 3.12. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (Q, u, P) - continuous.

(2) A is uniformly (Q, u, P) - continuous in L.

(3) For some point a, A is uniformly (Q, u, P) - continuous at the point a.

(4) A is L - uniformly (Q, u, P) - continuous.

Corollary 3.20. *A linear operator (mapping) A is (Q, u, P) - continuous if and only if it is (Q, u, P) - bounded at* $\mathbf{0}$.

Corollary 3.19 implies the following result.

Corollary 3.21. *Any (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - continuous.*

These results show that for linear operators, the concepts of (Q, u, P) - continuous at a point operators and (Q, u, P) - continuous operators coincide.

At the same time, taking $L = M = \mathbb{R}_\omega$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value of real hypernumbers and $u = \{(q, p)\}$, we see that Example 3.6 demonstrates that there are operators that are (Q, u, P) - continuous at one point and not (Q, u, P) - continuous at another point. A similar situation is also presented in Example 3.7.

Definitions and Lemma 3.10 imply the following result.

Lemma 3.13. *If $W \subseteq V \subseteq L$, then any V - uniformly (Q, u, P) - continuous operator is W - uniformly (Q, u, P) - continuous.*

Lemma 3.9 imply the following result.

Lemma 3.14. *For any $V \subseteq L$, any V - uniformly (Q, u, P) - continuous operator is (Q, u, P) - continuous in V.*

Let us study relations between relative continuity and relative boundedness.

Theorem 3.1. *A linear operator (mapping) $A : L \to M$ is (Q, u, P) - continuous if and only if it is (Q, u, P) - bounded.*

Proof. Sufficiency. Let us consider a (Q, u, P) - bounded linear operator (mapping) $A : L \to M$ and suppose that A is not (Q, u, P) - continuous. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not (q, p) - continuous. By Corollary 3.9, A is not (q, p) - continuous at $\mathbf{0}$. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from L for which $q(x_n) < 1/n$ while $p(A(x_n)) > k$.

Let us consider the set $Z = \{z_n; n = 1, 2, 3, \ldots\}$ where $z_n = n \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q(n \cdot x_n) = n \cdot q(x_n) < 1,$$

i.e., Z is a q - bounded set. At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A(n \cdot x_n)) = n \cdot p(A(x_n)) > kn.$$

Thus, the image of Z is not a p - bounded set and A is not a (Q, u, P) - bounded operator. This contradicts our assumption and by *reductio ad absurdum*, A is (Q, u, P) - continuous.

Necessity. Let us consider a (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ and suppose that A is not (Q, u, P) - bounded. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not (q, p) - bounded. By Corollary 3.3, A is not (q, p) - bounded at $\mathbf{0}$. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from L for which $q(x_n) < k$ while $p(A(x_n)) > n$.

Let us consider the set $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = (1/n) \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q((1/n) \cdot x_n) = (1/n) \cdot q(x_n) < k/n.$$

It means that the sequence $\{z_n;\ n = 1, 2, 3, \ldots\}$ q - converges to $\mathbf{0}$.

At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A((1/n) \cdot x_n)) = (1/n) \cdot p(A(x_n)) > k.$$

It means that the sequence $\{A(z_n);\ n = 1, 2, 3, \ldots\}$ does not p - converge to $\mathbf{0}$. This violates conditions from Definition 3.5 and shows A is not a (Q, u, P) - continuous operator. Thus, we have a contradiction with our assumption that A is a (Q, u, P) - continuous operator. By *reductio ad absurdum*, A is (Q, u, P) - bounded.

Theorem is proved. □

Corollary 3.22. *A linear operator (mapping) $A : L \to M$ is (q, p) - continuous if and only if it is (q, p) - bounded.*

Corollary 3.22 implies the following result.

Corollary 3.23. *A linear operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - continuous if and only if it is L - uniformly (Q, u, P) - bounded.*

As topology of topological vector spaces is determined by system of seminorms (Rudin, 1991), Theorem 3.1 gives us the following classical result ((Dunford & Schwartz, 1958); (Rudin, 1991)).

Corollary 3.24. *A linear mapping A of a topological vector space L into a topological vector space M is continuous if and only if it is bounded.*

As for linear operators (mappings) continuity at a point coincides with continuity and boundedness at a point coincides with boundedness, we have the following results.

Corollary 3.25. *A linear operator (mapping) $A : L \to M$ is (q, p) - continuous at a point a if and only if it is (q, p) - bounded at a.*

Corollary 3.26. *A linear operator (mapping) $A : L \to M$ is (Q, u, P) - continuous at a point a if and only if it is (Q, u, P) - bounded at a.*

Let us take a vector subspace V of L and consider uniform (Q, u, P) - continuity in V.

Theorem 3.2. *A linear operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - continuous in V if and only if it is uniformly (Q, u, P) - bounded in V.*

Proof. Sufficiency. Let us consider a vector subspace V of L and a uniformly (Q, u, P) - bounded in V linear operator (mapping) $A : L \to M$ and suppose that A is not uniformly (Q, u, P) - continuous in V. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not uniformly (q, p) - continuous. Consequently, there is a positive real number k such that for any natural number n, there are elements x_n and y_n from V for which $q(x_n - y_n) < 1/n$ while $p(A(x_n) - A(y_n)) > k$.

Let us consider two sets $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ and $U = \{u_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = n \cdot x_n$ and $u_n = n \cdot y_n$ for all $n = 1, 2, 3, \ldots$ As V is a vector subspace of L, then Z and U are subsets of V. Besides,

$$q(z_n - u_n) = q(n \cdot x_n - n \cdot y_n) = q(n \cdot (x_n - y_n)) = n \cdot q(x_n - y_n) < 1.$$

It means that the set $\{z_n - u_n;\ n = 1, 2, 3, \ldots\}$ is q - bounded.

At the same time, as A is a linear operator, we have

$$p(A(z_n - u_n)) = p(A(n \cdot x_n - n \cdot y_n)) = n \cdot p(A(x_n) - A(y_n)) > kn.$$

It means that the set $\{A(z_n - u_n);\ n = 1, 2, 3, \ldots\}$ is not p - bounded. Thus, A is not a uniformly (Q, u, P) - bounded in V operator. This contradicts our assumption and by *reductio ad absurdum*, A is uniformly (Q, u, P) - continuous in V.

Necessity. Let us consider a uniformly (Q, u, P) - continuous in V linear operator (mapping) $A : L \to M$ and suppose that A is not uniformly (Q, u, P) - bounded in V. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not uniformly (q, p) - bounded in V. By Corollary 3.3, A is not (q, p) - bounded in V at $\mathbf{0}$ as V is a vector subspace of L. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from V for which $q(x_n) < k$ while $p(A(x_n)) > n$.

Let us consider the set $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = (1/n) \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q((1/n) \cdot x_n) = (1/n) \cdot q(x_n) < k/n.$$

It means that the sequence $\{z_n;\ n = 1, 2, 3, \ldots\}$ q - converges to $\mathbf{0}$.

At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A((1/n) \cdot x_n)) = (1/n) \cdot p(A(x_n)) > k.$$

It means that the sequence $\{A(z_n);\ n = 1, 2, 3, \ldots\}$ does not p - converge to $\mathbf{0}$. This violates conditions from Definition 3.5 and shows A is not a uniformly (Q, u, P) - continuous in V operator. Thus, we have a contradiction with our assumption that A is a uniformly (Q, u, P) - continuous in V operator. By *reductio ad absurdum*, A is uniformly (Q, u, P) - bounded in V.

Theorem is proved. □

Corollary 3.27. *For any vector subspace V of L, a linear operator (mapping) $A : L \to M$ is uniformly (q, p) - continuous in V if and only if it is uniformly (q, p) - bounded in V.*

As before, V is a vector subspace of L and we study V - uniform (Q, u, P) - continuity.

Theorem 3.3. *A linear operator (mapping) $A : L \to M$ is V - uniformly (Q, u, P) - continuous if and only if it is V - uniformly (Q, u, P) - bounded.*

Proof. Sufficiency. Let us consider a vector subspace V of L and a V - uniformly (Q, u, P) - bounded linear operator (mapping) $A : L \to M$ and suppose that A is not V - uniformly (Q, u, P) - continuous. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not V - uniformly (q, p) - continuous. Consequently, there is a positive real number k such that for any natural number n, there are elements x_n from V and y_n from L for which $q(x_n - y_n) < 1/n$ while $p(A(x_n) - A(y_n)) > k$.

Let us consider two sets $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ and $U = \{u_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = n \cdot x_n$ and $u_n = n \cdot y_n$ for all $n = 1, 2, 3, \ldots$ As V is a vector subspace of L, then Z is a subset of V. Besides,

$$q(z_n - u_n) = q(n \cdot x_n - n \cdot y_n) = q(n \cdot (x_n - y_n)) = n \cdot q(x_n - y_n) < 1.$$

It means that the set $\{z_n - u_n;\ n = 1, 2, 3, \ldots\}$ is q - bounded.

At the same time, as A is a linear operator, we have

$$p(A(z_n - u_n)) = p(A(n \cdot x_n - n \cdot y_n)) = n \cdot p(A(x_n) - A(y_n)) > kn.$$

It means that the set $\{A(z_n - u_n);\ n = 1, 2, 3, \ldots\}$ is not p - bounded. Thus, A is not a V - uniformly (Q, u, P) - bounded operator. This contradicts our assumption and by *reductio ad absurdum*, A is V - uniformly (Q, u, P) - continuous.

Necessity. Let us consider a V - uniformly (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ and suppose that A is not V - uniformly (Q, u, P) - bounded. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not V - uniformly (q, p) - bounded. By Corollary 3.3, A is not (q, p) - bounded at $\mathbf{0}$. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from L for which $q(x_n) < k$ while $p(A(x_n)) > n$.

Let us consider the set $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = (1/n) \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q((1/n) \cdot x_n) = (1/n) \cdot q(x_n) < k/n.$$

It means that the sequence $\{z_n; n = 1, 2, 3, \ldots\}$ q - converges to $\mathbf{0}$.

At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A((1/n) \cdot x_n)) = (1/n) \cdot p(A(x_n)) > k.$$

It means that the sequence $\{A(z_n); n = 1, 2, 3, \ldots\}$ does not p - converge to $\mathbf{0}$. This violates conditions from Definition 3.6 and shows A is not a V - uniformly (Q, u, P) - continuous operator. Thus, we have a contradiction with our assumption that A is a V - uniformly (Q, u, P) - continuous operator. By *reductio ad absurdum*, A is V - uniformly (Q, u, P) - bounded.

Theorem is proved. □

Corollary 3.28. *For any subset V of L, a linear operator (mapping) $A : L \to M$ is V - uniformly (q, p) - continuous if and only if it is V - uniformly (q, p) - bounded.*

Let us take a subset V of the space L.

Definition 3.7. a) An operator (mapping) $A : L \to M$ is called uniformly (Q, u, P) - *continuous at a point a* from L if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, for any element b from L, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

b) An operator (mapping) $A : L \to M$ is called u - *uniformly* (Q, u, P) - *continuous* if it is uniformly (Q, u, P) - continuous at all points of L.

c) An operator (mapping) $A : L \to M$ is called u - *uniformly* (Q, u, P) - *continuous in V* if for any positive real number k, there is a positive real number h such that for any elements a and b from V and any hyperseminorms q and p with $(q, p) \in u$, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

d) An operator (mapping) $A : L \to M$ is called uV - *uniformly* (Q, u, P) - *continuous* if for any positive real number k, there is a positive real number h such that for any elements a from V and b from L, and any hyperseminorms q and p with $(q, p) \in u$, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Note that to be uL - *uniformly* (Q, u, P) - *continuous* or to be u - *uniformly* (Q, u, P) - *continuous in L* means the same for all operators.

It it possible to ask a question how u - uniform (Q, u, P) - continuity is connected to (Q, u, P) - continuity. The following example and Lemma 3.5 clarify this situation.

Example 3.8. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions. It is possible (Burgin, 2012) for all real numbers x, to define seminorms $q_{ptx} = p_{ptx}$ by the following formula

$$q_{ptx}(f) = p_{ptx}(f) = |f(x)|.$$

We define $A(f) = xf(x)$ for all real functions f and $u = \{(q_{ptx}, p_{ptx}); x \in \mathbb{R}\}$. Taking the function $f(x) = x$ as the point a from L, we see that $A(f) = x^2$. Thus, taking some positive real number k, e.g., $k = 1$, the corresponding h from Definition 3.2 always exists but it decreases with the growth of x. For instance, when $k = 1$, we have

$$q_{pt1}(f - g) < 1 \text{ implies } p_{pt1}(A(f) - A(g)) = p_{pt1}(xf - xg) < 1.$$

At the same time, $q_{pt10}(f - g) < 1$ does not imply $p_{pt10}(A(f) - A(g)) < 1$. It only implies $p_{pt10}(A(f) - A(g)) = p_{pt10}(xf - xg) < 10$. To have $p_{pt10}(A(f) - A(g)) < 1$, we need $q_{pt10}(f - g) < 0.1$.

It means that for any pair (q_{ptx}, p_{ptx}) of seminorms and a number k, we need to find a specific number h to satisfy Definition 3.7.a. Consequently, the operator A is (Q, u, P) - continuous at $f = x$ but it is not uniformly (Q, u, P) - continuous at f.

The same example shows that there are (Q, u, P) - continuous operators that are not u - uniformly (Q, u, P) - continuous.

Definitions imply the following result.

Lemma 3.15. *a) Any uniformly (Q, u, P) - continuous at a point a operator A is (Q, u, P) - continuous at the point a.*

b) Any u - uniformly (Q, u, P) - continuous operator A is (Q, u, P) - continuous.

Lemma 3.16. *Any u - uniformly (Q, u, P) - continuous in L operator is u - uniformly (Q, u, P) - continuous.*

For linear operators, the inverse of Lemma 3.15 is also true.

Proposition 3.13. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is u - uniformly (Q, u, P) - continuous.

(2) A is u - uniformly (Q, u, P) - continuous in L.

(3) For some point a, A is uniformly (Q, u, P) - continuous at the point a.

(4) A is uL - uniformly (Q, u, P) - continuous.

Corollary 3.29. *A linear operator (mapping) A is u - uniformly (Q, u, P) - continuous in L if and only if it is (Q, u, P) - continuous at $\mathbf{0}$.*

Corollary 3.20 implies the following result.

Corollary 3.30. *Any u - uniformly (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ is u - uniformly (Q, u, P) - continuous in L.*

These results show that for linear operators, the concepts of uniformly (Q, u, P) - continuous at a point operators and u - uniformly (Q, u, P) - continuous operators coincide.

At the same time, taking $L = M = \mathbb{R}_\omega$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value of real hypernumbers and $u = \{(q, p)\}$, we see that Example 3.6 demonstrates that there are operators that are (Q, u, P) - continuous at one point and not (Q, u, P) - continuous at another point. A similar situation is also presented in Example 3.7.

Definitions and Lemma 3.9 imply the following result.

Lemma 3.17. *If $W \subseteq V \subseteq L$, then any u - uniformly (Q, u, P) - continuous in V operator is u - uniformly (Q, u, P) - continuous in W.*

For finite relations u, different concepts of uniform continuity coincide.

Proposition 3.14. *If the relation u is finite, then, an operator (mapping) $A : L \to M$ is u - uniformly (Q, u, P) - continuous (u - uniformly (Q, u, P) - continuous at a point a) if and only if it is (Q, u, P) - continuous ((Q, u, P) - continuous at a point a).*

Proof. As any u - uniformly (Q, u, P) - continuous (u - uniformly (Q, u, P) - continuous at a point a) operator is (Q, u, P) - continuous ((Q, u, P) - continuous at the same point), we need only to show that when the relation u is finite, a (Q, u, P) - continuous (at a point a) operator $A : L \to M$ is uniformly (Q, u, P) - continuous (at the point a). At first, we consider local boundedness.

Indeed, by Definition 3.6, for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - continuous at the point a, that is, by Definition 3.4, the following condition is true:

Condition 2. For any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

This number h can be different for different pairs (q, p), but because u is finite, there is only a finite number of these pairs. So, we can take

$$l = min\{h : h \text{ satisfies Condition 2 for a pair } (q, p) \in u\},$$

and this number l will satisfy the condition from Definition 3.7 Thus, the operator A is u - uniformly (Q, u, P) - continuous at the point a.

The global case is proved in a similar way.

Proposition is proved. □

Corollary 3.31. *If systems of hyperseminorms Q and P are finite, then an operator (mapping) A is uniformly (Q, u, P) - continuous if and only if it is (Q, u, P) - continuous.*

There are connections between uniform with respect to systems of hyperseminorms continuity and uniform boundedness that are similar to the connections between nonuniform with respect to systems of hyperseminorms continuity and nonuniform boundedness described in Theorems 3.1 - 3.3. Namely, we have the following results.

Theorem 3.4. *A linear operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - continuous at a point a if and only if it is uniformly (Q, u, P) - bounded at a.*

Proof is similar to the proof of Theorem 3.1.

Let us take a vector subspace V of the space L.

Theorem 3.5. *A linear operator (mapping) $A : L \to M$ is u - uniformly (Q, u, P) - continuous in V if and only if it is u - uniformly (Q, u, P) - bounded in V.*

Proof is similar to the proof of Theorem 3.2.

Theorem 3.6. *A linear operator (mapping) $A : L \to M$ is uV - uniformly (Q, u, P) - continuous if and only if it is uV - uniformly (Q, u, P) - bounded.*

Proof is similar to the proof of Theorem 3.3.

4. Conclusion

Semitopological vector spaces are introduced and studied. Semitopological vector spaces are more general than conventional topological vector spaces, which have been very useful for solving many problems in functional analysis. Thus, we come to the following problems.

Problem 1. Study topology in semitopological vector spaces.

Problem 2. Study applications of semitopological vector spaces.

In addition, hypernorms and hyperseminorms are introduced and studied. In this paper, it is demonstrated that hyperseminormed and hypernormed spaces are semitopological vector spaces.

These results bring us to the following problems.

Problem 3. Study what kinds of topology it is possible to define with systems of seminorms, hypernorms or hyperseminorms.

It is proved (cf. (Rudin, 1991)) that systems of seminorms characterize locally convex spaces and thus, there are topological vector spaces topology in which is not defined by systems of seminorms. It is possible to ask if the same is true for semitopological vector spaces. Namely, we have the following problem.

Problem 4. Is the topology in a semitopological vector space always defined by a system of seminorms?

In this paper, hypermetrics and hyperpseudometrics are also introduced and it is demonstrated that hyperseminorms induce hyperpseudometrics, while hypernorms induce hypermetrics. Sufficient and necessary conditions for a hyperpseudometric (hypermetric) to be induced by a hyperseminorm (hypernorm) are found. Hyperpseudometrics and hypermetrics define definite topologies in vector spaces.

Problem 5. Study what kinds of topology it is possible to define with hyperpseudometrics and hypermetrics.

In this paper, boundedness and continuity are defined relative to systems of hyperseminorms or hypernorms. Inclusion of hyperseminorm sets is reflected in the strength of corresponding topologies, namely, the larger is the set Q of hyperseminorms (hypernorms), the weaker topology it defines. In such a way, we obtain a definite scalability of spaces ((Burgin, 2004); (Burgin, 2006)) with systems of hyperseminorms (hypernorms), coming to the following problem.

Problem 6. Study scalability of topological spaces defined by systems of hyperseminorms and hypernorms.

Topological vector spaces provide an efficient context for the development of integration (Choquet, 1969); (Edwards & Wayment, 1970); (Shuchat, 1972); (Kurzweil, 2000).

Problem 7. Study integration in semitopological (polyhyperseminormed) vector spaces.

At the same time, integration and hyperintegration in bundles with a hyperspace base are defined and studied in (Burgin, 2010) where the hyperspace is built by means of seminorms. The goal of this paper is to provide a base for developing the theory of extrafunction spaces in an abstract setting of algebraic systems and topological spaces, where integration plays an important role (Burgin, 2012). So, we naturally come to the following problem.

Problem 8. Study integration and hyperintegration in bundles with a hyperspace base where the hyperspace is built by means of hyperseminorms.

It is possible to define norms and seminorms with values not only in number or hypernumber spaces but in more general spaces, e.g., operator spaces.

Problem 9. Study vector spaces that have norms or/and seminorms with values in general spaces.

Problem 10. Study continuity of non-linear operators in (mappings of) polyhyperseminormed (semitopological) vector spaces.

Here we have proved (Theorem 2.3) that any hyperseminormed vector space is a semitopological vector space. It would be interesting to find if a more general statement is also true.

Problem 11. Is any polyhyperseminormed vector space a semitopological vector space?

Thus, the theory of semitopological vector spaces opens many new opportunities for research in mathematics.

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