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Visual Motif Patterns in Separation Spaces

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Abstract

This article introduces descriptive separation spaces useful in the discovery of what are known as motif patterns. The proposed approach presents the separation axioms in terms of descriptive proximities. Asymmetries arise naturally in the form of the separation of neighbourhoods of descriptively distinct points in what are known as Leader uniform topological spaces. A practical application of the proposed approach is given in terms of visual motif patterns, identification of nearness structures and pattern stability analysis in digital images.

Keywords: Descriptive proximity, near sets, visual motif patterns, separation spaces.
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1. Introduction

This article introduces separation spaces, useful in the study of set patterns. Various forms of separation in topological spaces are defined by what are known as separation axioms. The main purpose of a separation axiom is to make the points and sets in a space topologically distinguishable (Thron, 1966, §14.1). The earliest of such spaces comes from F. Hausdorff, where distinct points belong to disjoint neighbourhoods (Hausdorff, 1957*a*, §40.II). In this article, traditional separation spaces are extended to description-based separation spaces. The practical benefit of considering descriptive separation spaces is the generation of multiple patterns that are descriptively distinguishable. In a Hausdorff space, for example, a pair of descriptively distinct points become generators of distinguishable set patterns.

A form of set pattern (Grenander, 1993, §17.5) of particular interest in an approach to pattern recognition is given in terms of what are known as descriptive motif patterns. A *descriptive motif pattern* is a collection of sets such that each member of the collection is descriptively close to a motif. A *motif* is a set with members that are near one or more members of other sets. Motifs

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are a particular form set pattern generators. Visual motif patterns are found in pictures, geometric structures, and digital images. A *visual motif pattern* is a particular form of descriptive motif pattern that is a collection of sets such that each member of the collection is visually close to a set that is a motif. Visual motif patterns have a number of important applications (Naimpally & Peters, 2013; Peters, 2013*a*).

The study of visual patterns includes a consideration of S. Leader's uniform topology[1] in a metric space (Leader, 1959) and its extension to descriptive uniform topologies that provide a basis for new forms of asymmetric spaces. A *descriptive uniform topology* is determined by finding the collection of all sets that are descriptively near a given set.

Set descriptions result from the introduction of feature vectors that describe members of sets such as sets of pixels in digital images. These considerations lead in a straightforward way to a form of topology of digital images with considerable practical importance in solving image analysis and image classification problems. Since we are interested in patterns in separation spaces, we introduce stability criteria for the generation of multiple set patterns. A visual pattern is considered *stable*, provided the members of the pattern do not wander away from the pattern generator, neither spatially nor descriptively.

2. Preliminaries

Let X be a nonempty set of points, $\mathcal{P}(X)$ the powerset of X, $\mathcal{P}^2(X)$ the set of all collections of subsets of X. A single point $x \in X$ is denoted by a lowercase letter, a subset $A \in \mathcal{P}(X)$ by an uppercase letter, collection of subsets in $\mathcal{P}^2(X)$ by a round letter such as $\mathcal{B} \in \mathcal{P}^2(X)$. The *closure* of a subset $A \in \mathcal{P}(X)$ (denoted by cl$A$) is defined by

$$\mathrm{cl}A = \{x \in X : x\ \delta\ A\},$$

i.e., cl$A$ is the set of all points x in X that are near A. Let δ on a nonempty set X denote a spatial nearness (proximity) relation. For $A, B \in \mathcal{P}(X)$, $A\ \delta\ B$ (reads A is spatially near B), provided $A \cap B \neq \emptyset$, *i.e.*, the intersection of A and B is not empty (cl$A$ and cl$B$ have at least one point in common). The spatial proximity (nearness) relation δ is defined by

$$\delta = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset\}.$$

$A\ \underline{\delta}\ B$ (reads A far (remote) from B), provided cl$A$ and cl$B$ have no points in common such that $\underline{\delta} = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta$. Sets that are far from each other relative to the locations of the points in the sets (the points in one set are not among the points of the other set) are called *spatially remote* sets. The complement of a set $C \in \mathcal{P}(X)$ is denoted by C^c.

In the study of patterns, a descriptive form of EF-proximity is useful (Peters & Naimpally, 2012). Let X be a nonempty set endowed with a descriptive proximity relation δ_Φ, $x \in X$, $A, B \in \mathcal{P}(X)$, and let $\Phi = \{\phi_1, \ldots, \phi_i, \ldots, \phi_n\}$, a set of probe functions $\phi_i : X \to \mathbb{R}$ that represent features

[1] Metric space uniformity is logically equivalent to EF-proximity and the axioms given by Efremovič (Efremovič, 1951) (see Theorem 1.15, one of the most beautiful results in set-theoretic topology (Naimpally & Peters, 2013, §1.11, p. 27)). Many thanks to Som Naimpally for pointing this out.

of each x, where $\phi_i(x)$ equals a feature value of x. Let $\Phi(x)$ denote a feature vector for the object x, *i.e.*, a vector of feature values that describe x, where

$$\Phi(x) = (\phi_1(x), \ldots, \phi_i(x), \ldots, \phi_n(x)) \,.$$

A feature vector provides a description of an object. Let $A, B \in \mathcal{P}(X)$. Let $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. For example,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\} \,.$$

The expression $A \ \delta_{\Phi} \ B$ reads A *is descriptively near* B. The descriptive proximity of A and B is defined by

$$A \ \delta_{\Phi} \ B \Leftrightarrow \mathcal{Q}(\mathrm{cl}A) \cap \mathcal{Q}(\mathrm{cl}B) \neq \emptyset.$$

Descriptive remoteness of A and B (denoted by $A \ \underline{\delta}_{\Phi} \ B$) is defined by

$$A \ \underline{\delta}_{\Phi} \ B \Leftrightarrow \mathcal{Q}(\mathrm{cl}A) \cap \mathcal{Q}(\mathrm{cl}B) = \emptyset.$$

Early informal work on the descriptive intersection of disjoint sets based on the shapes and colours of objects in the disjoint sets is given by N. Rocchi (Rocchi, 1969, p.159). The *descriptive intersection* $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(\mathrm{cl}A) \text{ and } \Phi(x) \in \mathcal{Q}(\mathrm{cl}B)\} \,.$$

The descriptive intersection will be nonempty, provided there is at least one element of cl$A$ with a description that matches the description of a least one element of cl$B$. That is, a nonempty descriptive intersection of sets A and B is a set containing $a \in \mathrm{cl}A$ and $b \in \mathrm{cl}B$ such that $\Phi(a) = \Phi(b)$. Observe that A and B can be disjoint and yet $A \underset{\Phi}{\cap} B$ can be nonempty. In finding subsets $A, B \in \mathcal{P}(X)$ that are descriptively near, one considers descriptive intersection of the closure of A and the closure of B. That is, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B$ implies $A \ \delta_{\Phi} \ B$. The descriptive proximity (nearness) relation δ_{Φ} is defined by

$$\delta_{\Phi} = \left\{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B \neq \emptyset\right\}.$$

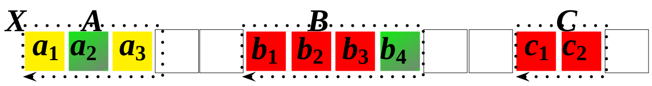

Figure 1. $\Phi = \{\text{colour probe fns}\}$, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B = \{a_2, b_4\}$, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}C = \emptyset$.

Example 2.1. Descriptive intersection of disjoint sets
The coloured and white squares in Figure 1 represent cells in a weave. A *cell* in a fabric is that part of a weave strand that overlaps another weave strand. The parallel strands of each layer in a weave are perpendicular to those strands in the other layer, making the cells square (Thomas, 2009). Choose Φ to be a set of probe functions representing weave cell colours. Let the set of cells X in Figure 1 be endowed with δ_{Φ}. Notice that sets $A, B \in \mathcal{P}(X)$ are disjoint but the descriptive intersection is nonempty. That is, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B = \{a_2, b_4\}$. Similarly, for $B, C \in \mathcal{P}(X)$, $\mathrm{cl}B \underset{\Phi}{\cap} \mathrm{cl}C = \{b_1, b_2, b_3, c_1, c_2\}$. ■

The descriptive remoteness of A and B (denoted by $A \underline{\delta}_\Phi B$) such that $\underline{\delta}_\Phi = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta_\Phi$ is defined by

$$A \underline{\delta}_\Phi B \Leftrightarrow \mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B = \emptyset.$$

Example 2.2. Descriptively remote disjoint sets
Choose Φ to be a set of probe functions representing weave cell colours. In Figure 1, sets $A, C \in \mathcal{P}(X)$ are disjoint. In addition, there are no cells in A with descriptions that resemble cells in C. Hence, the descriptive intersection is empty. That is, $A \underline{\delta}_\Phi C$ (A and C are remote), since $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}C = \emptyset$. ■

2.1. *Descriptive EF-proximity*

A binary relation δ_Φ is a *descriptive EF-proximity*, provided the following axioms are satisfied for $A, B, C \in \mathcal{P}^2(X)$.

($\boldsymbol{EF_\Phi}$.1) $A \delta_\Phi B$ implies $A \neq \emptyset, B \neq \emptyset$.
($\boldsymbol{EF_\Phi}$.2) $A \underset{\Phi}{\cap} B \neq \emptyset$ implies $A \delta_\Phi B$.
($\boldsymbol{EF_\Phi}$.3) $A \delta_\Phi B$ implies $B \delta_\Phi A$ (descriptive symmetry).
($\boldsymbol{EF_\Phi}$.4) $A \delta_\Phi (B \cup C)$, if and only if, $A \delta_\Phi B$ or $A \delta_\Phi C$.
($\boldsymbol{EF_\Phi}$.5) Descriptive Efremovič axiom:

$$A \underline{\delta}_\Phi B \text{ implies } A \underline{\delta}_\Phi C \text{ and } B \underline{\delta}_\Phi C^c \text{ for some } C \in \mathcal{P}(X).$$

The structure (X, δ_Φ) is a *descriptive EF-proximity space* (or, briefly, *descriptive EF space*).

Theorem 2.1. *Let $(X, \delta), (X, \delta_\Phi)$ be spatial and descriptive EF-spaces, respectively, with nonempty sets $A, B \in \mathcal{P}(X)$, $A \cap B \neq \emptyset$. Then $A \cap B \subseteq A \underset{\Phi}{\cap} B$.*

Proof. Let $A, B \in \mathcal{P}(X)$ and assume $A \cap B \neq \emptyset$. If $x \in A \cap B$, then, by definition, $\Phi(x) \in \mathcal{Q}(A)$ and $\Phi(x) \in \mathcal{Q}(B)$. By assumption $x \in A \cap B \subseteq A \cup B$. Then, $x \in A \underset{\Phi}{\cap} B$. Hence, $A \cap B \subseteq A \underset{\Phi}{\cap} B$. □

Descriptive EF-proximity is useful in describing, analysing and classifying the parts within a single digital image or the parts in either near or remote sets in separate digital images. The basic approach to the study of set patterns introduced in this article reflects recent work on descriptively near sets (see, *e.g.*, (Peters & Naimpally, 2012; Peters, 2012; Peters *et al.*, 2013)). Applications of descriptive EF-proximity are numerous (see, *e.g.*, (Naimpally & Peters, 2013; Peters, 2013*c*)).

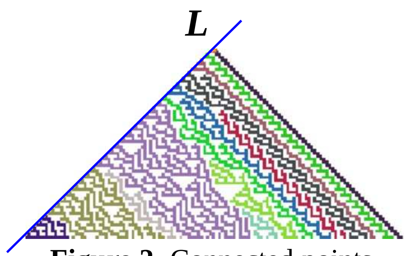

Figure 2. Connected points.

2.2. *Shape set patterns*

Shape descriptors are useful in representing, extracting and quantifying shape information from images. In general, a digital image *shape descriptor* is an expression that describes, identifies or indexes an image region. Shape descriptors are usually mathematical expressions used to extract image region shape feature values. In this section, we briefly consider picture points in terms of adjacency, connectedness, and edges.

Let $p, q \in \mathbb{Z} \times \mathbb{Z}$ be points in a grid X. Points p and q are *spatially adjacent*, provided they are joined by an edge (Klette & Rosenfeld, 2004). For example, pairs of magenta pixels in the grid in Figure 2 are spatially adjacent, since each pair of magenta pixels is joined by an edge.

Remark 2.1. Points vs. cells.
Points are the standard elements in standard topological spaces. In some discrete cases, the base elements are *cells* (indivisible collections of points) (Düntsch & Vakarelov, 2007). ■

In keeping with an interest in descriptive proximity, points p, q in a grid are *descriptively adjacent*, provided p, q have matching descriptions and there is an edge connecting p, q such that the description of the points on a connecting edge match the descriptions of p, q. For example, for the blue line $L \subset X$ along the northwest edge of the weave in Figure 2, each pair of pixels $p, q \in L$ are descriptively adjacent but pixels below L are not descriptively adjacent to any pixel in L, since L contains only blue pixels in Figure 2. Let p, q be magenta points, then the descriptive closure of L is descriptively far from p, q and the closure of any point $r \in L$ is descriptively far from either p or q, *i.e.*,

$$\Phi = \{\phi : \phi(x) = \text{ colour brightness of } x \text{ for } x \in X\},$$
$$\mathrm{cl}_\Phi L = \left\{x \in X : x \underset{\Phi}{\cap} L \neq \emptyset\right\},$$
$$\mathrm{cl}_\Phi r = \{y \in X : \Phi(y) = \Phi(r)\},$$
$$\mathrm{cl}_\Phi L \ \underline{\delta}_\Phi \ \{p, q\},$$
$$\mathrm{cl}_\Phi r \ \underline{\delta}_\Phi \ p,$$
$$\mathrm{cl}_\Phi r \ \underline{\delta}_\Phi \ q.$$

Descriptive adjacency is the heartbeat (main influence) in the study of visual motif patterns in pictures that are descriptive proximity spaces (Peters *et al.*, 2013; Peters, 2013*a*,*c*; Peters & Naimpally, 2012; Naimpally & Peters, 2013) (for the underlying near set theory, see, also, (Peters, 2013*b*; Henry, 2010)).

A 2D digital image (also called a picture) is defined on a finite, rectangular array of point samples called a *grid*. An element of a grid is a point sample or pixel. In terms of a digitized optical sensor value, a *point sample* (briefly, point) is a single number in a greyscale image or a set of 3 numbers in a colour image (Smith, 1995). In a 2D model of an image, a pixel is a point sample that exists only at a point in the plane. For a colour image, each pixel is defined by three point samples, one for each colour channel.

Let M be a set of grid points in a picture and let

$$S = p_0, p_1, \ldots, p_{i-1}, p_i, \ldots, p_n$$

be a sequence of points in M. The sequence S is called a *path*. Further, let $p = p_0, q = p_n$. Then M is *connected*, provided, for all points $p, q \in M$, point p_i is adjacent to p_{i-1} in a path between p

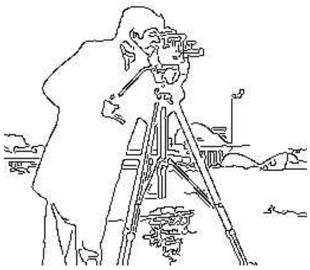

Figure 3. Sample straight edges.

and q in M. Maximally connected subsets of M are called *connected components* of M (Klette & Rosenfeld, 2004, §1.1.4).

The set of points in L in Figure 2 are connected and the remainder of the points in this weave are also connected. Let $X = L \cup W$, where W is the set of points in the threads in the weave in Figure 2. The set X is not a connected component, since there are pairs of points in separate threads with no path between the points. However, taken separately, any thread W containing pixels with the same colour is a connected component.

Let M be a grid that is connected and let points $p, q \in M$. A path between p and q defines an *edge*. A path between p and q defines a *straight edge*, provided every point in the path has the same gradient orientation. The penultimate example of a picture edge is a straight line segment such as the edges along the contour of the camera tripod legs in Figure 3. Hence, straight edges in a picture are distinguished from ridges, valleys and, in general, arcs, where the points in the paths defining non-straight edges have unequal gradient orientation.

A *shape set pattern* is a set pattern that results from the choices of shape descriptors used in comparing descriptions of picture elements. For example, the pairs of points along the diagonal in the northeast corner of Figure 2 are both spatially adjacent (each pair points along the upper northeast diagonal are joined by an edge) and descriptively adjacent (each pair points p, q along the diagonal are joined by an edge containing points that descriptively match p, q). Spatial adjacency and descriptive adjacency shape descriptors are important in separating spatially connected points from descriptively connected points in a picture and deriving spatial and descriptive set patterns in pictures.

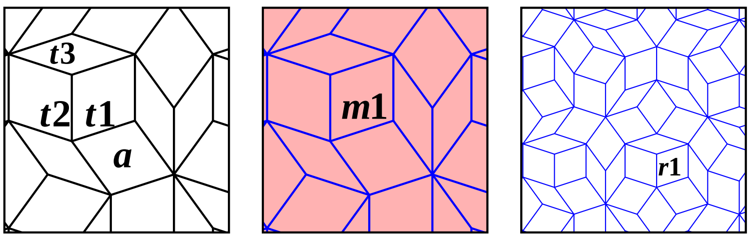

Figure 4. Spatial $\mathfrak{P}(t1) = \{t1, t2, t3, a\}$ & descriptive $\mathfrak{P}_{\Phi}(t1) = \{t1, m1, r1\}$.

Example 2.3. Descriptive penrose tiling shape pattern.
Choose Φ to be a set of probe functions representing shape features such as connected, edge gradient, and edge gradient orientation as well as colour and intensity features. Also, for example, choose tile $t1$ in Figure 4 as a shape pattern generator[2]. Tile $t1$ in the penrose tiling in Figure 4 is descriptively near tiles $m1$ (in the middle tiling) and $r1$ (in the righthand tiling) as well as a number of other unlabelled tiles that are descriptively near some part of $t1$. In generating descriptive shape patterns, we use the descriptive closure of a set A in a picture X (denoted by $\mathrm{cl}_\Phi A$), defined by

$$\mathrm{cl}_\Phi A = \left\{ x \in X : \{x\}\ \delta_\Phi\ A,\ \textit{i.e.}, \{x\} \underset{\Phi}{\cap} A \neq \emptyset \right\}.$$

In effect, $x\ \mathrm{cl}_\Phi A$ for $x \in X$ means $\Phi(x) \in \mathcal{Q}(A)$. Then

$$\mathfrak{P}_\Phi(t1) = \{t1, m1, r1, \ldots\}.$$

For example, $\mathrm{cl}_\Phi t1\ \cap\ \mathrm{cl}_\Phi m1 \neq \emptyset$, since the gradient orientation of edges along the border of $t1$ match the gradient orientation of the edges along the border of $m1$. Similarly, $\mathrm{cl}_\Phi t1\ \cap\ \mathrm{cl}_\Phi r1 \neq \emptyset$, and so on. ■

3. Descriptive uniform topology on digital images

It was S. Leader who pointed out in 1959 that it is possible to determine what he called a uniform topology in a metric space (Leader, 1959). By introducing a metric on a nonempty set of points, one obtains a metric space. Then a topology in the metric space results from observing which points are close to each given set of points. A point x in a set X is *close* to a set A, provided the distance between x and A is zero. A *digital uniform topology* in a metric space on a digital image is determined by observing which sets of pixels are close to a given set of pixels.

A useful alternative form of uniform topology (called a discrete uniform topology) arises in a proximity space by defining the nearness of sets in terms of set intersection. A discrete uniform topology in a proximity space is determined by observing which sets have nonempty intersection with a given set. In a discrete uniform topology, sets that are close to a given set are called *near sets*.

A descriptive form of either the Leader form of uniform topology or discrete uniform topology arises when the nearness of sets is based on the descriptions of members of one set matching the descriptions of members of another set. A *descriptive uniform topology* in a metric space is determined by finding which sets are descriptively close to each given set. In a descriptive uniform topology, nonempty disjoint sets can be descriptively near each other. The introduction of a uniform topology in a metric space or discrete uniform topology or descriptive uniform topology on a digital image provides a basis for the study of visual patterns in a image. In the sequel, it is assumed that each of the traditional separation spaces is defined in the context of a discrete uniform topology and that each descriptive separation space is defined in the context of a descriptive uniform topology on a nonempty set. From an application point-of-view, the focus in this article is on the introduction of uniform topologies that provide a basis for the introduction of asymmetric spaces on digital images.

[2]Regular structures known as *pattern generators* in pattern theory, are described in U. Grenander (Grenander, 1993, 1996), in building patterns from simple building blocks.

4. Antisymmetric spaces

During the 1930s, separation axioms were discovered and called *Trennungsaxiome* (*Trennung* is German for *separation*) by P. Alexandroff and H. Hopf (Alexandroff & Hopf, 1935, 58ff, §4). Hence, these axioms are named with a subscripted T as $T_n, n = 0, 1, 2, 3, 4, 5$. Often these axioms have alternate names such as Hausdorff, normal, regular, Tychonoff, and so on and there is no unanimity in the nomenclature. In this article, we consider only axioms T_0, T_1, T_2. Each of these separation axioms concern the distinctness of points.

Remark 4.1. Distinct points.
Let X be a nonempty set endowed with a proximity relation δ. Points $x, y \in X$ are *spatially distinct*, provided the closures of x and y are not near, *i.e.*, $\text{cl}\{x\} \underline{\delta} \text{cl}\{y\}$. ■

The anti-symmetric axiom T_0 (discovered by A. Kolmogorov) is defined as follows.

> $\boldsymbol{T_0}$: (a) For every pair of distinct points, at least one of them is far from the other, or
> (b) For every pair of distinct points in a topological space X, there exists an open set containing one of the points but not the other point (*cf.* (Alexandroff & Hopf, 1935, p. 58)).

The discovery of T_0 topologies in digital images hinges on what is meant by the observation that points are descriptively distinct.

Remark 4.2. Descriptively distinct points.
Let Φ be a set of probe functions that represent features of points x in a nonempty set X. Then let X be endowed with a descriptive proximity relation δ_Φ. Points x, y are *descriptively distinct*, provided x and y are spatially distinct and the feature vectors $\Phi(x)$ and $\Phi(y)$ are not equal. For example, points x, y in a digital image X are descriptively distinct (*descriptively far*), provided x and y are spatially distinct and have different descriptions, *i.e.*, $x \underline{\delta}_\Phi y$. ■

Let Φ be a set of probe functions representing features of members of a set and let $\varepsilon > 0$. There is a descriptive form of T_0 space (denoted by T_0^Φ). Let a *descriptive open neighbourhood* $N_{\Phi(x)}$ be defined by

$$N_{\Phi(x)} = \{y \in X : \Phi(x) = \Phi(y) \text{ and } |x - y| < \varepsilon\}.$$

That is, the description of each point in $N_{\Phi(x)}$ matches the description of x. Due to the spatial restriction $|x - y| < \varepsilon$, $N_{\Phi(x)}$ is also called a *bounded descriptive neighbourhood* (Peters, 2013*d*, §1.19.3).

> $\boldsymbol{T_0^\Phi}$: For every pair x, y of descriptively distinct points in a topological space X, there exists a descriptive open neighbourhood containing one of the points but not the other point.

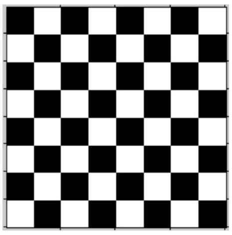

Figure 5. Sample visual space.

Example 4.1. $\boldsymbol{T_0^\Phi}$ **Visual space**.
Let X be represented by the checkerboard in Figure 5 and let x, y be black and white points in X. It is easily verified that X is a topological space. Then let $N_{\Phi(x)}$ be a descriptive open neighbourhood of x. The point y is excluded from $N_{\Phi(x)}$, since $\Phi(x) \neq \Phi(y)$. This is true for every pair of descriptively distinct points in X. Hence, X is a T_0^Φ space. ■

$\boldsymbol{T_1}$: A topological space is T_1 if, and only if, distinct points are not near.

$\boldsymbol{T_1^\Phi}$: A topological space is T_1^Φ if, and only if, descriptively distinct points are not descriptively near.

Example 4.2. Checkerboard $\boldsymbol{T_1^\Phi}$ **space**. Choose Φ to be a set of probe functions that represent greyscale and colour intensities of points in an image. Let a topological space X be represented by the checkerboard in Figure 5. X is an example of a visual T_1^Φ space. To see this, let $x, y \in X$ be points in black and white squares, respectively. The points x and y are descriptively distinct and $x \, \underline{\delta}_{\Phi} \, y$. In general, black and white pixels in X are descriptively distinct and not near, descriptively. Hence, the checkerboard is an example of a T_1^Φ space. ■

Lemma 4.1. *A digital image X endowed with a descriptive proximity δ_Φ such that X contains descriptively distinct points is a T_1^Φ space.*

Proof. Let X be a digital image (a set of points called pixels) endowed with a descriptive proximity δ_Φ. Choose Φ, a set of probe functions that represent features of points in X. Let points $x, y \in X$ be descriptively distinct. Then $x \, \underline{\delta}_{\Phi} \, y$, *i.e.*, x is descriptively not near y. Hence, X is a T_1^Φ space. □

Hausdorff observed that it is possible for a pair of distinct points to have distinct neighbourhoods and used this axiom in his work. The corresponding space with *pairs of distinct points belong to disjoint neighbourhoods* (Hausdorff, 1957*b*, §40.II) is now named after him and is called the T_2 or Hausdorff space.

T_2: A topological space is T_2, if and only if, distinct points have disjoint neighbourhoods (distinct points live in disjoint *houses*[4]).

There is a descriptive counterpart of a traditional T_2 space (denoted by T_2^{Φ}), introduced in (Peters, 2013*a*) (see, also, (Naimpally & Peters, 2013)). In a T_2^{Φ} space, one can observe that descriptively distinct points belong to disjoint descriptive neighbourhoods.

T_2^{Φ}: A topological space is T_2^{Φ} if, and only if, descriptively distinct points have disjoint descriptive neighbourhoods.

Example 4.3. A T_2^{Φ} Visual space. Choose Φ to be a set of probe functions that represent greyscale and colour intensities of points in an image. Let a topological space X again be represented by the checkerboard in Figure 5. X is an example of a visual T_2^{Φ} space. To see this, let $x, y \in X$ be points in black and white squares, respectively. Then consider a pair of descriptive neighbourhoods $N_{\Phi(x)}, N_{\Phi(y)}$ of x and y, respectively. Neighbourhood $N_{\Phi(x)}$ contains only points with descriptions that match the description of x, *i.e.*, $N_{\Phi(x)}$ contains only black points. Similarly, neighbourhood $N_{\Phi(y)}$ contains only points with descriptions that match the description of y, *i.e.*, $N_{\Phi(y)}$ contains only white points. Hence, $N_{\Phi(x)}, N_{\Phi(y)}$ are disjoint. ■

Figure 6. Manitoba dragonfly.

Observe that a T_2^{Φ} space is also a T_1^{Φ} space, since, by definition, descriptively distinct points are not near. The dragonfly in Figure 6 provides an illustration of a biology-based T_2^{Φ} space (see Example 4.4 for details). Also observe that a T_1^{Φ} space is also a T_0^{Φ} space, since, for every pair of descriptively distinct points, one can find a descriptive open set containing of the points and not containing the other point. The penultimate example of a T_1^{Φ} space that is also a T_0^{Φ} space is a

[4]A partition is a T_2 space if, and only if, every class has no more than one point, *i.e.*, every class is single tenant "house".

space where descriptively distinct points belong to open descriptive neighbourhoods. From these observations, observe that $T_2^\Phi \Rightarrow T_1^\Phi \Rightarrow T_0^\Phi$.

Let $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. A *bounded descriptive neighbourhood* $N_{\Phi(x)}$ of a point x in a set X is defined by

$$N_{\Phi(x)} = \{y \in X : d(\Phi(x), \Phi(y)) = 0 \text{ and } |x - y| < \varepsilon\},$$

where d is the taxicab distance between the descriptions of x and y, *i.e.*,

$$d(\Phi(x), \Phi(y)) = \sum_{i=1}^{n} |\phi_i(x) - \phi_i(y)| : \phi_i \in \Phi.$$

Theorem 4.1. *A digital image X endowed with a descriptive proximity δ_Φ such that X contains two or more descriptively distinct points is a T_2^Φ space.*

Proof. Let X be a digital image (a set of points called pixels) endowed with a descriptive proximity δ_Φ. Choose Φ, a set of probe functions that represent features of points in X. Let points $x, y \in X$ be descriptively distinct. Let $N_{\Phi(x)}, N_{\Phi(y)}$ be descriptive neighbourhoods of x, y, respectively. If $a \in N_{\Phi(x)}$, then $d(\Phi(a), \Phi(x)) = 0$, *i.e.*, each member of $N_{\Phi(x)}$ must descriptively match x. Similarly, each $b \in N_{\Phi(y)}$ descriptively matches y. Then, $N_{\Phi(x)} \cap N_{\Phi(y)} = \emptyset$. Hence, X is a T_2^Φ space. □

Figure 7. Dragonfly edges.

Example 4.4. Dragonfly T_2^Φ Shape space.
Choose Φ to be a set of probe functions that represent the gradient orientation of the points in an image. Let a topological space X be represented by the dragonfly in Figure 6, endowed with a descriptive proximity relation δ_Φ. X is an example of a complex visual T_2^Φ shape space. To see this, let $x, y \in X$ be points along the edges of the filtered dragonfly image in Figure 7. The points x and y are descriptively distinct, since these points have different gradient orientations. In addition, points x, y are centers of disjoint descriptive neighbourhoods $N_{\Phi(x)}, N_{\Phi(y)}$, respectively, in a T_2^Φ Shape Space.

Proof. We assume that $\Phi(x) \neq \Phi(y)$, *i.e.*, x and y have different gradient orientations in Figure 7. The descriptive neighbourhood $N_{\Phi(x)}$ of point x (with no spatial restriction) is defined by

$$N_{\Phi(x)} = \{a \in X : \Phi(x) = \Phi(a)\},$$

i.e., the gradient orientation of x matches the gradient orientation of each point a in $N_{\Phi(x)}$. Hence, $y \notin N_{\Phi(x)}$, since the gradient orientation of y does not match the gradient orientation of x. Similarly, observe that $x \notin N_{\Phi(y)}$. Then $N_{\Phi(x)}, N_{\Phi(y)}$ are disjoint. This is true of every pair of points in X that have unequal gradient orientations. Hence, X is an example of a descriptive T_2^Φ shape space. $\square$

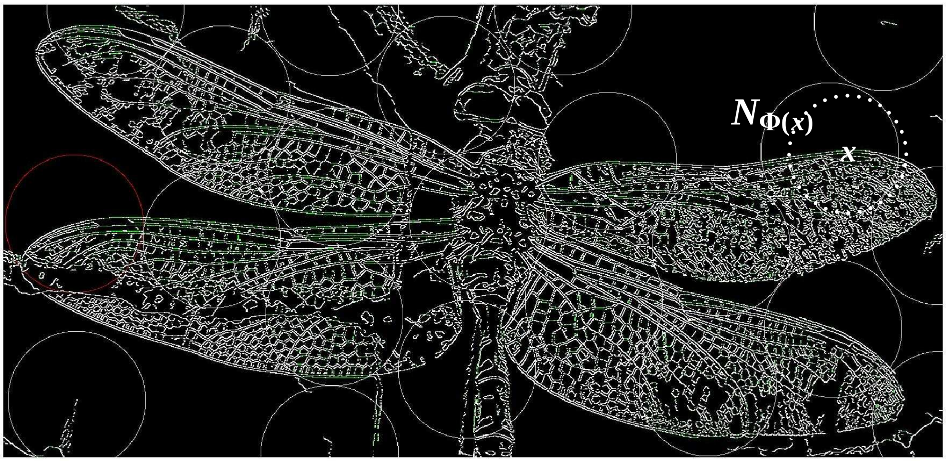

Figure 8. $\boldsymbol{T_2^\Phi}$ Shape space.

Remark 4.3. $\boldsymbol{N_{\Phi(x)}}$, $\boldsymbol{T_2^\Phi}$ **Implementation details**.
A Matlab® 7.10.0 (R2010a) script written by C. Uchime has been used on the dragonfly image in Figure 6 to extract the edges shown in Figure 7. From Example 4.4, we know that the dragonfly in Figure 7 provides a basis for a T_2^Φ shape space. Next, bounded descriptive neighbourhoods $N_{\Phi(x)}, N_{\Phi(y)}$ of points x, y, respectively, are found by selecting x, y, radius ε, and pixel gradient orientation as the shape descriptor. For simplicity, only $N_{\Phi(x)}$ is shown in Figure 8.

Using C. Uchime's Matlab script, the selection of x, y is done manually by clicking on two points of interest on the dragonfly wings (see Figure 8). Starting with $N_{\Phi(x)}$, for example, the construction of the shape pattern $\mathfrak{P}_\Phi(N_{\Phi(x)})$ is carried out by using Matlab to search through the image for points (outside the motif neighbourhood) with gradient orientations that match the gradient orientation of x. For each pixel $x' \notin N_{\Phi(x)}$ such that $\Phi(x') = \Phi(x)$, a new neighbourhood is constructed. In practice, only a restricted number of neighbourhoods are found, namely, those neighbourhoods with centers that are reasonably close to the motif neighbourhood center x. ■

4.1. *Descriptive nearness structures*

Herrlich nearness structures are extended to descriptive nearness structures in this section. One begins the study of such structures by choosing Φ, a set of probe functions that represent features of members of a nonempty set X. Let X be endowed with a descriptive proximity relation Φ. By way of illustration, the honey bee in Figure 9 provides a basis for a shape nearness space relative to the bee image edges shown in Figure 10 (see Example 4.5 for details).

Figure 9. Bee

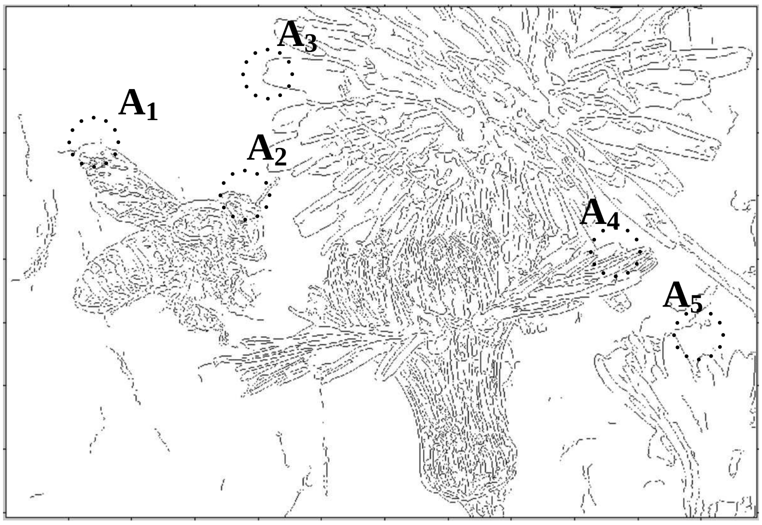

Figure 10. $\xi_{\Phi} = \{A_1, \ldots, A_5\}$

For descriptive nearness, we use the following notation.

$$
\begin{aligned}
X &= \text{ nonempty set of points},\\
\Phi &= \{\text{probe functions representing features of } x \in X\},\\
\mathcal{A}, \mathcal{B} &\text{ denote collections of subsets in } X, \ i.e., \ \mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X),\\
\mathcal{Q}(\mathcal{A}) &= \{\mathcal{Q}(A) : A \in \mathcal{A}\},\\
\eta_{\Phi}\mathcal{A}, \ \text{or } \mathcal{A} \in \eta, &\ i.e., \text{ members of } \mathcal{A} \text{ are descriptively near},\\
\underline{\eta}_{\Phi}\mathcal{A}, &\ i.e., \text{ members of } \mathcal{A} \text{ are not descriptively near},\\
A \ \eta_{\Phi} \ B &= \eta_{\Phi}\{A, B\} \ (A \text{ descriptively near } B),\\
\mathcal{A} \vee \mathcal{B} &= \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\},\\
cl_{\eta_{\Phi}} E &= \{x \in X : \{x, E\} \in \eta_{\Phi}\} \ (x \text{ descriptively near } E),\\
cl_{\eta_{\Phi}} \mathcal{A} &= \{cl_{\eta} A : \mathcal{Q}(A) \in \mathcal{Q}(\mathcal{A})\}.
\end{aligned}
$$

A descriptive nearness structure (denoted by ξ_{Φ}) is defined by

$$\xi_{\Phi} = \left\{ \mathcal{A} \in \mathcal{P}^2(X) : \bigcap_{\Phi} \{A : A \in \mathcal{A}\} \neq \emptyset \right\}.$$

In the following axioms, let $\mathcal{A} \in \xi_{\Phi}$. It can be shown that the descriptive nearness structure ξ_{Φ} satisfies (dN.1)-(dN.5):

(dN.1) $\bigcap_{\Phi} \{A : A \in \mathcal{A}\} \neq \emptyset \Rightarrow \eta_{\Phi}\mathcal{A}$ is not empty,

(dN.2) $\underline{\eta}_{\Phi}\mathcal{A}$ and $\underline{\eta}_{\Phi}\mathcal{B} \Rightarrow \underline{\eta}(\mathcal{A} \vee \mathcal{B})$,

(dN.3) $\eta_{\Phi}\mathcal{A}$ and, for each $B \in \mathcal{B}$, there is an $A \in \mathcal{A} : A \subset B \Rightarrow \eta_{\Phi}\mathcal{B}$,

(dN.4) $\emptyset \in \mathcal{A} \Rightarrow \underline{\eta}_{\Phi}\,\mathcal{A}$,
(dN.5) $\eta_{\Phi}(cl_{\eta}\mathcal{A}) \Rightarrow \eta_{\Phi}\mathcal{A}$ (descriptive Herrlich axiom).

Example 4.5. Descriptive Herrlich nearness.
Let the set X be represented by the set of edge pixels in Figure 10 and let Φ contain a single probe function representing pixel orientation. Each member of the collection of subsets $\mathcal{A}$ contains ridge pixels, where

$$\xi_{\Phi} = \mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\},$$

since each pair of sets in $\mathcal{A}$ contain pixels with matching orientation. Observe that there are other collections of subsets $\mathcal{B}$ in Figure 10 containing pixels with matching orientations that are not the same as the pixels orientations in the subsets in $\mathcal{A}$. Hence, ξ_{Φ} contains more collections of descriptively near subsets that are not shown in Figure 10. ■

5. Visual patterns in descriptive separation spaces

Figure 11. T_2^{Φ}.

Visual patterns arise naturally from the different forms of descriptive separation spaces. We illustrate this in terms of patterns that naturally occur in T_1^{Φ} and T_2^{Φ} spaces. Let $\mathcal{P}^2(X)$ denote the set of collections of subsets in X and let pattern $\mathfrak{P} \in \mathcal{P}^2(X)$, motif $M \in \mathcal{P}(X)$. Let Φ be a set of probe functions that represent features of members of X and let X be endowed with a descriptive proximity δ_{Φ}. For example, the 1870 Punch dancing delivery boy image in Figure 11 provides a basis for a visual pattern (see Example 5.2 for details). Further, a visual pattern $\mathfrak{P}_{\Phi}$ is a descriptive motif pattern, provided the following axioms are satisfied.

(**motif**.1) Sets in $\mathfrak{P}_{\Phi}$ are pairwise disjoint.
(**motif**.2) A is descriptively near M ($A\ \delta_{\Phi}\ M$) for each $A \in \mathfrak{P}_{\Phi}$.
(**motif**.3) If there are pairs $A, B \in \mathfrak{P}_{\Phi}$ that are copies of M, there is an isometry[5] of the plane that maps A onto B.

A descriptive motif pattern is an example of what is known as a discrete pattern in the study of patterns in tilings and weaving (see, *e.g.*, (Grünbaum & Shepard, 1987)). Observing visual patterns in an image is aided by various forms of image filtering, sharpening the features of pixel neighbourhoods, making it more possible to detect those parts of an image that are either close or remote from each other.

[5]Let A and B be sets of pixels in digital images endowed with metrics d_X and d_Y. An *isometry* is a distance-preserving map (Beckman & Quarles, 1953). For any pair pixels $x, y \in A$ with descriptions $\Phi(x), \Phi(y)$ found in B (*i.e.*, $f(\Phi(x)), f(\Phi(y)) \in B$), a map $f : A \to B$ is an isometry, provided

$$d_Y(f(\Phi(x)), f(\Phi(y))) = d_X(\Phi(x), \Phi(y)).$$

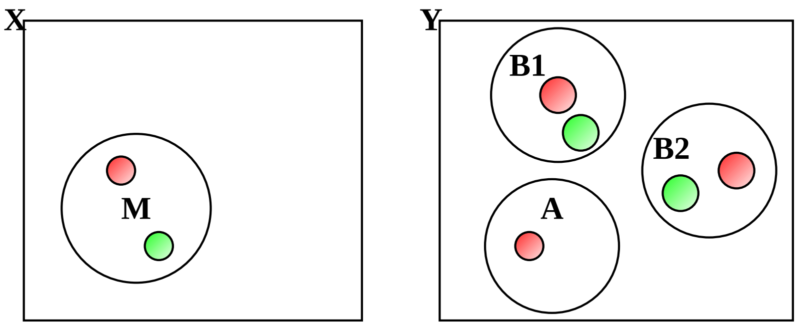

Figure 12. $\mathfrak{P}_\Phi = \{A, B1, B2\}$.

Example 5.1. Sample descriptive motif pattern.
Let sets of points X, Y endowed with a proximity relation δ be represented by Figure 12. Choose Φ to be a set of probe functions that represent greyscale and colour features of points in X, Y. The set M in Figure 12 represents a motif in a set pattern. Observe that $A, B1, B2$ are pairwise disjoint and each of the sets $A, B1, B2$ is descriptively near M. For example, M is descriptively near $B1$, since M and $B1$ contain subsets with red and green pixels. Again, for example, M is descriptively near A, since M and A contain subsets with red pixels. There is also an isometry between the descriptions of points in X and the descriptions of the points in Y. From these observation, we obtain the descriptive motif pattern $\mathfrak{P}_\Phi = \{A, B1, B2\}$. ■

5.1. Visual patterns in descriptive T1 spaces

To find visual patterns in descriptive T_1 spaces, do the following:

(1) Choose Φ, a set of probe functions representing features of points in a T_1^Φ space X.
(2) Select a pair of descriptively distinct points $x, y \in X$. By definition, $x \underline{\delta}_\Phi y$. Hence, the T_1^Φ space property is satisfied.
(3) Let M_1, M_2 denote point sets $\{x\}, \{y\}$, respectively.
(4) Determine all subsets of X containing points that descriptively match M_1 and then determine all subsets of points that descriptively match M_2.

As a result of the above steps, we can identify a pair of descriptive motif patterns $\mathfrak{P}_\Phi(M_1), \mathfrak{P}_\Phi(M_2)$ in a T_1^Φ space X. In addition, each such a motif pattern is a member of a descriptive Herrlich topology ξ_Φ defined on X.

Let X be endowed with a proximity δ_Φ such that X is a T_1^Φ space and let $M = \{x\}$ be a motif containing a single point $x \in X$, which defines a descriptive motif pattern $\mathfrak{P}_\Phi(M)$. If $A, B \in \mathfrak{P}_\Phi(M)$, then $A \underset{\Phi}{\cap} B \neq \emptyset$. From this, we obtain the following result.

Theorem 5.1. *Let (X, δ_Φ) be a T_1^Φ space with nearness structure ξ_Φ on X and let $\mathfrak{P}_\Phi(M)$ be a descriptive motif pattern determined by a motif M containing a single point x in X. Then $\mathfrak{P}_\Phi(M) \in \xi_\Phi$.*

5.2. Visual patterns in descriptive T2 spaces

To find visual patterns in descriptive T_2 spaces, do the following:

(1) Choose Φ, a set of probe functions representing features of points in a T_2^Φ space X.

(2) Select a pair of descriptively distinct points $x, y \in X$. By definition, $N_{\Phi(x)} \underline{\delta}_\Phi N_{\Phi(y)}$, since the description of each point in a descriptive neighbourhood matches the description of the point at the centre of the neighbourhood and $x \underline{\delta}_\Phi y$). That is, neighbourhoods $N_{\Phi(x)} \underline{\delta}_\Phi N_{\Phi(y)}$ are descriptively disjoint. Hence, the T_2^Φ space property is satisfied.

(3) Let M_1, M_2 denote neighbourhoods $N_{\Phi(x)} \underline{\delta}_\Phi N_{\Phi(y)}$, respectively.

(4) Determine all subsets of X that are descriptively near M_1 and then determine all subsets of X such that descriptively near M_2.

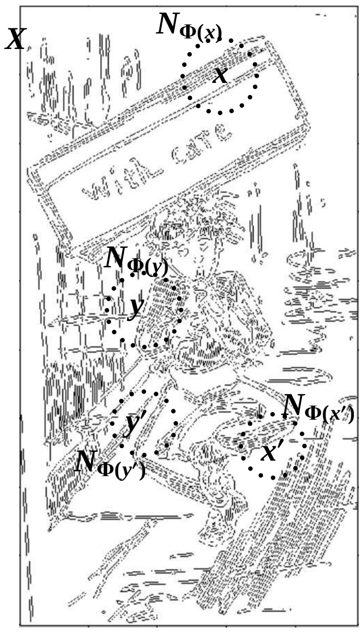

Figure 13. Sample T_2^Φ visual edge patterns.

As a result of the above steps, we can identify a pair of descriptive motif patterns $\mathfrak{P}_\Phi(M_1)$, $\mathfrak{P}_\Phi(M_2)$ in a T_2^Φ space X. In general, each such a motif pattern is not a member of the same descriptive Herrlich nearness structure ξ_Φ defined on X. To see this, consider a pair of neighbourhoods $N_{\Phi(x')}, N_{\Phi(x'')}$ that are descriptively near $N_{\Phi(x)}$. We know that $N_{\Phi(x)} \delta_\Phi N_{\Phi(x')}$ but it is possible that $N_{\Phi(x')} \underline{\delta}_\Phi N_{\Phi(x'')}$, if, for example, we compare pixel colours. $N_{\Phi(x)}$ may have a mixture of red and green colours, where $N_{\Phi(x')}$ has pixels with red colours but no green colours and $N_{\Phi(x'')}$ has pixels with green colours but no red colours. In other words, many different Herrlich nearness structures can be found in the same digital image.

Example 5.2. Edge motif pattern in T_2^Φ space.
Let X be the set of edge points in Figure 13, extracted from the 1870 Punch image in Figure 11,

using the edge function with the Canny filter[6] available in Matlab. Choose Φ to be a set of probe functions representing the orientation (gradient direction) of edge pixels in X. Observe that if pixels x, y in X have different orientations (*i.e.*, x and y are descriptively distinct), then $x \underline{\delta}_{\Phi}\, y$. Then $N_{\Phi(x)}, N_{\Phi(y)}$ are descriptively disjoint neighbourhoods. Hence, X is an example of T_2^{Φ} space.

Then let M_1, M_2 denote motif neighbourhood edge point sets $N_{\Phi(x)}, N_{\Phi(y)}$ of points x, y, respectively. An indication of the descriptive motif patterns $\mathfrak{P}_{\Phi}(M_1), \mathfrak{P}_{\Phi}(M_2)$ determined by M_1, M_2 is suggested by the edge regions containing points x', y'. In the pattern representing $\mathfrak{P}_{\Phi}(M_1)$, for example, notice that x' is the centre of bounded descriptive neighbourhood $N_{\Phi(x')}$ containing points with matching orientations. And the M_1 edge point neighbourhood is descriptively near $N_{\Phi}(x')$, since the orientation of one or more edges in $N_{\Phi(x)}$ match the orientation of one or more edges in $N_{\Phi(x')}$, *i.e.*,

$$N_{\Phi(x)}\ \delta_{\Phi}\ N_{\Phi(x')} : M_1 \doteqdot N_{\Phi(x)}.$$

Similarly, there is a descriptive neighbourhood $N_{\Phi(y')}$ in the edge pattern $\mathfrak{P}_{\Phi}(M_2)$ so that

$$N_{\Phi(y)}\ \delta_{\Phi}\ N_{\Phi(y')} : M_2 \doteqdot N_{\Phi(y)}.$$

Continuing this process, one can observe many other edge motif patterns in this particular T_2^{Φ} space. ■

Theorem 5.2. *A descriptive T_2^{Φ} space contains distinct descriptive motive patterns.*

Proof. Immediate from Lemma 4.1 and the definition of descriptive motif patterns. □

6. Stability in pattern constructions

A meaningful theory of stable pattern selection requires models of pattern-forming mechanisms that are simple enough to be understood in detail (Dee & Langer, 1983). An approach to achieving pattern selection stability in propagating patterns in either T_1 or T_2 spaces is introduced in this section. Basically in this study of descriptive patterns in a pair of digital images A, B, it is necessary to propagate a pattern in image B with some assurance that the pattern generated in B will belong to the class of images containing the image A and each new set added to a pattern does not wander or drift away from the pattern generator. That is, given a pattern generator M, each new set A added to pattern $\mathfrak{P}_{\Phi}(M)$ must be sufficiently near M, *spatially*.

P.E. Forsseén and D. Lowe observe that shape descriptors are reliable in detecting maximally stable extremal regions in digital images (Forsseén & Lowe, 2007, 1-8). In this work, descriptive motif set pattern growth is stable, provided the shape-based description of each set added to the pattern matches the shape-based description of the pattern motif. This interpretation of pattern stability is comparable to U. Grenander's notion of configuration transformation stability (Grenander, 1993, §4.1.1). To arrive at a formal definition of pattern stability, we introduce the descriptive distance between collections in terms of the Čech distance between sets.

[6]The Matlab canny filter is based on J.F. Canny's approach to edge detection introduced in his M.Sc. thesis completed in 1983 at the MIT Artificial Intelligence Laboratory (Canny, 1983). For details about this considered in the context of a topology of digital images, see (Peters, 2013*d*, §6.2).

Let $A, B \in C$ be nonempty sets in a space C and let

$$D(A, B) = \inf \{|a - b| : a \in A, b \in B\}$$

be the Čech distance between A and B. That is, a configuration transformation T on a configuration space C is stable, if, for any $\varepsilon > 0$, there exists a δ such that

$$D(A, B) \leq \delta \Rightarrow D(T(A), T(B)) \leq \varepsilon.$$

Let (X, δ_Φ) be a descriptive proximity Hausdorff space and let $A, B \in \mathcal{P}, \mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$. Next, consider a descriptive form of a Grenander configuration transformation, namely, T_Φ. That is, the transformation $T_\Phi \doteqdot \mathfrak{P}_\Phi : \mathcal{P}(X) \to \mathcal{P}^2(X)$ is defined by

$$\mathfrak{P}_\Phi(M) = \mathcal{A} : M\ \delta_\Phi\ B \text{ for } B \in \mathcal{A}, \text{and } D(M, B) \leq \varepsilon.$$

Definition 6.1. Pattern stability sufficiently near criterion.
Let $\mathfrak{P}_\Phi(M)$ be a descriptive motif pattern constructed on a nonempty set X, $\varepsilon > 0$ and let $A \in \mathfrak{P}_\Phi(M)$. The pattern $\mathfrak{P}_\Phi(M)$ is stable, provided the distance requirement $D(M, A) < \varepsilon$ is satisfied. That is, $\mathfrak{P}_\Phi(M)$ is stable, provided A is *sufficiently near* M for each A added to $\mathfrak{P}_\Phi(M)$. ■

Let $B \ll A$ denote the fact that B is a *proximal neighbourhood* of A, provided $A \subset B$. From Def. 6.1, we obtain the following result.

Lemma 6.1. *Let $M \subset X$, a T_2^Φ space and let $\mathfrak{P}_\Phi(M)$ be a descriptive motif pattern. Let $A, B \in \mathfrak{P}_\Phi(M)$. $\mathfrak{P}_\Phi(M)$ is stable, if and only if, $D(M, A) < \varepsilon$ and $B \ll A$ implies $D(M, B) < \varepsilon$.*

From Def. 6.1 and Lemma 6.1, we obtain the following result.

Theorem 6.1. Descriptive pattern stability.
Let $\mathfrak{P}_\Phi$ be a pattern configuration transformation used to construct collections of patterns on X, a T_2^Φ space endowed with a descriptive proximity δ_Φ such that Φ is a set of probe functions representing shape descriptors, let $M \in \mathcal{P}(X)$, $\varepsilon > 0$. Then the following are equivalent.
(1) $\mathfrak{P}_\Phi(M)$ is stable.
(2) $D(M, A) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M)$.
(3) $D(M, A) < \varepsilon$ and $B \ll A$ implies $D(M, B) < \varepsilon$.

Proof.
(1) ⇔ (2): $\mathfrak{P}_\Phi(M)$ is stable, if and only if, from Def. 6.1, $D(M, A) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M)$.
(1) ⇔ (3): $\mathfrak{P}_\Phi(M)$ is stable, if and only if, from Lemma 6.1, $D(M, A) < \varepsilon$ and $B \ll A$ implies $D(M, B) < \varepsilon$.
(2) ⇔ (3): $D(M, A) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M)$, if and only if, $B \in \mathfrak{P}_\Phi(M)$, provided $B \ll A$. □

Remark 6.1. Pattern stability and clustering stability.
Observe that descriptive pattern generation is a form of clustering. Recall that data clustering is a *natural grouping* of a set of patterns or points or objects (Jain, 2010). Let X be a T_2^Φ space and let $M \in \mathcal{P}(X)$. Then the use of M to generate the pattern $\mathfrak{P}_\Phi(M)$ can be considered a natural grouping

of sets in the pattern relative to the pattern generator M. That is, $A \in \mathfrak{P}_\Phi(M)$, provided $A\ \delta_\Phi\ M$. Hence, an obvious research path in the study of descriptive pattern generation is to consider the parallel between clustering stability (*e.g.*, (Ben-Hur *et al.*, 2002; Wang, 2010; Reizer, 2011)) and descriptive pattern generation stability. For example, it has been found (Ben-Hur *et al.*, 2002) that pairwise similarity between clusterings of sub-samples in a dataset provides a basis for clustering stability. A partial guarantee that descriptive pattern generation is stable, stems from the fact that $A \in \mathfrak{P}_\Phi(M)$, if and only if, A is descriptively near M. But this is only a partial guarantee of pattern generation stability, since subset similarity in a descriptive pattern does not prevent subsets from drifting or wandering away *spatially* from the pattern generator M. To achieve full descriptive pattern generation stability, we consider distance-based pattern generation in keeping with recent work on the stability of distance-based clustering (see,*e.g.*, (Wang, 2010)). In the distance-based approach to descriptive pattern stability, we introduce the *sufficiently near* criterion in Def. 6.1. ■

6.1. Multiple pattern generation stability

Since we are interested in constructing multiple patterns across disjoint regions of digital images that resemble each other in a T_2^Φ space, we introduce a stability criterion for the generation of multiple patterns. Again, the goal is to arrive at a view of stability of multiple patterns in a T_2^Φ space such that patterns do not wander or drift away from each other. Let $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ be collections containing sets $A, B \in \mathcal{P}(X)$, respectively. To complete the definition of pattern stability, we introduce the descriptive distance D_Φ, which a descriptive form of the distance between sets introduced by E. Čech (Čech, 1966, §18.A.2). The distance D_Φ is used to define the descriptive distance $\mathbb{D}_\Phi$ between collections of sets. The descriptive distance $\mathbb{D}_\Phi : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \to \mathbb{R}$ between collections $\mathcal{A}, \mathcal{B}$ is defined by

$$\mathbb{D}_\Phi(\mathcal{A}, \mathcal{B}) = \inf \{D_\Phi(A, B) : A \in \mathcal{A}, B \in \mathcal{B}\}, \text{ where,}$$
$$D_\Phi(A, B) = \inf \{d(\Phi(a), \Phi(b)) : a \in A, b \in B\}.$$

The descriptive distance $\mathbb{D}_\Phi$ can be used to measure the distance between descriptive motif set patterns, since such patterns are collections of nonempty sets that are descriptively near each other. Let $\{A\}, \{B\}$ denote collections, each containing one set. Then $\mathfrak{P}_\Phi$ is a *stable descriptive pattern*, if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\mathbb{D}_\Phi(\{A\}, \{B\}) \leq \delta \ \Rightarrow \mathbb{D}_\Phi(\mathfrak{P}_\Phi(A), \mathfrak{P}_\Phi(B)) < \varepsilon.$$

That is, whenever sets A and B are descriptively near, then the corresponding patterns $\mathfrak{P}_\Phi(A), \mathfrak{P}_\Phi(B)$ are descriptively near. This form of set pattern stability works well in comparing regions of pairs of digital images, where we need to guarantee that the transformation that produces the descriptive set patterns in separate image regions is stable.

Definition 6.2. Multiple pattern stability criterion.
Let $\mathfrak{P}_\Phi$ be a pattern configuration transformation used to construct collections of patterns on X, a T_2^Φ space endowed with a descriptive proximity δ_Φ and let $\varepsilon > 0, \delta > 0$. Let $x, y \in X$ be distinct

points and let M_1, M_2 be disjoint neighbourhoods of x, y, respectively. Further, let $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$. Patterns $\mathfrak{P}_\Phi(M_1) \in \mathcal{A}, \mathfrak{P}_\Phi(M_2) \in \mathcal{B}$ are stable, provided

$$\mathbb{D}_\Phi(\mathcal{A}, \mathcal{B}) \leq \delta \ \Rightarrow \ \mathbb{D}_\Phi\left(\mathfrak{P}_\Phi(M_1), \mathfrak{P}_\Phi(M_2)\right) < \varepsilon. \qquad \blacksquare$$

To achieve stability in comparing image regions in the same digital image regions in pairs of images, it is necessary to consider pixel features that can be reliably matched, regardless of the appearance of the surroundings of a region. In this article, the focus is on constructing motif set patterns containing neighbourhoods of points defined by connected point sets that are straight edges. Neighbourhood selection is determined by the gradient orientation of the focal point of a pattern motif neighbourhood. The construction of a pattern motif (a descriptive neighbourhood of point) reduces to finding a connected set of points along an edge such that the edge points have matching gradient orientation. Hence, a gradient orientation-based motif set pattern results from finding neighbourhoods of points containing straight edges with pixel gradient orientations that match the gradient orientation of the points in the motif neighbourhood of the pattern.

Keeping in mind the underlying descriptive uniform topology in a Hausdorff T_2^Φ space X endowed with a descriptive proximity δ_Φ, an image pixel y belongs to a neighbourhood of point x, provided the gradient orientation of y matches the gradient orientation of x. Let Φ be a set of shape descriptors that includes pixel gradient orientation. In addition, let the descriptive neighbourhood $N_{\Phi(x)}$ be a pattern motif M that is a connected set of points belonging to a straight edge, *i.e.*, $y \in N_{\Phi(x)}$, provided $\Phi(y) = \Phi(x)$. Then the pattern $\mathfrak{P}_\Phi(M)$ is a collection of straight edges defined by

$$\mathfrak{P}_\Phi(M) = \left\{ N_{\Phi(y)} \in \mathcal{P}(X) : N_{\Phi(y)} \ \delta_\Phi \ M \right\}.$$

Pattern stability is achieved by guaranteeing that only matching straight edges belong to the pattern $\mathfrak{P}_\Phi(M)$. In comparing regions across pairs of digital images, stability is achieved by comparing straight edge patterns. Let $x, y \in X, Y$ be a pixels in a pair of digital images X, Y, respectively. Further, let $\mathfrak{P}_\Phi(M_1), \mathfrak{P}_\Phi(M_2)$ be straight edge shape patterns in images X, Y, respectively, such that $M_1 = N_{\Phi(x)}, M_2 = N_{\Phi(y)}$. Pattern $\mathfrak{P}_\Phi(M_x)$ is close to pattern $\mathfrak{P}_\Phi(M_y)$, provided the straight edges represented by neighbourhoods in the patterns have matching edge-neighbourhood motifs, *i.e.*,

$$\begin{aligned}
&\mathfrak{P}_\Phi(M_1) \ \delta_\Phi \ \mathfrak{P}_\Phi(M_2), \text{ if and only if,}\\
&N_{\Phi(x)} \ \delta_\Phi \ N_{\Phi(y)}, \text{ if and only if,}\\
&\Phi(x) = \Phi(y). \qquad \blacksquare
\end{aligned}$$

From Def. 6.2 and Theorem 6.1, we obtain the following result.

Theorem 6.2. Multiple pattern generation stability.
Let $\mathfrak{P}_\Phi$ be a pattern configuration transformation used to construct collections of patterns on X, a T_2^Φ space endowed with a descriptive proximity δ_Φ such that Φ is a set of probe functions representing shape descriptors, let $M_1, M_2 \in \mathcal{P}(X)$, and let $\varepsilon > 0$. Further, let $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$. Then the following are equivalent.
(1) $\mathfrak{P}_\Phi(M_1) \in \mathcal{A}, \mathfrak{P}_\Phi(M_2) \in \mathcal{B}$ *are stable.*
(2) $D(M_1, A) < \varepsilon, D(M_2, B) < \varepsilon$ *for each* $A \in \mathfrak{P}_\Phi(M_1)$ *and for each* $B \in \mathfrak{P}_\Phi(M_2)$.

6.2. Comparison with existing clustering stability analysis

One of the most widely used clustering techniques is k-means clustering. This is a non-hierarchical clustering approach, which aims to partition n p-dimensional observations into k clusters ($k \leq n$) by minimizing a measure of dispersion within the clusters. In k-means clustering, the selection of the number of clusters affects the clustering stability significantly (Ben-Hur *et al.*, 2002). Let k be the true number of clusters in an image. If the number of clusters is greater than k, then some of the true clusters will be split into smaller clusters during clustering. On the other hand, if the number of clusters is less than k, then some of the true clusters will be merged into bigger clusters during clustering. Both cases will lead to unstable clusterings. Hence, clustering stability can be used as a quality measure of the clustering algorithm.

Ben-Hur, Elisseeff and Guyon propose distribution of pairwise similarity between clusterings of sub-samples of a dataset as a stability measure of a partition (Ben-Hur *et al.*, 2002). Another notion of stability as proposed in (Lange *et al.*, 2004) is based on the average dissimilarity of solutions computed on two different data sets. While the aforementioned approaches focus on maximizing the within-cluster similarity and within-cluster dissimilarity, Wang proposes a new measure of the quality of clusterings based on the clustering instability from sample to sample (Wang, 2010). On the other hand, Reizer proposes to measure the quality of clustering through stability from sample to sample (Reizer, 2011).

In contrast to the traditional clustering methods, the descriptive-based pattern generation method proposed in this article does not require the number of clusters to be pre-determined. The pattern $\mathfrak{P}_\Phi(M)$ may grow as long as it satisfies the condition that each new set A added to pattern $\mathfrak{P}_\Phi(M)$ is sufficiently near M, both spatially and descriptively. However, similar to clustering stability, we may say that the pattern generation is stable, provided it produces similar patterns on data originating from the same source. Based on this argument, a definition for pattern stability can be derived from the clustering stability model given in (Reizer, 2011).

Since we are interested in determining when a generated pattern in a sample digital image Y serves as an indicator that Y belongs to the class of digital images represented by a pattern generated in a query image X, we define pattern stability in terms of the expected descriptive distance between $\mathfrak{P}_\Phi(M, X)$ (pattern generated in X) and $\mathfrak{P}_\Phi(M, Y)$ (pattern generated in Y).

Definition 6.3. Pattern Stability.
Let $th > 0$ denote an expectation threshold and let $E[\cdot]$ denote the expected value of $\cdot$. Further, let $\mathfrak{P}_\Phi(M, X)$ be a pattern generated by M in X and $\mathfrak{P}_\Phi(M, Y)$, pattern generated by M in Y. The *stability of any description-based pattern* $\mathfrak{P}_\Phi(M)$ (denoted by $Stab(\mathfrak{P}_\Phi(M))$) is defined by

$$Stab(\mathfrak{P}_\Phi(M)) = \begin{cases} 1, & \text{if } E\left[\mathbb{D}_\Phi(\mathfrak{P}_\Phi(M, X), \mathfrak{P}_\Phi(M, Y))\right] \leq th, \\ 0, & \text{otherwise } \mathfrak{P}_\Phi(M) \text{ is unstable.} \end{cases}$$

where X and Y are two independent samples from some unknown distribution. Pattern $\mathfrak{P}_\Phi(M)$ is stable, provided $Stab(\mathfrak{P}_\Phi(M)) = 1$. ■

Furthermore, given two patterns $\mathfrak{P}_\Phi(M_1)$ and $\mathfrak{P}_\Phi(M_2)$, pattern generation will be stable, provided $M_1 \underline{\delta}_\phi M_2$ *and* $Stab(\mathfrak{P}_\Phi(M_1)) = Stab(\mathfrak{P}_\Phi(M_2)) = 1$. In addition, for any set A, $A\ \delta_\Phi\ M_1$ and $A\ \underline{\delta}_\phi\ M_2$ will ensure that set A will always be added to pattern $\mathfrak{P}_\Phi(M_1)$. This is advantageous in achieving pattern stability for the method proposed in this article compared to the traditional clustering methods such as k-means clustering, since pattern stability, in our case, derives its strength from the fact that each set A added to a pattern has *descriptive proximity* to the pattern generator M in a descriptive proximity space.

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