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Hadamard Product of Simple Sets of Polynomials in $\mathbb{C}^n$

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Abstract

In this paper we give some convergence properties of Hadamard product set of polynomials defined by several simple monic sets of several complex variables in complete Reinhardt domains and in hyperelliptical regions too.

Keywords: Basic sets of polynomials, Hadamard product, complete Reinhardt domains, hyperelliptical regions.
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1. Introduction

In 1933, Whittaker (Whittaker, 1933), (Whittaker, 1949) introduced the subject of basic sets of polynomials of a single complex variable. This subject is developed by several authors using one and several complex variables. It is of fundamental importance in the theory of basic sets of polynomials of several complex variables to define some kinds of basic sets of polynomials in $\mathbb{C}^n$. This is the main aim of this paper. We will define and study Hadamard products of basic sets of polynomials in complete Reinhardt domains and in hyperelliptical regions.

We start with basic concepts, notations and terminology on this paper.

Let $\mathbb{C}$ represent the field of complex variables. In the space $\mathbb{C}^2$ of the two complex variables z and w, the successive monomial $1, z, w, z^2, zw, w^2, ...$ are arranged so that the enumeration number of the monomial $z^j w^k$ in the above sequence is

$$\frac{1}{2}(j+k)(j+k)+k; \quad j,k \geq 0.$$

The enumeration number of the last monomial of a polynomial $P(z,w)$ in two complex variables is called the degree of the polynomial. A sequence $\{P_i(z;w)\}_0^\infty$ of polynomials in two complex variables in which the order of each polynomial is equal to its degree is called a simple set

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(see (Kishka, 1993), (Kumuyi & Nassif, 1986) and (Sayyed & Metwally, 1998)). Such a set is conveniently denoted by $\{P_i(z; w)\}$, where the last monomial in $P_{m,n}(z, w)$ is $z^m w^n$.

If further, the coefficient of this last monomial is 1, the simple set is termed monic. Thus, in the simple monic set $\{P_{m,n}(z; w)\}$ the polynomial $P_{m,n}(z, w)$ is represented as follows.

$$P_{m,n}(z, w) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} P^{m,n}_{k-j,j} z^{k-j} w^j \quad (P^{m,n}_{m,n} = 1; P^{m,n}_{m+n-j,j} = 0, j > n).$$

Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables. The following definition is introduced in (Mursi & Makar, 1955*a*,*b*).

Definition 1.1. A set of polynomials $\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0, P_1, P_2, ..., P_n, ...\}$ is said to be basic when every polynomial in the complex variables z_s; $s \in I = \{1, 2, 3, ..., n\}$ can be uniquely expressed as a finite linear combination of the elements of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to (Mursi & Makar, 1955*b*), the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and only if there exists a unique row-finite matrix $\overline{P}$ such that $\overline{P}P = P\overline{P} = \mathbf{I}$, where $P = [P_{\mathbf{m},\mathbf{h}}]$ is the matrix of coefficients, $\overline{P}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and $\mathbf{I}$ is the infinite unit matrix.

Similar definition for a simple monic set can be extended to the case of several complex variables by replacing m, n by $(\mathbf{m}) = (m_1, m_2, m_3, ..., m_n)$, j, k by $(\mathbf{h}) = (h_1, h_2, h_3, ..., h_n)$ and z, w by $\mathbf{z}$, where each of $(\mathbf{m})$ and $(\mathbf{h})$ be multi-indices of non-negative integers.

The fact that the simple monic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of several complex variables is necessarily basic follows from the observation that the matrix $[P_{\mathbf{m},\mathbf{h}}]$ of coefficients of the polynomials of the set is a lower triangular matrix with non-zero diagonal elements. (These elements are each equal to 1 for monic sets).

Definition 1.2. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be algebraic of degree ℓ when its matrix of coefficients P satisfies the usual identity

$$\alpha_0 P^\ell + \alpha_1 P^{\ell-1} + ... + \alpha_\ell I = 0.$$

Thus, we have a relation of the form

$$\overline{P}_{\mathbf{m},\mathbf{h}} = \delta_{\mathbf{m},\mathbf{h}}\gamma_0 + \sum_{s_1=1}^{\ell-1} \gamma_{s_1} P^{(s_1)}_{\mathbf{m},\mathbf{h}},$$

where $P^{(s_1)}_{\mathbf{m},\mathbf{h}}$ are the elements of the power matrix P^{s_1} and $\gamma_{s_1}, \quad s_1 = 0, 1, 2, ..., \ell - 1$ are constant numbers. In the space of several complex variables $\mathbb{C}^n$. Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables, a closed complete Reinhardt domain of radii $\alpha_s r (> 0)$; $s \in I = \{1, 2, 3, ..., n\}$ is here denoted by $\bar{\Gamma}_{[\alpha \mathbf{r}]}$ and is given by

$\bar{\Gamma}_{[\alpha \mathbf{r}]} = \bar{\Gamma}_{[\alpha_1 r, \alpha_2 r, ..., \alpha_n r]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| \leq \alpha_s r \quad ; s \in I\}$, where α_s are positive numbers. The open complete Reinhardt domain is here denoted by $\Gamma_{[\alpha \mathbf{r}]}$ and is given by

$\Gamma_{[\alpha \mathbf{r}]} = \Gamma_{[\alpha_1 r, \alpha_2 r, ..., \alpha_n r]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| < \alpha_s r \quad ; s \in I\}$.

Consider unspecified domain containing the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$. This domain will be of radii $\alpha_s r_1$; $r_1 > r$, then making a contraction to this domain, we will get the domain $\bar{D}([\alpha\mathbf{r}^+]) = \bar{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$, where r^+ stands for the right-limit of r_1 at r.

Now let $\mathbf{m} = (m_1, m_2, ..., m_n)$ be multi-indices of non-negative integers. The entire function $f(\mathbf{z})$ of several complex variables has the following representation:

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}.$$

Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ and

$$M[f; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left| f(\mathbf{z}) \right|.$$

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\left\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\right\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \overline{P}_{\mathbf{h}}[\mathbf{z}].$$

Let $N_{\mathbf{m}} = N_{m_1, m_2, ..., m_n}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m},\mathbf{h}}$ in the last equality.

A basic set satisfying the condition

$$\lim_{\langle \mathbf{m} \rangle \to \infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle \mathbf{m} \rangle}} = 1, \tag{1.1}$$

is called, as in (Mursi & Makar, 1955*a*,*b*) and (Kishka & El-Sayed Ahmed, 2003) a Cannon set.

Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be a basic set of polynomials of the several complex variables z_s; $s \in I$, then the Cannon sum for this set in the complete Reinhardt domains is given as follows:

$$\Omega(P_{\mathbf{m}}, [\alpha\mathbf{r}]) = \prod_{s=1}^{n} (\alpha_s r)^{<\mathbf{m}>-\mathbf{m}_s} \sum_{\mathbf{h}} \left| \bar{P}_{\mathbf{m},\mathbf{h}} \right| M(P_{\mathbf{m}}, [\alpha\mathbf{r}]),$$

where

$$M(P_{\mathbf{m}}, [\alpha\mathbf{r}]) = \max_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left| P_{\mathbf{m}}[\mathbf{z}] \right|.$$

The Cannon function is defined by:

$$\Omega(P, [\alpha\mathbf{r}]) = \lim_{\langle \mathbf{m} \rangle \to \infty} \left\{ \Omega(P_{\mathbf{m}}, [\alpha\mathbf{r}]) \right\}^{\frac{1}{\langle \mathbf{m} \rangle}}.$$

When this associated series converges uniformly to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain; in other words, as in the classical terminology of Whittaker for a single complex variable (see (Whittaker, 1949)), the basic set $P_{\mathbf{m}}[\mathbf{z}]$ will be effective in that domain. For more information about basic sets of polynomials we refer to ((Abul-Ez, 2000)-(Whittaker, 1949)).

The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which are represented.

Concerning the effectiveness of the basic set of polynomials of several complex variables in complete Reinhardt domains, we have the following results from (Mursi & Makar, 1955*a*,*b*).

Theorem 1.1. *(Mursi & Makar, 1955a,b) The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed complete Reinhardt $\overline{\Gamma}_{[\alpha_s \mathbf{r}]}$ is that*

$$\Omega(P; r_s) = \prod_{s=1}^{n} \alpha_s r. \tag{1.2}$$

In the space of several complex variables $\mathbb{C}^n$, an open elliptical region $\sum_{s=1}^{n} \frac{|z_s|^2}{r_s{}^2} < 1$ is here denoted by $\mathbf{E}_{r_s}$ and its closure $\sum_{s=1}^{n} \frac{|z_s|^2}{r_s{}^2} \leq 1$; is denoted by $\overline{\mathbf{E}}_{r_s}$, where r_s; $s \in I$ are positive numbers. In terms of the introduced notations these regions satisfy the following inequalities:

$$\mathbf{E}_{r_s} = \{\mathbf{w} : |\mathbf{w}| < 1\}$$

$$\overline{\mathbf{E}}_{r_s} = \{\mathbf{w} : |\mathbf{w}| \leq 1\},$$

where $\mathbf{w} = (w_1, w_2, w_3, ..., w_n)$, $w_s = \frac{z_s}{r_s}$; $s \in I$. Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\overline{\mathbf{E}}_{\mathbf{r_s}}$ and

$$M[f; r_s] = \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}} \left| f(\mathbf{z}) \right|.$$

Then it follows that $\{|z_s| \leq r_s t_s; \quad |t_s| = 1\} \subset \overline{\mathbf{E}}_{\mathbf{r_s}}$; hence

$$|a_{\mathbf{m}}| \leq \frac{M[f; \rho_s]}{\rho^{\mathbf{m}} t^{\mathbf{m}}} = \frac{M[f; \rho_s]}{\prod_{s=1}^{n} \rho_s^{m_s} t_s^{m_s}} \leq \inf_{|t|=1} \frac{M[f; \rho_s]}{\prod_{s=1}^{n} (\rho_s t_s)^{m_s}}$$
$$= \sigma_{\mathbf{m}} \frac{M[f; \rho_s]}{\prod_{s=1}^{n} \rho_s^{m_s}}$$

for all $0 < \rho_s < r_s$; $s \in I$, where

$$\sigma_{\mathbf{m}} = \inf_{|t|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{\langle \mathbf{m} \rangle\}^{\frac{\langle \mathbf{m} \rangle}{2}}}{\prod_{s=1}^{n} m_s^{\frac{m_s}{2}}}$$

and $1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{\langle \mathbf{m} \rangle}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0$; $s \in I$. Thus, it follows that

$$\lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} (r_s)^{\langle \mathbf{m} \rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \frac{1}{\prod_{s=1}^{n} \rho_s} \quad ; \quad \rho_s < r_s; s \in I$$

and since ρ_s can be chosen arbitrary near to r_s; $s \in I$, we conclude that

$$\lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} (r_s)^{\langle \mathbf{m} \rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \frac{1}{\prod_{s=1}^{n} r_s}.$$

Now, write

$$G(P_{\mathbf{m}}; r_s) = \max_{\mu,\nu} \sup_{\overline{\mathbf{E}}_{\mathbf{r_s}}} \left| \sum_{\mathbf{j}=\mu}^{\nu} \overline{P}_{\mathbf{m};\mathbf{j}} P_{\mathbf{j}}[\mathbf{z}] \right|,$$

where, r_s; $s \in I$ are positive numbers.

The Cannon sum of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for $\overline{\mathbf{E}}_{r_s}$ will be

$$\Omega(P_{\mathbf{m}}; r_s) = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{r_s\}^{\langle \mathbf{m} \rangle - m_s} G(P_{\mathbf{m}}; r_s)$$

and the Cannon function for the same set is

$$\Omega(P; r_s) = \lim_{\langle \mathbf{m} \rangle \to \infty} \{\Omega(P_{\mathbf{m}}; r_s)\}^{\frac{1}{\langle \mathbf{m} \rangle}}.$$

Concerning the effectiveness of the basic set of polynomials of several complex variables in hyperellipse, we have the following results from (El-Sayed Ahmed & Kishka, 2003).

Theorem 1.2. *(El-Sayed Ahmed & Kishka, 2003) The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\overline{\mathbf{E}}_{r_s}$ is that*

$$\Omega(P; r_s) = \prod_{s=1}^{n} r_s.$$

Convergence properties (effectiveness) for Hadamard product set simple monic sets of polynomials of a single complex variable is introduced by Melek and El-Said in (Melek & El-Said, 1985). In (Nassif & Rizk, 1988) Nassif and Rizk introduced an extension of this product in the case of two complex variables using spherical regions. In (El-Sayed Ahmed, 2006), the same author has studied this problem in $\mathbb{C}^n$ using hepespherical regions. It should be mentioned here the study of this problem in Clifford analysis (see (Abul-Ez, 2000)). For more details on basic sets of polynomials in Clifford setting, we refer to (Abul-Ez, 2000; Abul-Ez & De Almeida, 2013; Abul-Ez & Constales, 2003; Aloui *et al.*, 2010; Aloui & Hassan, 2010; Hassan, 2012; Saleem *et al.*, 2012) and others. In the present paper, we aim to investigate the extent of a generalization of this Hadamard product set in $\mathbb{C}^n$ using hyperspherical regions.

In (Nassif & Rizk, 1988), Nassif and Rizk introduced the following definition.

Definition 1.3. Let $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ be two simple monic sets of polynomials, where

$$P_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} P_{i,j}^{m,n} z^i w^j,$$

$$q_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} q_{i,j}^{m,n} z^i w^j.$$

Then the Hadamard product of the sets $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ is the simple monic set $\{U_{m,n}(z,w)\}$ given by

$$U_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} U_{i,j}^{m,n} z^i w^j,$$

where

$$U_{i,j}^{m,n} = \frac{\sigma_{m,n}}{\sigma_{i,j}} P_{i,j}^{m,n} q_{i,j}^{m,n}; \quad ((i,j) \leq (m,n)),$$

and

$$\sigma_{m,n} = \inf_{|t|=1} \frac{1}{t^{m+n}} = \frac{\{m+n\}^{\frac{m+n}{2}}}{m^{\frac{m}{2}} n^{\frac{n}{2}}}.$$

In this paper, we give an inevitable modification in the definition of Hadamard product of basic sets of polynomials of two complex variables as to yield favorable results in the case of several complex variables in complete Reinhardt domains in $\mathbb{C}^n$, by using k basic sets of polynomials instead of two sets.

Now, we are in a position to extend the above product by using k basic sets of polynomials of several complex variables in complete Reinhardt domains, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}, \{P_{2,\mathbf{m}}[\mathbf{z}]\}, ..., \{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$.

Definition 1.4. Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, where

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}. \tag{1.3}$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{1.4}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \left(\prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}}\right). \tag{1.5}$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in (Metwally, 2002).

2. Effectiveness in complete Reinhardt domains

In this section, we will study the effectiveness of the extended Hadamard product of simple monic sets of polynomials of several complex variables defined by (1.4) and (1.5) in closed complete Reinhardt domains and at the origin.

Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables z_s; $s \in I$, so that we can write

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{2.1}$$

where

$$P_{s_2,m_1,m_2,...,m_n}^{m_1,m_2,...,m_n} = 1; \quad s_2 = 1, 2, ..., k\,.$$

The normalizing functions of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are defined by (see (Nassif & Rizk, 1988))

$$\mu(P_{s_2}; \alpha_s r) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup\Big\{M[P_{s_2,\mathbf{m}}; \alpha_s r]\Big\}^{\frac{1}{\langle\mathbf{m}\rangle}}, \tag{2.2}$$

where $M[P_{s_2,\mathbf{m}}; \alpha_s r]$ are defined as follows:

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha r]}} \big|P_{s_2,\mathbf{m}}[\mathbf{z}]\big|.$$

Notice that the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality in (2.2), we have

$$|P_{s_2,\mathbf{m},\mathbf{h}}| \leq \frac{1}{\prod_{s=1}^{n} (\alpha_s r)^{\langle\mathbf{m}\rangle}} \sup_{\bar{\Gamma}_{[\alpha r]}} \big|P_{s_2,\mathbf{m}}[\mathbf{z}]\big|,$$

which implies that

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] \geq \prod_{s=1}^{n} (\alpha_s r)^{\langle\mathbf{m}\rangle}.$$

It follows from (2.2) that

$$\mu(P_{s_2}; \alpha_s r) \geq \prod_{s=1}^{n} \alpha_s r. \tag{2.3}$$

Next, we show if ρ is positive number greater than r, then

$$\mu(P_{s_2}; \alpha_s \rho) \leq \frac{\prod_{s=1}^{n} \alpha_s \rho}{\prod_{s=1}^{n} \alpha_s r} \mu(P_{s_2}; \alpha_s r), \quad \alpha_s \rho > \alpha_s r. \tag{2.4}$$

In fact, this relation follows by applying (2.2) to the inequality

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] \leq K \left(\frac{\prod_{s=1}^{n} \alpha_s \rho}{\prod_{s=1}^{n} \alpha_s r} \right)^{\langle\mathbf{m}\rangle} M[P_{s_2,\mathbf{m}}; \alpha_s r],$$

which in its turn, is derivable from (2.2), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = O(\langle \mathbf{m} \rangle + 1)$.

Now, let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, and that $\{H^*_{\mathbf{m}}[\mathbf{z}]\}$ is the set defined as follows

$$H^*_{\mathbf{m}}[\mathbf{z}] = \prod_{s_2=1}^{k} P_{s_2,\mathbf{m}}[\mathbf{z}]. \tag{2.5}$$

The following fundamental result is proved.

Theorem 2.1. *If, for any $\alpha_s r > 0$*

$$\mu(P_{s_2}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r, \tag{2.6}$$

then

$$\mu(H^*; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r. \tag{2.7}$$

Proof. We first observe that, if ρ be any finite number greater than r , then by (2.1), (2.2) and (2.9), we obtain that

$$\mu(P_{s_2}; \alpha_s \rho) = \prod_{s=1}^{n} \alpha_s \rho. \tag{2.8}$$

Now, given $r^* > r$, we choose finite number r' such that

$$\alpha_s r < \alpha_s r' < \alpha_s r^*. \tag{2.9}$$

Then by (2.1) and (2.6), we obtain that

$$M(P_{s_2,\mathbf{h}}; \alpha_s r) < \eta \prod_{s=1}^{n} \alpha_s (r')^{\langle \mathbf{h} \rangle} \quad \text{where} \quad \eta > 1, \tag{2.10}$$

where $\langle \mathbf{h} \rangle = h_1 + h_2 + h_3 + \ldots h_n$. Also from (2.4), we can write

$$H^*_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} P_{s_2,\mathbf{h}}[\mathbf{z}].$$

Hence (2.9) and (2.10) lead to

$$M[H^*_{\mathbf{m}}; \alpha_s r] \leq \eta K \Big(1 - \Big(\frac{\prod_{s=1}^{n} \alpha_s r'}{\prod_{s=1}^{n} \alpha_s r^*}\Big)^{n}\Big)^{-n} M[P_{s_2,\mathbf{m}}; \alpha_s r^*],$$

Making $\langle \mathbf{m} \rangle \to \infty$ and applying (2.7), we get

$$\mu(H^*; \alpha_s r) = \lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \sigma_{\mathbf{m}} M[H^*_{\mathbf{m}}; \alpha_s r] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \mu(P_{s_2}; \alpha_s r^*) = \prod_{s=1}^{n} \alpha_s r^*,$$

which leads to the equality (2.6), by the choice of r^* near to r, and our theorem is therefore proved. □

Remark. From Theorem 2.1, if we consider the simple monic sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ accord to condition (2.6), then it is not hard to prove by induction for the j-power sets $\{P^{(j)}_{s_2,\mathbf{m}}[\mathbf{z}]\}$ that

$$\mu(P^{(j)}_{s_2}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r. \tag{2.11}$$

Now, we give the following result.

Theorem 2.2. *Let* $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ *be simple monic algebraic sets of polynomials of several complex variables, which accord to condition (10). Then the set will be effective in the closed complete Reinhardt domain* $\bar{\Gamma}_{[\alpha r]}$.

Proof. Suppose that the monomial $\mathbf{z}^{\mathbf{m}}$ admit the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}].$$

Since the set $\{P_{1,\mathbf{m}}[\mathbf{z}]\}$ is algebraic, we find there exists a relation of the form

$$\overline{P}_{1,\mathbf{m},\mathbf{h}} = \sum_{j=1}^{k} a_j P^{(j)}_{1,\mathbf{m},\mathbf{h}}; \quad ((\mathbf{h}) \leq (\mathbf{m})), \tag{2.12}$$

where k is a finite positive integer which together with the coefficients $(a_j)_{j=1}^{k}$, is independent of the indices $(\mathbf{m})$, $(\mathbf{h})$. The coefficients $P^{(j)}_{1,\mathbf{m},\mathbf{h}}$ are defined by

$$P^{(j)}_{1,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=1}^{(\mathbf{m})} P^{(j)}_{1,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}; \quad 1 \leq j \leq k.$$

It follows that

$$\left| P^{(j)}_{1,\mathbf{m},\mathbf{h}} \right| (\alpha_s r)^{\langle \mathbf{m} \rangle} \leq \sigma_{\mathbf{h}} M[P^{(j)}_{1,\mathbf{m}}; \alpha_s r]. \tag{2.13}$$

According to (2.11) for given $r^* > r$ and from the definition corresponding to $\mu(P^{(j)}_1; \alpha_s r)$, we deduce that

$$M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r] < K(\alpha_s r^*)^{\langle \mathbf{h} \rangle}. \tag{2.14}$$

Applying (2.13) and (2.14) in (2.12), we obtain that

$$\left| \overline{P}^{(j)}_{1,\mathbf{m},\mathbf{h}} \right| < \zeta \beta K \frac{\prod_{s=1}^{n} (\alpha_s r^*)^{\langle \mathbf{m} \rangle}}{\prod_{s=1}^{n} (\alpha_s r)^{\langle \mathbf{m} \rangle}}, \tag{2.15}$$

where

$$\beta = \max\{|a_j|; 0 \leq j \leq k\} \quad \text{and} \quad \zeta \quad \text{is a constant.} \tag{2.16}$$

In view of the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}],$$

the Cannon sum of the set $\{P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]\}$ will be

$$\Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |\overline{P}^{(j)}_{\mathbf{m},\mathbf{h}}| M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r], \tag{2.17}$$

where,

$$M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha \mathbf{r}]}} |P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]|. \tag{2.18}$$

Therefore (2.14), (2.15) and (2.17) (for $r^* > r$) give

$$\Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) < \zeta K \beta \prod_{s=1}^{n} (\alpha_s r^*)^{\langle \mathbf{m} \rangle}. \tag{2.19}$$

Hence the Cannon function of the set $\{P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]\}$ turns out to be

$$\Omega(P^{(j)}_{1}; \alpha_s r) = \lim_{\langle \mathbf{m} \rangle \to \infty} \left\{ \Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} = \prod_{s=1}^{n} \alpha_s r^*,$$

which, by the choice of r^*, implies that

$$\Omega(P^{(j)}_{1}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r.$$

As very similar, we can obtain that the sets $\{P^{(j)}_{\nu,\mathbf{m}}[\mathbf{z}]\}$; $\nu = 2, 3, 4, ..., k$ will be effective in the losed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$. Our theorem is therefore proved. □

3. Effectiveness in hyperelliptical regions

Now, we are in a position to extend the above product by using k basic sets of polynomials of several complex variables, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}, \{P_{2,\mathbf{m}}[\mathbf{z}]\}, ..., \{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$.

Definition 3.1. Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, where

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}. \tag{3.1}$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}}\mathbf{z}^{\mathbf{h}}, \tag{3.2}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \left(\frac{\sigma_{\mathbf{m}}}{\sigma_{\mathbf{h}}}\right)^{k-1}\left(\prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}}\right). \tag{3.3}$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in (Metwally, 2002).

Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables z_s; $s \in I$, so that we can write

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}}\mathbf{z}^{\mathbf{h}}, \tag{3.4}$$

where

$$P^{m_1,m_2,...,m_n}_{s_2,m_1,m_2,...,m_n} = 1; \quad s_2 = 1, 2, ..., k\,.$$

The normalizing functions of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are defined by (see (Nassif & Rizk, 1988))

$$\mu(P_{s_2}; r_s) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup\left\{\sigma_{\mathbf{m}} M[P_{s_2,\mathbf{m}}; r_s]\right\}^{\frac{1}{\langle\mathbf{m}\rangle}}, \tag{3.5}$$

where $M[P_{s_2,\mathbf{m}}; r_s]$ are defined as follows:

$$M[P_{s_2,\mathbf{m}}; r_s] = \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}}\left|P_{s_2,\mathbf{m}}[\mathbf{z}]\right|.$$

Notice that the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality, we deduce

$$|P_{s_2,\mathbf{m},\mathbf{h}}| \le \frac{\sigma_{\mathbf{m}}}{[\prod_{s=1}^{n} r_s]^{\langle\mathbf{m}\rangle}} \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}}\left|P_{s_2,\mathbf{m}}[\mathbf{z}]\right|,$$

which implies that

$$M[P_{s_2,\mathbf{m}}; r_s] \ge \frac{[\prod_{s=1}^{n} r_s]^{\langle\mathbf{m}\rangle}}{\sigma_{\mathbf{m}}}.$$

It follows from (3.4) that

$$\mu(P_{s_2}; r_s) \ge r_s. \tag{3.6}$$

Next, we show if ρ_s are positive numbers greater than r_s, then

$$\mu(P_{s_2}; \rho_s) \le \frac{\prod_{s=1}^{n}\rho_s}{\prod_{s=1}^{n} r_s}\mu(P_{s_2}; r_s), \quad \rho_s > r_s. \tag{3.7}$$

In fact, this relation follows by applying (3.4) to the inequality

$$M[P_{s_2,\mathbf{m}};\rho_s] \le K\Big(\frac{\prod_{s=1}^{n}\rho_s}{\prod_{s=1}^{n} r_s}\Big)^{\langle\mathbf{m}\rangle} M[P_{s_2,\mathbf{m}};r_s],$$

which in its turn, is derivable from (3.4), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = O(\langle\mathbf{m}\rangle + 1)$.

Now, let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, and that $\{H^*_{\mathbf{m}}[\mathbf{z}]\}$ is the set defined as follows

$$H^*_{\mathbf{m}}[\mathbf{z}] = \prod_{s_2=1}^{k} P_{s_2,\mathbf{m}}[\mathbf{z}]. \tag{3.8}$$

The following fundamental result is proved.

Theorem 3.1. *If, for any* $r_s > 0$

$$\mu(P_{s_2};r_s) = \prod_{s=1}^{n} r_s, \tag{3.9}$$

then

$$\mu(H^*;r_s) = \prod_{s=1}^{n} r_s. \tag{3.10}$$

Proof. We first observe that, if ρ be any finite number greater than r , then by (3.4), (3.5) and (3.7), we obtain that

$$\mu(P_{s_2};\rho_s) = \prod_{s=1}^{n} \rho_s. \tag{3.11}$$

Now, given $r^*_s > r_s$, we choose finite number r'_s such that

$$r_s < r'_s < r^*. \tag{3.12}$$

Then by (3.4) and (3.8), we obtain that

$$M(P_{s_2,\mathbf{h}};r_s) < \frac{\eta}{\sigma_{\mathbf{h}}}[\prod_{s=1}^{n} r'_s]^{\langle\mathbf{h}\rangle} \quad \text{where} \quad \eta > 1, \tag{3.13}$$

where $\langle\mathbf{h}\rangle = h_1 + h_2 + h_3 + \ldots h_n$. Also from (3.7), we can write

$$H^*_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} P_{s_2,\mathbf{h}}[\mathbf{z}].$$

Hence (3.9) and (3.10) lead to

$$\mu(H^*; r_s) = \lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \sigma_{\mathbf{m}} M[H^*_{\mathbf{m}}; r_s] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \mu(P_{s_2}; r^*_s) = \prod_{s=1}^{n} r^*_s,$$

which leads to the equality (3.8), by the choice of r^*_s near to r_s, and our theorem is therefore proved. □

Remark. From Theorem 3.1 if we consider the simple monic sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ accord to condition (3.8), then it is not hard to prove by induction for the j-power sets $\{P^{(j)}_{s_2,\mathbf{m}}[\mathbf{z}]\}$ that

$$\mu(P^{(j)}_{s_2}; r_s) = \prod_{s=1}^{n} r_s. \tag{3.14}$$

Remark. It should be remarked that the results of this paper improve some results in (El-Sayed Ahmed, 2006, 2013).

4. Conclusion

We have obtained some essential and important results for the effectiveness of the Hadamard product set of polynomials in complete Reinhardt domains and in heperelliptical regions. From the established theorems, representations and convergence of power set of the the Hadamard product set are introduced in complete Reinhardt domains and in heperelliptical regions too. Various problems relating to the properties of the Hadamard set of simple basic sets of polynomials are treated with particular emphasis on distinction between the single and several complex variables cases. An important result is established for the relationship between the Cannon functions of simple sets of polynomials in several complex variables and those of the directly Hadamard sets.

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