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# Common Fixed Points of Hardy and Rogers Type Fuzzy Mappings on Closed Balls in a Complete Metric Space

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## Abstract

In this paper we obtain some common fixed point theorems for Hardy and Rogers type fuzzy mappings on closed balls in a complete metric space. Our investigation is based on the fact that fuzzy fixed point results can be obtained simply from the fixed point theorem of multi-valued mappings with closed values.In real world problems there are various mathematical models in which the mappings are contractive on the subset of a space under consideration but not on the whole space itself. Our results generalize several results of literature.

*Keywords:* Fuzzy fixed point, Hardy and Rogers mapping, contraction, closed balls, continuous mapping.
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## 1. Introduction

It is a well-known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models.In 1922,Banach a Polish mathematician proved a theorem under appropriate of a fixed point this result is called Banach fixed point theorem.This theorem is also applied to prove the existence and uniqueness of the solutions of differential equations. Many authors have made different generalization of Banach fixed point theorem. The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity, and it has a wide range of applications in different areas such as nonlinear and adoptive control systems, parameterize estimation problems, fractal image decoding,computing magneto static fields in a nonlinear medium and convergence of recurrent networks.

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The notion of fixed points for fuzzy mappings was introduced by Weiss (Weiss, 1975) and Butnariu (Butnariu, 1982). Fixed point theorems for fuzzy set valued mappings have been studied by Heilpern (Heilpern, 1981) who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces which is a fuzzy extension of Banach fixed point theorem and Nadlers (Nadler, 1969) theorem for multi-valued mappings. Park and Jeong (Park & Jeong, 1997) proved some common fixed point theorems for fuzzy mappings satisfying in complete metric space which are fuzzy extensions of some theorems in (Azam, 1992; Park & Jeong, 1997). In this paper we obtain some common fixed point theorems of Hardy and Rogers type fuzzy mappings on closed balls.

## 2. Basic concepts

Let $(X, d)$ be a metric space, then we use the following notations: Let

$$2^X = \{A : A \text{ is a subset of } X\},$$

$$CL(2^X) = \{A \in 2^X : A \text{ is nonempty and closed}\},$$

$$C(2^X) = \{A \in 2^X : A \text{ is nonempty and compact}\},$$

$$CB(2^X) = \{A \in 2^X : A \text{ is nonempty, closed and bounded}\},$$

For $A, B \in CB(2^X)$, $d(x, A) = \inf_{y \in A} d(x, y)$, $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$ then the Hausdroff metric $d_H$ on $CB(2^X)$ induced by $d$ is defined as: $d_H(A, B) = \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$.
A fuzzy set in $X$ is a function with domain $X$ and values in $[0, 1]$ and $I^X$ is the collection of all fuzzy sets in $X$ . If $A$ is a fuzzy set and $x \in X$ then the function values $A(x)$ is called the grade of membership of $x$ in $A$. The $\alpha$-level set of a fuzzy set $A$, is denoted by $[A]_\alpha$, and is defined as:

$$[A]_\alpha = \{x : A(x) \geq \alpha \text{ if } \alpha \in (0, 1]\} \text{ and } [A]_0 = \overline{\{x : A(x) \geq 0\}}.$$

For $x \in X$, we denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where $\chi_A$ is the characteristic function of the crisp set $A$. Now we define a sub-collection of $I^X$ as follows: $\tau(X) = \{A \in I^X : [A]_1 \text{is nonempty and closed}\}$, for A,B $\in I^X$, $A \subset B$ means $A(x) \leq B(x)$ for each $x, y \in X$. For $A, B \in \tau(X)$ then define $D_1\{A, B\} = d_H([A]_1, [B]_1)$.
A point $x^* \in X$ is called a fixed point of a fuzzy mappings $T : X \to I^X$ if $x^* \subset Tx^*$ see (Heilpern, 1981)

**Lemma 2.1.** *(Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space* $(X, d)$ *. If* $a \in A$ *, then* $d(a, B) \leq d_H(A, B)$.

**Lemma 2.2.** *(Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space* $(X, d)$ *and* $0 < \xi \in \Re$ *then for* $a \in A$ *there exists* $b \in B$ *such that* $d(a, B) \leq d_H(A, B) + \xi$.

**Lemma 2.3.** *(Nadler, 1969) The completeness of* $(X, d)$ *implies that* $(CB(2^X), d_H)$ *is complete.*

**Theorem 2.1.** *(Hardy & Rogers, 1973) Let $(X, d)$ be a complete metric space and a mapping $T: X \to X$ suppose there exists non-negative constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that for each $x, y \in X$*

$$d(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, Fx) + a_3 d(y, Fy) + a_4 d(x, Fy) + a_5 d(y, Fx)$$

*holds then $F$ has a unique fixed point in $X$.*

## 3. Main Results

The mapping satisfies the contractive condition in Theorem (2.1) is called Hardy and Rogers type mapping. It is mentioned that Hardy and Rogers contractive condition does not implies that the mapping T is continuous, which differentiates it from Banach contractive condition for $c \in X$ and $0 < r < R$. Let $S_r(c) = \{x \in X/d(c, x) < r\}$ be the ball of radius r centered at c, the closure of $S_r(c)$ is denoted by $\overline{S_r(c)}$. We present a result regarding the existence of common fixed point for fuzzy mappings satisfying Hardy and Rogers type contractive condition on closed balls. The theorem is as follows:

**Theorem 3.1.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F, T: \overline{S_r(x_0)} \to \tau(X)$. Suppose there exist a constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Ty) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Ty]_1) + a_4 d(x, [Ty]_1) + a_5 d(y, [Fx]_1) \quad (3.1)$$

*for all $x, y \in \overline{S_r(x_0)}$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{(1 - a_3 - a_4)} \quad (3.2)$$

*holds. Then $F$ and $T$ has a common fuzzy fixed point in $\overline{S_r(x_0)}$ that is there exists $x^* \in \overline{S_r(x_0)}$ with $\{x^*\} \subseteq Fx^* \cap Tx^*$.*

*Proof.* Choose $x_1 \in X$ such that $\{x_1\} \subseteq Fx_0$ and

$$d(x_0, x_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{(1 - a_3 - a_4)} \quad (3.3)$$

since $[Fx_0]_1 \neq \phi$ for the sake of simplicity chooses $\lambda = \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)}$ this gives us $d(x_0, x_1) < (1 - \lambda)r$ which implies that $x_1 \in \overline{S_r(x_0)}$. Now choose $\varepsilon > 0$ such that

$$\lambda d(x_0, x_1) + \frac{\varepsilon}{(1 - a_3 - a_4)} < \lambda(1 - \lambda)r. \quad (3.4)$$

Then choose $\varepsilon > 0$ such that $\{x_2\} \subseteq Tx_1$ and by using inequality (3.1) and Lemma 2.1 we have

$$\begin{aligned}
d(x_1, x_2) &\leq D_1(Fx_0, Tx_1) + \varepsilon \\
&\leq a_1 d(x_0, x_1) + a_2 d(x_0, [Fx_0]_1) + a_3 d(x_1, [Tx_1]_1) + a_4 d(x_0, [Tx_1]_1) + a_5 d(x_1, [Fx_0]_1 + \varepsilon \\
&\leq a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + a_5 d(x_1, x_1) + \varepsilon \\
&= (a_1 + a_2) d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + \varepsilon
\end{aligned}$$

i.e. $d(x_1, x_2) \leq \lambda d(x_0, x_1) + \frac{\varepsilon}{(1-a_3-a_4)}$ where $\lambda = \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)}$. Now by inequality (3.4) we get $d(x_1, x_2) < \lambda(1-\lambda)r$. Note that $x_2 \in \overline{S_r(x_0)}$ since

$$
\begin{aligned}
d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) < (1-\lambda)r + \lambda(1-\lambda)r = (1-\lambda)r(1+\lambda) \\
&< (1-\lambda)(1 + \lambda + \lambda^2 + \lambda^3 + ...)r = r
\end{aligned}
$$

continue this process and having chosen $\{x_n\}$ in $X$ such that $\{x_{2k+1}\} \subseteq Fx_{2k}$ and $\{x_{2k+2}\} \subseteq Tx_{2k+1}$ with $d(x_{2k+1}, x_{2k+2}) < \lambda^{2k+1}(1-\lambda)r$ where $k = 0, 1, 2, \ldots$

Notice that $\{x_n\}$ is cauchy sequence in $\overline{S_r(x_0)}$ which is complete. Therefore a point $x^* \in \overline{S_r(x_0)}$ exists with $lim_{n\longrightarrow\infty} x_n = x^*$. It remains to show that $\{x^*\} \subseteq Tx^*$ and $\{x^*\} \subseteq Fx^*$. Now by using Lemma 2.1 and inequality (3.1) we get

$$
\begin{aligned}
d(x^*, [Tx^*]_1) &\leq d(x^*, x_{2n+1}) + d(x_{2n+1}, [Tx^*]_1) \\
&\leq d(x^*, x_{2n+1}) + D_1(Fx_{2n+2}, Tx^*) \\
&\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, [Fx_{2n+2}]_1) + a_3 d(x^*, [Tx^*]_1) \\
&\quad + a_4 d(x_{2n+2}, [Tx^*]_1) + a_5 d(x^*, [Fx_{2n+2}]_1) \\
&\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) + a_3 d(x^*, [Tx^*]_1) \\
&\quad + a_4 d(x_{2n+2}, [Tx^*]_1) + a_5 d(x^*, x_{2n+1}) \\
&\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) + a_4 d(x_{2n+2}, x^*) \\
&\quad + a_4 d(x^*, [Tx^*]_1) + a_5 d(x^*, x_{2n+1}) \\
&\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) \\
&\quad + a_4 d(x_{2n+2}, x^*) + a_5 d(x^*, x_{2n+1}) \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty
\end{aligned}
$$

This implies that $d(x^*, [Tx^*]_1) = 0$, which implies that $\{x^*\} \subseteq Tx^*$. Similarly consider that $d(x^*, [Fx^*]_1) \leq d(x^*, x_{2n+2}) + d(x_{2n+2}, [Fx^*]_1)$ to show that $\{x^*\} \subseteq Fx^*$. This implies that the mappings **F** and **T** have a common fixed point $\overline{S_r(x_0)}$, i.e. $\{x^*\} \subseteq Fx^* \cap Tx^*$. □

**Corollary 3.1.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F \colon \overline{S_r(x_0)} \to \tau(X)$. Suppose there exist a non negative constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Fy]_1) + a_4 d(x, [Fy]_1) + a_5 d(y, [Fx]_1)$$

*for all $x, y \in \overline{S_r(x_0)}$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

*holds. Then $F$ has a common fuzzy fixed point in $\overline{S_r(x_0)}$ that is there exists $x^* \in \overline{S_r(x_0)}$ with*

$$\{x^*\} \subseteq Fx^*.$$

*Proof.* Put $T = F$ in Theorem 3.1 we get $x^* \in \overline{S_r(x_0)}$ such that $\{x^*\} \subseteq Fx^*$. □

**Theorem 3.2.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F, T : X: \rightarrow \tau(X)$.Suppose there exist a constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Ty) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Ty]_1) + a_4 d(x, [Ty]_1) + a_5 d(y, [Fx]_1)$$

*for all $x, y \in X$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

*holds.Then $F$ and $T$ has a common fuzzy fixed point in $X$ that is there exists $x^* \in X$ with*

$$\{x^*\} \subseteq Fx^* \cap Tx^*.$$

Proof: Fix $x_0 \in X$ and choose $r > 0$ such that

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

Now Theorem 3.1 guarantees that there exists $x^* \in X$ with

$$\{x^*\} \subseteq Fx^* \cap Tx^*.$$

**Corollary 3.2.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F : X \rightarrow \tau(X)$.Suppose there exist a constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Fy]_1) + a_4 d(x, [Fy]_1) + a_5 d(y, [Fx]_1)$$

*for all $x, y \in X$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

*holds.Then $F$ has a common fuzzy fixed point in $X$ that is there exists $x^* \in X$ with*

$$\{x^*\} \subseteq Fx^*.$$

*Proof: In Theorem 3.2 take T=F we get $x^* \in X$ such that $\{x^*\} \subseteq Fx^*$.*

## 4. The importance and future of this theory:

Fuzzy sets and mappings play important roles in the fuzzification of systems. In particular, in the recent years the fixed point theory for fuzzy mappings and for a family of these mappings obtained via implicit functions named Hardy and Rogers type mappings. In this article can further be used in the process of finding the solution of functional equations in fuzzy systems. As far as the application of contraction mapping is concerned the situation is not fully exploited. It is quite possible that a contraction $T$ is defined on the whole space $X$ but it is contractive on the subset $Y$ of the subset of the space rather on the whole space $X$. Moreover the contraction mapping under consideration may not be continues. If $Y$ is closed, then it is complete,so that a mapping $T$ has a fixed point $x$ in $Y$,and $x_n \rightarrow x$ as in the case of whole space X provided we improve a simple restriction on the choice of $x_0$, so that $x_n' s$ remains in $Y$. In this paper, we have discussed this concept for fuzzy Hardy and Rogers mappings on a complete metric space $x$ which generalize/improves several classical tendon with (Azam *et al.*, 2013) will become the foundation of fuzzy theory on closed balls.

An example of a fuzzy mapping which is contractive on the subset of a space but not on the whole space is as follows:

**Example 4.1.** Let $X = \mathbb{R}$ and $d : XxX \to \mathbb{R}$ is defined by $d(x, y) = |x - y|$ where $x, y \in X$ consider the mapping $F : X \to \tau(X)$ is defined by

$$F(x) = \begin{cases} \chi_{(1-x)}, & \text{if } x \text{ is irrational;} \\ \chi_{(\frac{1+x}{3})}, & \text{if } x \text{ is rational.} \end{cases}$$

then $F$ is Hardy and Rogers type fuzzy mapping on the closed balls $\overline{S_{(\frac{1}{2})}(\frac{1}{2})} = [0, 1]$ but not on the whole space $X$.

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