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Banach Frames, Double Infinite Matrices and Wavelet Coefficients

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Abstract

In this paper we study the action of a double infinite matrix A on $f \in H_v^p$ (weighted Banach space, $1 \leq p \leq \infty$) and on its wavelet coefficients. Also, we find the frame condition for $A-$transform of $f \in H_v^p$ whose wavelet series expansion is known.

Keywords: Frames, Riesz basis, wavelet coefficients, Banach space and frame operators.
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1. Introduction

The mathematical background for today's signal processing applications are Gabor (Feichtinger & Strohmer, 1998), wavelet (Daubechies, 1992) and sampling theory (Benedetto & Ferreira, 2001). Without signal processing methods several modern technologies would not be possible, like mobile phone, UMTS, xDSL or digital television. In other words, we can say that any advance in signal processing sciences directly leads to an application in technology and information processing. A signal is sampled and then analyzed using a Gabor respectively wavelet system. Many applications use a modification on the coefficients obtained from the analysis and synthesis operations. If the coefficients are not changed, the result of synthesis should be the original signal, i.e., perfect reconstruction is needed. One way is to analyze the signal using orthonormal basis. For practical point of view it is noted that the concept of an orthonormal basis is not always useful. Sometimes it is more important for a decomposing set to have other special properties rather than guaranteing unique coefficients. This led to the concept of frames introduced by Duffin and Schaeffer (Duffin & Schaeffer, 1952). Now a days it is one of most important foundations of Gabor (Moricz & Rhoades, 1989), wavelet (S.T. Ali & Gazeau, 2000) and sampling theory (Aldroubi & Gröchenig, 2001). In signal processing applications frames have received more and more attention

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(H. Bölcskei & Feichtinger, 1998; Kronland-Martinet & Grossmann, 1991; Munch, 1992; Sheikh & Mursaleen, 2004).

Frame provide stable expansions in Hilbert spaces, but they may be over complete and the coefficients in the frame expansion need not be unique unlike in orthogonal expansions. This redundancy is useful for the application point of view that is to noise reduction or for the reconstruction from lossy data (Daubechies, 1992; Duffin & Schaeffer, 1952; Matz & Hlawatsch, 2002). The construction of stable orthonormal basis are often difficult in a numerical efficient way than the construction of frames which are more flexible. Sometimes it is reasonable to use the frames to analyze additional properties of functions beyond the Hilbert space. These properties are encoded in the frame coefficients. Wavelet frames encode information on the smoothness and decay properties or phase space localization of functions by means of the magnitudes of the frame coefficients. The aim is to study these properties in Banach space norms. Moreover, to characterize an associated family of Banach spaces of functions by the values of the frame coefficients which play an important role in non-linear approximation and in compression algorithms (DeVore & Temlyakov, 1996). However, in (Gröchenig, 2004) Gröchenig showed that certain frames for Hilbert spaces extend automatically to Banach frames. Using this theory he derived some results on the construction of non-uniform Gabor frames and solved a problem about non-uniform sampling in shift-invariant spaces. Recently, Kumar (Kumar, 2013) studied the convergence of wavelet expansions associated with dilation matrix in the variable L^p spaces using the approximate identity. In an another paper Kumar (Kumar, 2009) studied the convergence of non-orthogonal wavelet expansions in $L^p(R), 1 < p < \infty$.

The space $L^2(R)$ of measurable function f is defined on the real line R, that satisfies $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$. The inner product of two square integrable functions $f, g \in L^2(R)$ is defined as

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \ \ ||f||^2 =< f, f >^{1/2} .$$

Every function $f \in L^2(R)$ can be written as $f(x) = \sum_{j,k \in z} c_{j,k}\varphi_{j,k}(x)$ (z is the set of integers).

This series representation of f is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $c_{j,k}$ are given by $c_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{j,k}(x)}dx =< f, \varphi_{j,k} >, \ \ \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$.

Now, if we define continuous wavelet transform as $(W_\varphi(f))(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\overline{\varphi\left(\frac{x-b}{a}\right)}dx, f \in L^2(R)$ then the wavelet coefficients are given by $c_{j,k} = (W_\varphi(f))\left(\frac{k}{2^j}, \frac{1}{2^j}\right)$.

A sequence $\{x_n\}$ in a Hilbert space H is a frame if there exist constant c_1 and $c_2, 0 < c_1 \leq c_2 < \infty$, such that $c_1||f||^2 \leq \sum_{n \in z} | < f, x_n > |^2 \leq c_2||f||^2$, for all $f \in H$. The supremum of all such numbers c_1 and infimum of all such numbers c_2 are called the frame bounds of the frame. The frame is called tight frame when $c_1 = c_2 = 1$. Any orthonormal basis in a Hilbert space H is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by (Grossmann *et al.*, 1985). In 1985, they defined that a wavelet function $\varphi \in L^2(R)$, constitutes a frame with frame bounds c_1 and c_2, if any $f \in L^2(R)$ such that $c_1||f||^2 \leq \sum_{j,k \in z} | < f, \varphi_{j,k} > |^2 \leq c_2||f||^2$. Again, it is said to be tight if $c_1 = c_2$ and is said to be exact if it ceases to be frame by removing any of its element. For more details see (Chui, 1992; Daubechies *et al.*, 1986).

2. Notations and Auxiliary Results

Let N and χ be countable index sets in some R^2 and both χ and N are separated i.e., $\inf_{m,n\in\chi; m\neq n} |m-n| \geq \delta > 0$, and likewise for N.

Weight Functions of Polynomial Growth. A weight is a non-negative continuous function on R^d. An $s-$moderate weight v is called polynomially grows, if there are constants $C, s \geq 0$ such that $v(t) \leq C(1+|t|)^s$.

Lemma 2.1. *If $f(x) = \sum_{j,k\in N} c_{j,k}\varphi_{j,k}(x)$ is a wavelet expansion of $f \in L^2(R^d)$ with wavelet coefficients $c_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{j,k}(x)}dx =< f, \varphi_{j,k} >$ and $A(a_{mnjk}) = [(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}$ for some $\varepsilon > 0$ and $j,k \in N, m,n \in \chi$, then the operator A defined on finite sequences $(c_{j,k})_{j,k\in N}$ by matrix multiplication $(Ac)_{m,n} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}c_{j,k}$ extends to a bonded operator from $l_v^p(N)$ to $l_v^p(\chi)$ for all $p \in [1,\infty]$ and all $s-$moderate weights v.*

Proof. To prove the result we have to show the boundedness of A from $l_v^1(N)$ to $l_v^1(\chi)$ and from $l_v^\infty(N)$ to $l_v^\infty(\chi)$. Then using the interpolation technique of [4] for weighted L^p-space, the lemma holds for all $p \in [1,\infty]$.

First we consider

$$\begin{aligned}
\|Ac_{j,k}\|_{l_v^1(\chi)} &= \sum_{m,n\in\chi}\left|\sum_{j,k\in N} a_{mnjk}c_{j,k}\right| v(m,n) \leq \sum_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}|c_{j,k}|v(m,n)\\
&\leq \sup_{j,k\in N}\left(\sum_{m,n\in\chi}[(1+|m-j|)(1+|n-k|)]^{-d-\varepsilon}\right)\times\\
&\quad \left(\sup_{m,n\in\chi; j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s}[v(j,k)]^{-1}v(m,n)\right)\times\sum_{j,k\in N}|c_{j,k}|v(j,k).
\end{aligned}$$

Using (Gröchenig, 2004), Lemma 2.2 in above inequality we obtain

$$\begin{aligned}
&\leq \sup_{j,k\in N}\left(C(1+|j-k|)^{-d-\varepsilon}\right)\left(\sup_{j,k\in N} C(1+|j-k|)^{-s}\right)\times\\
&\quad [v(j,k)]^{-1}v(m,n)\times\sum_{j,k\in N}|c_{j,k}|v(j,k) = C\|c_{j,k}\|_{l_v^1(N)}.
\end{aligned}$$

The first supremum in right hand side of above inequality is finite by (Gröchenig, 2004), Lemma 2.1] and second supremum in finite due to $s-$moderate and sub multiplicavity of the weights. Now we have

$$\begin{aligned}
\|Ac_{j,k}\|_{l_v^\infty(\chi)} &= \sup_{m,n\in\chi}\left|\sum_{j,k\in N} a_{mnjk}c_{j,k}\right| v(m,n)\\
&\leq \sup_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}|c_{j,k}|v(m,n)\\
&\leq \left(\sup_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k)]^{-d-\varepsilon}\right)\times
\end{aligned}$$

$$\left(\sup_{\substack{m,n \in \chi \\ j,k \in N}} [(1+|m-j|)(1+|n-k|)]^{-s}.\nu(m,n)\nu(j,k)^{-1} \right) \times \left(\sup_{j,k \in N} |c_{j,k}|\nu(j,k) \right).$$

Again, using (Gröchenig, 2004), Lemma 2.2 in above inequality we get

$$\leq \left(C \sup \sum_{j,k \in N} (1+|j-k|)^{-d-\varepsilon} \right) \left(\sup(1+|j-k|)^{-s}\nu(m,n)\nu(j,k)^{-1} \right) \times$$
$$\left(\sup_{j,k \in N} |c_{j,k}|\nu(j,k) \right) \leq CC' \|c_{j,k}\|_{l_\nu^\infty(N)}.$$

Let $\{\phi_{j,k} : j,k \in N\}$ be a Riesz basis of H with dual basis $\{\widetilde{\phi}_{j,k} : j,k \in N\}$ and ν be a weight function on R^d of polynomial type. □

Definition 2.1. Assume that $l_\nu^p(N) \subseteq l_\nu^2(N)$. Then the Banach space H_ν^p is defined to be

$$H_\nu^p = \{f \in H : f = \sum_{j,k \in N} c_{j,k}\phi_{j,k} \texttt{ for } c_{j,k} \in l_\nu^p(N)\}$$

with norm $\|f\|_{H_\nu^p} = \|c_{j,k}\|_{l_\nu^p}$. It should be noted that $c_{j,k}$ is uniquely determined, in fact, $c_{j,k} =< f, \widetilde{\phi}_{j,k} >$.

By assumption $l_\nu^p(N) \subseteq l_\nu^2(N)$, it means H_ν^p is a (dense) subset of H. On the other hand, if $l_\nu^p \nsubseteq l_\nu^2$ and $p < \infty$, we define H_ν^p to be the completion of the subspace H_0 of finite linear combinations, i.e., $H_0 = \{f = \sum_{j,k \in N} c_{j,k}\phi_{j,k} : supp$ c is finite }, with respect to the norm $\|f\|_{H_\nu^p} = \|c\|_{l_\nu^p}$. If $p = \infty$ and $l_\nu^p \nsubseteq l^2$, we take the weak completion of H_0 to define H_ν^∞. In these cases $H_\nu^p \nsubseteq H$.

Frame Operators and Localization of Frames. Let $\mathcal{F} = \{\varphi_{m,n} : m,n \in \chi\}$ be a frame for H and $Sf = \sum_{m,n \in \chi} < f, \varphi_{m,n} > \varphi_{m,n}$ be the corresponding frame operator. Each frame element has a natural expansion with respect to the given Riesz basis as

$$\varphi_{m,n} = \sum_{j,k \in N} \left\langle \varphi_{m,n}, \widetilde{\phi}_{j,k} \right\rangle \phi_{j,k} = \sum_{j,k \in N} < \varphi_{m,n}, \phi_{j,k} > \widetilde{\phi}_{j,k}.$$

The frame operator S is invertible on H. Our problem is how to extend the mapping properties of S on Banach spaces H_ν^p. For this purpose we take $f = \sum_{j,k} f_{j,k}\phi_{j,k}$ such that

$$\begin{aligned} Sf &= \sum_{m,n \in \chi} < f, \varphi_{m,n} > \varphi_{m,n} = \sum_{m,n \in \chi} \sum_{j,k \in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} > \varphi_{m,n} \\ &= \sum_{m,n \in \chi} \sum_{i,l \in N} \sum_{j,k \in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle \phi_{i,l} \\ &= \sum_{i,l} \left(\sum_{j,k} \left(\sum_{m,n} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle \right) f_{j,k} \right) \phi_{i,l}. \end{aligned}$$

Now let $A = a_{iljk}$ be infinite matrix defined as

$$a_{iljk} = \sum_{m,n\in\chi} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle = \left\langle S\phi_{j,k}, \widetilde{\phi}_{i,l} \right\rangle. \tag{2.1}$$

Define a mapping Γ such that $\Gamma : H \to l^2(N), (\Gamma f)_{j,k} = \left\langle f, \widetilde{\phi}_{j,k} \right\rangle$.

Since $\{\phi_{j,k}\}$ is a Riesz basis, Γ is invertible and an isometric isomorphism between H^p_ν and $l^p_\nu(N)$. Therefore, $S = \Gamma^{-1}A\Gamma$ carries over to the Banach spaces H^p_ν. To study the behavior of frame operator S on H^p_ν, it is sufficient to study the infinite matrix A on sequence space $l^p_\nu(N)$. For this purpose we will use the following fundamental theorem of Jaffard [14].

Theorem A. Assume that the matrix $G = (G_{k,l})_{k,l\in N}$ satisfies the following properties:

(a) G is invertible as an operator on $l^2(N)$, and

(b) $|G_{kl}| \le C(1 + |k - l|)^{-l}, k, l \in N$ for some constant $C > 0$ and some $r > d$. Then the inverse matrix $H = G^{-1}$ satisfies the same off-diagonal decay, that is

$$|H_{kl}| \le C^{'}(1 + |k - l|)^{-r}, k, l \in N.$$

Using above theorem we can prove:

Theorem 2.1. *Assume that the matrix $A = (a_{iljk})_{i,l,j,k\in N}$ satisfies the following properties:*

(a) A is invertible as an operator on $l^2(N)$, and

(b) $|a_{iljk}| \le C[(1 + |i - j|)(1 + |l - k|)]^{-r}, i, l, j, k \in N$ for some constant $C > 0$ and some $r > d$.

Then the inverse matrix $T = A^{-1}$ satisfies the same off-diagonal decay, i.e.,

$$\left|T_{iljk}\right| \le C^{'}[(1 + |i - j|)(1 + |l - k|)]^{-r}, i, l, j, k \in N.$$

Definition 2.2. The frame $\mathcal{F} = \{\varphi_{m,n} : m, n \in \chi\}$ is said to be polynomially localized with respect to Riesz basis $\{\phi_{j,k}\}$ with decay $s > 0$ (or simply s−localized), if

$$| < \varphi_{m,n}, \phi_{j,k} > | \le C[(1 + |m - j|)(1 + |n - k|)]^{-s}$$

and

$$| < \varphi_{m,n}, \widetilde{\phi}_{j,k} > | \le C[(1 + |m - j|)(1 + |n - k|)]^{-s} \forall i, k \in N \text{ and } m, n \in \chi.$$

Now we prove:

Proposition 2.1. *Let $\mathcal{F} = (\varphi_{m,n} : m, n \in \chi)$ is an $(s + d + \varepsilon)$−localized frame for $\varepsilon > 0, r \ge 0$ and $1 \le p \le \infty$.Then*

(i) the analysis operator defined by $C_\varepsilon f = (< f, \varphi_{m,n} >)_{m,n\in\chi}$ is bounded from H^p_ν to $l^p_\nu(\chi)$.

(ii) the synthesis operator defined on finite sequences by $D_\varepsilon c = \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}$ extends to a bounded mapping from $l^p_\nu(\chi)$ to H^p_ν.

(iii) the frame operator $S = S_\varepsilon = D_\varepsilon C_\varepsilon = \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ *maps* H^p_ν *into* H^p_ν*, and the series converges unconditionally for* $1 \le p \le \infty$.

Proof. (i) Assume that $f = \sum_{j,k\in N} f_{j,k}\phi_{j,k}$, $| < f, \varphi_{m,n} > | = \left|\sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} >\right|$. In view of Definition 2.4, we get $\le C\sum_{j,k\in N}\left|f_{j,k}\right|[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon} \le CC'\sum_{j,k\in N}|f_{j,k}|(1+|j-k|)^{-s-d-\varepsilon}$.

If $f \in H^p_\nu$, then $\|f\|_{H^p_\nu} = \left\|(f_{j,k})_{j,k\in N}\right\|_{l^p_\nu(N)}$ and Lemma 2.1 gives that $\|C_\varepsilon f\|_{l^p_\nu(\chi)} \le CC'\left\|(f_{j,k})_{j,k\in N}\right\|_{l^p_\nu(N)} = CC'\|f\|_{H^p_\nu}$.

(ii) Now we have $(D_\varepsilon c)_{j,k\in N} = \left\langle \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}, \widetilde{\phi}_{j,k}\right\rangle$ or

$$\begin{aligned}|(D_\varepsilon c)_{j,k\in N}| &\le \sum_{m,n\in\chi}|c_{m,n}|\left|\left\langle\varphi_{m,n},\widetilde{\phi}_{j,k}\right\rangle\right| \le C\sum_{m,n\in\chi}|c_{m,n}|[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}\\ &\le CC'\sum_{m,n\in\chi}|c_{m,n}|(1+|j-k|)^{-s-d-e} = CC'(A^*|c|)_{j,k}.\end{aligned}$$

Now Lemma 2.1 (by interchanging N and χ) gives $\|D_\varepsilon c\|_{H^p_\nu} = \|A^*|c|\|_{l^p_\nu(N)} \le \|A^*\|_{op}\|c\|_{l^p_\nu(\chi)}$.

(iii) The boundlessness of frame operator S follows by combining (1) and (ii). For unconditional convergence of the series defining S, let $\varepsilon > 0$, choose $N_0 = N_0(\varepsilon)$, such that $\| < f, \varphi_{m,n} >_{m,n(not\in N_0)} \|_{l^p_\nu} \le \varepsilon$. Then for any finite set $N_1 \supseteq N_0$, from (i) and (ii), we obtain

$$\left\|Sf - \sum_{m,n\in N} < f, \varphi_{m,n} > \varphi_{m,n}\right\|_{H^p_\nu} \le \|C_\varepsilon\|_{op}\| < f,\varphi_{m,n} > \| \le \|C_\varepsilon\|_{op}.\varepsilon.$$

Which implies that the series $\sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ converges unconditionally in H^p_ν. □

Proposition 2.2. *Assume that* $\mathcal{F} = \{\varphi_{m,n} : m, n \in \chi\}$ *is polynomially localized with respect to the Riesz basis* $\{\phi_{j,k}\}$ *with decay* $s > d$*. Then*

$$|A| = |a_{iljk}| \le C(1+|j-k|)^{-s} \quad for\ i,l,j,k \in N.$$

Proposition 2.3. *From (2.1) we get*

$$\begin{aligned}|a_{iljk}| &\le C\sum_{m,n\in\chi}[(1+|m-j|)(1+|n-k|)(1+|i-m|)(1+|l-n|)]^{-s}\\ &\le CC'\sum_{i,l\in N}[(1+|i-j|)(1+|l-k|)]^{-s} \le CC'C''(1+|j-k|)^{-s}.\end{aligned}$$

3. Main Results

The following definition is due to Moricz and Rhoades (Moricz & Rhoades, 1989).

Definition 3.1. Let $A = (a_{iljk})$ be a double non-negative infinite matrix of real numbers. Then, $A-$transform of a double sequence $x = \{x_{j,k}\}$ is $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}x_{j,k}$ which is called $A-$means or $A-$transform of the sequence $x = \{x_{j,k}\}$.

Sheikh and Mursaleen (Sheikh & Mursaleen, 2004) study the frame condition by using the action of frame operator A on non-negative infinite matrix in Hilbert space. In this paper our aim is to extend these results on weighted Banach space in R^d.

Now we prove our main results:

Theorem 3.1. *Let $A = (a_{iljk})$ be a double non-negative infinite matrix. If $f(x) = \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}(x)$ is a wavelet expansion of $f \in H^p_\nu$ with wavelet coefficients $c_{m,n} =< f, \varphi_{m,n} >$, then the frame condition for $A-$transform of $f \in H^p_\nu$ is*

$$c_1\|f\|_{H^p_\nu} \le \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \le c_2\|f\|_{H^p_\nu}$$

where $\{\varphi_{m,n} : m, n \in \chi\}$ is an $(s + d + \varepsilon)-$localized frame for $\varepsilon > 0$, $s \ge 0$ and $1 \le p \le \infty$.

Proof. We take $f = \sum_{j,k\in N} f_{j,k}\phi_{j,k}$, then

$$\sum_{m,n\in\chi} | < Af, \varphi_{m,n} > | \le \left| \sum_{j,k\in N} \sum_{m,n\in\chi} Af_{j,k} < \phi_{j,k}, \varphi_{m,n} > \right| \le \sum_{j,k\in N} |Af_{jk}|| < \phi_{j,k}, \varphi_{m,n} > |$$
$$\le c \sum_{j,k\in N} |Af_{j,k}|((1 + |m - j|)(1 + |n - k|))^{-s-d-\varepsilon} \le CC' \sum_{j,k\in N} |Af_{j,k}|(1 + |j - k|)^{-s-d-\varepsilon}.$$

If $f \in H^p_\nu$, then $\|f\|_{H^p_\nu} = \|(f_{j,k})_{j,k\in N}\|_{l^p_\nu}$. Hence we get

$$\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \le CC'\|A\|_{op}\|f\|_{H^p_\nu} \le c_2\|f\|_{H^p_\nu}.$$

Now, for any $f \in H^p_\nu$, define

$$\widetilde{f} = \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|^{-1}_{l^p_\nu} f < A\widetilde{f}, \varphi_{m,n} >= \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|^{-1}_{l^p_\nu} < Af, \varphi_{m,n} >$$

then

$$\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \le 1.$$

Hence, if there exists a positive constant α, such that

$$\|Ac_{m,n}\|_{l^p_\nu} \le \alpha \left[\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right]^{-1} \|Ac_{m,n}\|_{l^p_\nu} \le \alpha \left[\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right]^{-1} \|f\|_{H^p_\nu} \le \left(\frac{\alpha}{\|A\|_{op}} \right)$$

it follows that $\left[\left\|\sum_{m,n\in\chi} < Af, \varphi_{m,n} >\right\|_{l^p_\nu}\right] \ge c_1\|f\|_{H^p_\nu}$.

Hence the proof is completed. □

Theorem 3.2. *If* $f = \sum_{j,k \in N} c_{j,k}\phi_{j,k}$ *and* $\{\varphi_{m,n} : m, n \in \chi\}$ *forms a frame with respect to Riesz basis* $\{\phi_{j,k}\}$*, then the* $\alpha_{j,k}$ *are the wavelet coefficients of* Af*, where* $\{d_{i,l}\}$ *is defined as the* $A-$*transform of* $\{c_{j,k}\}$ *such that*

$$\begin{aligned} d_{i,l} &= \sum_{j,k \in N} a_{il\,jk} c_{j,k}, \\ \alpha_{j,k} &= \sum_{i,l \in \chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > . \end{aligned}$$

Proof. Using the definition of $A-$transform of $f = \sum_{il \in \chi} c_{i,l}\varphi_{i,l}$ by assumption we get

$$< Af, \varphi_{i,l} >= \sum_{j,k \in N} a_{il\,jk} c_{j,k} < \phi_{j,k}, \varphi_{i,l} >$$

or

$$\sum_{i,l \in \chi} < Af, \varphi_{i,l} >= \sum_{i,l \in \chi} (Ac)_{i,l} < \phi_{j,k}, \varphi_{i,l} >= \sum_{i,l \in \chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > .$$

Therefore, the wavelet coefficients of Af with respect to Riesz basis $\{\phi_{j,k}\}$ are given by

$$\alpha_{j,k} = \sum_{i,l \in \chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > .$$

Hence the proof is completed. □

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