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Proximate Growth and Best Approximation in L^p-norm of Entire Functions

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Abstract

Let $0 < p \leq +\infty$ and $V_K = \sup\left\{\frac{1}{d}\ln|P_d|, P_d \text{ polynomial of degree} \leq d, \|P_d\|_K \leq 1\right\}$ the Siciak extremal function of a L-regular compact K. The aim of this paper is the characterization of the proximate growth of entire functions of several complex variables by means of the best polynomial approximation in L_p-norm on a L-regular compact K.

Keywords: Extremal function, L-regular, proximate growth, best approximation of entire function, L^p-norm.
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1. Introduction

The classical growth have been characterized in term of approximation errors for a function continuous on $[-1, 1]$ by A.R. Reddy(see (Reddy, 1972*a*), and a compact K of positive capacity by T. Winiarski (see (Winiarski, 1970) with respect to maximum norm. For a nonconstant entire function $f(z) = \sum_{k=0}^{+\infty} a_k.z^{\lambda_k}$ and $M(f, r) = \max_{|z|=r} |f(z)|$, it is well known that the function $r \to \log(M(f, r))$ is indefinitely increasing convex function of $\log(r)$. To estimate the growth of f precisely, R.P. Boas, (see (Boas, 1954)), has introduced the concept of order, defined by the number ρ $(0 \leq \rho \leq +\infty)$:

$$\rho = \limsup_{r\to+\infty} \frac{\log\log(M(f, r))}{\log(r)}.$$

The concept of type has been introduced to determine the relative growth of two functions of same nonzero finite order. An entire function, of order ρ $(0 < \rho < +\infty)$, is said to be of type σ $(0 \leq \sigma \leq +\infty)$ if

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$\sigma = \limsup_{r\to+\infty} \frac{\log(M(f,r))}{r^\rho}.$

If f is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function cannot be precisely measured by the above concept. However S.K. Bajpai, O.P. Juneja and G.P. Kapoor (see (Bajpai *et al.*, 1976)) have introduced the concept of index-pair of an entire function. Thus, for $p \geq q \geq 1$, they defined also the number

$$\rho(p,q) = \limsup_{r\to+\infty} \frac{\log^{[p]}(M(f,r))}{\log^{[q]}(r)}$$

$b \leq \rho(p,q) \leq +\infty$ where $b = 0$ if $p > q$ and $b = 1$ if $p = q$.

The function f is said to be of index-pair (p,q) if $\rho(p-1,q-1)$ is nonzero finite number. The number $\rho(p,q)$ is called the (p,q)-order of f.

S.K. Bajpai, O.P. Juneja and G.P. Kapoor defined also the concept of the (p,q)-type $\sigma(p,q)$, for $b < \rho(p,q) < +\infty$, by

$$\sigma(p,q) = \limsup_{r\to+\infty} \frac{\log^{[p-1]}((M(f,r)))}{\left(\log^{[q-1]}(r)\right)^{\rho(p,q)}}$$

In their works, the authors established the relationship of (p,q)-growth of f in term of the coefficients a_k in the Maclaurin series of f.

We have also many results in terms of polynomial approximation in classical case. Let K be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity and f be a complex function defined and bounded on K. For $k \in \mathbb{N}$ put

$$E_k(K,f) = \left\|f - T_k\right\|_K$$

where the norm $\left\|.\right\|_K$ is the maximum on K and T_k is the $k-th$ Chebytchev polynomial of the best approximation to f on K.

S.N. Bernstein showed (see (Bernstein, 1926), p. 14), for $K = [-1,1]$, that there exists a constant $\rho > 0$ such that

$$\lim_{k\to+\infty} k^{1/\rho}\sqrt[k]{K_k(K,f)}$$

is finite, if and only if, f is the restriction to K of an entire function of order ρ and some finite type.

This result has been generalized by A.R. Reddy (see (Reddy, 1972*a*) and (Reddy, 1972*b*)) as follows:

$$\lim_{k\to+\infty} \sqrt[k]{E_k(K,f)} = (\rho.e.\sigma)2^{-\rho}$$

if and only if f is the restriction to K of an entire function g of order ρ and type σ for $K = [-1,1]$.

In the same way T. Winiarski (see (Winiarski, 1970)) generalized this result for a compact K of the complex plane $\mathbb{C}$, of positive logarithmic capacity noted $c = cap(K)$ as follows:

If K be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity then

$$\lim_{k\to+\infty} k^{\frac{1}{\rho}}\sqrt[k]{E_k(K,f)} = c(e\rho\sigma)^{\frac{1}{\rho}}$$

if and only if f is the restriction to K of an entire function of order ρ $(0 < \rho < +\infty)$ and type σ.

Recall that the capacity of $[-1, 1]$ is $cap([-1, 1]) = \frac{1}{2}$ and the capacity of a unit disc is $cap(D(O, 1)) = 1$.

The authors considered respectively the Taylor development of f with respect to the sequence $(z_n)_n$ and the development of f with respect to the sequence $(W_n)_n$ defined by

$$W_n(z) = \prod_{j=1}^{j=n} (z - \eta_{nj}), \quad n = 1, 2, ...$$

where $\eta^{(n)} = (\eta_{n0}, \eta_{n1}, ..., \eta_{nn})$ is the $n - th$ extremal points system of K (see (Winiarski, 1970), p. 260). We remark that the above results suggest that rate at which the sequence $\left(\sqrt[k]{K_k(K, f)}\right)_k$ tends to zero depends on the growth of the entire function (order and type). For a compact K the Siciak's extremal function of K (see (Siciak, 1962) and (Siciak, 1981)) is defined by:

$$V_K = \sup\left\{\frac{1}{d} \log |P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_K \leq 1\right\}.$$

It is known that the regularity of a compact K (we say K is L-regular) is equivalent to the continuity of V_K in $\mathbb{C}^n$.

Let K be a compact L-regular of $\mathbb{C}^n$. For an entire function f in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see (Zeriahi, 1983)), M. Harfaoui (see (Harfaoui, 2010) and (Harfaoui, 2011)) generalized growth in term of coefficients with respect the sequence $(A_k)_k$. The growth used by M. Harfoui was defined according to the functions α and β (see (Harfaoui, 2010), pp. 5, eq. (2.14)), with respect to the set:

$$\Omega_r = \left\{z \in \mathbb{C}^n, \ exp(V_K)(z) < r\right\}.$$

M. Harfaoui (see (Harfaoui, 2010) and (Harfaoui, 2011)) obtained a result of generalized order and generalized type ((α,β)-order and (α,β)-type) in term of approximation in L^p-norm for a compact of $\mathbb{C}^n$. Later M. Harfaoui an M. El Kadiri (see (Kadiri & Harfaoui, 2013)) obtained the results in term of (p, q)-order and (p, q)-type for the entire functions .

These results will bee used to establish the generalized growth in terms of best approximation in L_p-norm for $p \geq 1$.

Let f be a function defined and bounded on K. For $k \in \mathbb{N}$ put

$$\pi_k^p(K, f) = \inf\left\{\left\|f - P\right\|_{L^p(K,\mu)}, \ P \in \mathcal{P}_k(\mathbb{C}^n)\right\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and μ the well-selected measure (The equilibrium measure $\mu = (dd^c V_K)^n$ associated to a L-regular compact K) (see (Zeriahi, 1987)) and $L^p(K, \mu)$, $p \geq 1$, is the class of all functions such that:

$$\left\|f\right\|_{L^p(K,\mu)} = \left(\int_K |f|^p \, d\mu\right)^{1/p} < \infty.$$

For an entire function $f \in \mathbb{C}^n$, M. Harfaoui and M. El Kadiri established a precise relationship between the (p, q)-growth and the general growth ((α, α)-growth) with respect to the set (see

((Harfaoui, 2010), (Harfaoui, 2011), (Kadiri & Harfaoui, 2013) and (Harfaoui & Kumar, 2014)) and the coefficients of the development of f with respect to the sequence $(A_k)_k$. He used these results to give the relationship between the generalized growth of f and the sequence $(\pi_k^p(K, f))_k$.

To our knowledge no work is discussed in term of best approximation in L_p-norm with respect to the proximate growth.

The aim of this paper is to give the proximate growth and the $(m, 1)$- proximate growth of entire functions in $\mathbb{C}^n$ $(m \in \mathbb{N}^*)$ by means of the best polynomial approximation in term of L^p-norm, with respect to the set

$$\Omega_r = \left\{ z \in \mathbf{C}^n ; \exp V_K(z) \leq r \right\}.$$

In the paper of A. R. Reddy and T. Winiarski (see (Reddy, 1972*a*), (Reddy, 1972*b*) and (Winiarski, 1970)) the authors use the development of f in the basis $(z_n)_n$ and $(W_n)_n$ and used the Cauchy inequality.

In our work we use a new basis of extremal polynomial and we replace the the Cauchy inequality by an inequality given by A. Zeriahi (see (Zeriahi, 1983)).

So we establish relationship between the rate at which $\left(\pi_k^p(K, f)\right)^{1/k}$, for $k \in \mathbb{N}$, tends to zero in terms of best approximation in L^p-norm, and the proximate growth growth of entire functions of several complex variables for a L-regular compact K of $\mathbb{C}^n$.

2. Notations aned auxiliary results

Before we give some definitions and results which will be frequently used.

For $p \in \mathbb{Z}$ put

$$\log^{[p]}(x) = \log(\log^{[p-1]}(x)); \quad \log^{[0]}(x) = x; \Lambda_{[p]} = \prod_{k=1}^{p} \log^{[k]}(x).$$

$$\exp^{[p]}(x) = \exp(\exp^{[p-1]}(x)); \quad \exp^{[0]}(x) = x \text{ and } \; E_{[p]}(x) = \prod_{k=0}^{p} \exp^{k}(x).$$

Lemma 2.1. *(see (Bajpai et al., 1976))*

With the above notations we have the following results

$$(RR_1) \quad E_{[-p]}(x) = \frac{x}{\bigwedge_{[p-1]}(x)} \text{ and } \bigwedge_{[-p]}(x) = \frac{x}{E_{[p-1]}(x)}$$

$$(RR_2) \quad \frac{d}{dx} \exp^{[p]}(x) = \frac{E_{[p]}(x)}{x} = \frac{1}{\bigwedge_{[-p-1]}(x)}$$

$$(RR_3) \quad \frac{d}{dx} \log^{[p]}(x) = \frac{E_{[-p]}(x)}{x} = \frac{1}{\bigwedge_{[p-1]}(x)}$$

$$(RR_4) \quad E_{[p]}^{-1}(x) = \begin{cases} x, & \text{if } p = 0 \\ \log^{[p-1]}\{\log(x) - \log^{[2]}(x) + o(\log_{[3]}(x))\}, & \text{if } p = 1, 2, \ldots \end{cases}.$$

(RR₅) $\lim_{x\to+\infty} \exp\left(E_{[p-2]}(x)\right) = \begin{cases} e & if\ p = 2 \\ 1 & if\ p \geq 3 \end{cases}$

(RR₆) $\lim_{x\to+\infty} \left[\exp^{[p-1]}\left(E^{-1}_{[p-2]}(x)\right)\right]^{\frac{1}{x}} = \begin{cases} e & if\ p = 2 \\ 1 & if\ p \geq 3 \end{cases}$

It is known that if K is a compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see (Siciak, 1962) and (Siciak, 1981)), in particular, we have:

(P_1) Bernstein-Markov inequality:

$\forall\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \left\|P_d\right\|_K = C(1+\varepsilon)^{s_k}\left\|P_d\right\|_{L^2(K,\mu)}, \tag{2.1}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Waish (B.W) inequality:

For every set L-regular K and every real $r > 1$ we have:

$$\left\|f\right\|_K \leq M.r^{deg(f)}\left(\int_K |\ f\ |^p\ .d\mu\right)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

For $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ..., s_n(k))$ be a bijection such that

$$|s(k+1)| \geq |s(k)| \ \text{ where } \ |s(k)| = s_1(k) + ... + s_n(k),$$

A. Zeriahi (see (Zeriahi, 1983)) constructed according to the Hilbert Schmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see (Siciak, 1962)), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)} \tag{2.3}$$

such that

$$\left\|A_k\right\|_{L^p(E,\mu)} = \left[\inf\left\{\left\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\right\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}.$$

We need the following notations which will be used in the sequel:

(N_1) $\nu_k = \nu_k(K) = \left\|A_k\right\|_{L^2(K,\mu)}$.

(N_2) $a_k = a_k(K) = \left\|A_k\right\|_K = \max_{z\in K}\left|A_k(z)\right|$ and $\tau_k = \left(a_k\right)^{1/s_k}$,

where $s_k = deg(A_k)$.

With that notations and (B.W) inequality we have

$$\left\|A_k\right\|_{\overline{\Omega}_r} \leq a_k.r^{s_k} \tag{2.4}$$

where $s_k = deg(A_k)$.

Lemma 2.2. *(see (Zeriahi, 1983))*
Let K be a compact L-regular subset of $\mathbb{C}^n$. Then

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_K(z)), \tag{2.5}$$

for every $z \in \mathbb{C}^n \setminus \widehat{K}$ the connected component of $\mathbb{C}^n \setminus K$,

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_K}{\nu_k}\right]^{1/s_k} = 1. \tag{2.6}$$

3. Growth with respect to the proximate order and coefficient with respect to extremal polynomial.

Before we give some definitions and results which will be frequently used in this paper.

Definition 3.1.
Let ρ be a positive real such that $0 < \rho < +\infty$. A proximate order for ρ is a function $\rho(r)$ defined in $\mathbb{R}^+$ and verified:

1. $\lim\limits_{r\to+\infty} \rho(r) = \rho$;
2. $\lim\limits_{r\to+\infty} r\rho' \log(r) = 0$.

Example 3.1. The function $\rho(r)$ defined by

$$r^{\rho(r)} = r^{\rho}(\ln(r))^{\beta_1}.(\ln^{[2]}(r))^{\beta_2})...(\ln^{[m](r)})^{\beta_m}$$

is a proximate order for ρ, where $\log^{[m](r)}$ is defined by:

$$\log^{[0]}(r) = r, \ \ \log^{[m](r)} = \ln^+(\log^{[m-1]}(r)) \ \text{ and } \ \ln^+(t) = 1_{[1;+\infty[}\ln(t)$$

Theorem 3.1. *If $h(r)$ is a positive function for $r > 0$ such that*

$$\lim_{r\to+\infty}\frac{\log(h(r))}{\log(r)} = \rho < +\infty,$$

then the proximate order $\rho(r)$ maybe chosen such that for every $r > 0$: $h(r) \leq r^{\rho(r)}$, and for some sequence $r_n^{\rho(r_n)}$, $h(r_n) \leq r_n^{\rho(r_n)}$, for n sufficiently large.

For an entire function in $\mathbb{C}^n$ we define the K-type for the proximate order as follows:

Definition 3.2.
Let K be a L-regular of $\mathbb{C}^n$. If for an entire function in $\mathbb{C}^n$

$$\limsup_{r\to+\infty} \frac{\log\left(M_K(f,r)\right)}{r^{\rho(r)}} \tag{3.1}$$

is finite not zero then the function $\rho(r)$ is called proximate order

$$\sigma_K = \lim_{r\to+\infty} \frac{\ln\left(M_K(f,r)\right)}{r^{\rho(r)}} \tag{3.2}$$

is called K-type of f with respect to the proximate order $\rho(r)$, where

$$M_K(f,r) = \sup_{z\in\Omega_r} \left|f(z)\right|.$$

Let K be a compact L-regular and f an entire function of several variables and $f(z) = \sum_{k=0}^{+\infty} f_k.A_k$ the development of f with respect to the sequence of extremal polynomials.

2.1. K-type of f with respect to the proximate order

Theorem 3.2.
If $\rho(r)$ is a proximate order for ρ then the K-type of f with respect to the proximate order is given by the formula:

$$\sigma_K = \frac{1}{e.\rho} \limsup_{k\to+\infty} \left(\varphi(s_k)\tau_k\right)^{\rho}.\left|f_k\right|^{\rho/s_k}, \tag{3.3}$$

where φ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.
We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.

Lemma 3.1. *[7, p.42(1.58)]*
For every $k > 0$ we have

$$\limsup_{t\to+\infty} \frac{\varphi(k.t)}{\varphi(t)} = k^{1/\rho}.$$

Proof of theorem 3.2.
Put $\sigma = \frac{1}{e.\rho} \limsup_{k\to+\infty} \left(\varphi(s_k)\tau_k\right)^{\rho}.\left|f_k\right|^{\rho/s_k}$ and show that $\sigma = \sigma_K$.
Show that $\sigma \leq \sigma_K$.

We have for every $\theta > 1$ $\sigma_K = \lim_{r\to+\infty} \frac{\ln\left(M_K(f,r\theta)\right)}{r^{\rho(r\theta)}}$, then for every $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for every $r > r(\varepsilon)$

$$\log\left(\|f\|_{\overline{\Omega}_{r\theta}}\right) \leq (r\theta)^{r\theta}(\sigma(K,f) + \varepsilon). \tag{3.4}$$

But $(r+1)^{N_\theta}\|f\|_{\overline{\Omega}_{r\theta}} \leq \exp\left((\sigma_{K,f} + \varepsilon)(r\theta)^{r\theta}\right)$, where $N_\theta \in \mathbb{N}$ such that

$$\left|f_k\right|\nu_k \leq C_\theta.r^{-s_k}.\frac{(r+1)^{N_\theta}}{(r-1)^{2n-1}}\|f\|_{\overline{\Omega}_{r\theta}} \tag{3.5}$$

then

$$\left|f_k\right|\nu_k \leq C_\theta.r^{-s_k}.\exp\left((\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}\right),$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$ or

$$\log\left(\left|f_k\right|\nu_k\right) \leq \log(C_\theta) - s_k\log(r) + \left((\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}\right), \tag{3.6}$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$.

Chose r such that $s_k = [(\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}.]$, where $[x]$ means the integer part of x. Then $s_k \leq (\sigma(K,f)+\varepsilon)(r\theta)^{r\theta} < s_k + 1$. Replacing in the relation (3.6) we get

$$\log\left(\left|f_k\right|\nu_k\right) \leq \log(C_\theta) - s_k\log(r) + s_k\log(\theta) + \frac{s_k+1}{\rho}. \tag{3.7}$$

Since $\frac{s_k}{\rho(\sigma(K,f)+\varepsilon)} \leq (r\theta)^{r\theta}$, then $\varphi(\frac{s_k}{\rho(\sigma(K,f)+\varepsilon)}) \leq r\theta$, thus

$$\log\left[(\tau_k.\varphi(s_k))^\rho(\left|f_k\right|)^{\rho/s_k}\right] \leq \frac{\rho}{s_k}\log(C_\theta) + \rho\log\Big(\frac{\varphi(s_k)}{\varphi(\frac{s_k}{\rho.(\sigma(K,f)+\varepsilon)})}\Big) + 1 + \frac{1}{s_k}.$$

After passing to the upper limit and applying the lemma 2.1, the relation (2.6) of the lemma 2.2 and the lemma 3.1 we get

$$\limsup_{k\to+\infty}\log\left[(\varphi(s_k))^\rho(\nu_k.\left|f_k\right|)^{\rho/s_k}\right] \leq \log\big(\rho.\sigma(K,f)\big) + 1 = \log\big(e.\rho.(\sigma(K,f))\big). \tag{3.8}$$

which gives the result

$$\limsup_{r\to+\infty}(\tau_k.\varphi(s_k))^\rho(\left|f_k\right|)^{\rho/s_k} \leq e.\rho.(\sigma(K,f)). \tag{3.9}$$

Show that $\sigma \geq \sigma_K$. If $\sigma < \sigma_K$ let σ_1 and σ_2 such that $\sigma < \sigma_1 < \sigma_2 < \sigma_K$. There exists $k_1$such that for every $k > k_1$:

$$(\tau_k)^{s_k}.|f_k| \leq \frac{e.\rho.(\sigma_1)^{1/\rho}}{\varphi(s_k)} \tag{3.10}$$

as we have also for k sufficiently large ($k > q_2$), $(\sigma_1.\rho)^{1/\rho}.\frac{\varphi(\frac{s_k}{\sigma_1.\rho})}{\varphi(s_k)}$, then for $k_0 = \max(q_1, q_2)$ we have

$$M_K(f,r) \leq \sum_{k=0}^{k_0}|f_k|.\|A_k\|_{\overline{\Omega}_r} + \sum_{k=k_0+1}^{+\infty}|f_k|.\|A_k\|_{\overline{\Omega}_r}. \tag{3.11}$$

According to the Bernstein-Walsh inequality we have

$$\|A_k\|_{\overline{\Omega}_r} \leq a_k(K).r^{s_k},$$

and according to the Bernstein-Markov inequality we have

$$a_k(K) \leq A_\epsilon.(1+\epsilon)^{s_k}a_k(K).\tau_k^{s_k}.$$

Thus

$$M_K(f,r) \leq C_0.r^{s_k} + A_\epsilon. \sum_{k=k_0+1}^{+\infty} \Big(\frac{e^{1/\rho}}{\varphi(s_k/\sigma_1.\rho)}\Big)^{s_k}.((1+\epsilon))^{s_k}. \tag{3.12}$$

If we put $\delta = \dfrac{\sigma_1}{\sigma_2}$ $(\delta < 1)$ then

$$M_K(f,r) \leq C_0.r^{s_k} + A_\epsilon. \sum_{k=k_0+1}^{+\infty} \Big(\frac{1+\epsilon}{1-\epsilon}\Big)^{s_k}. \sup_{k>k_0} e^{\Psi(s_k)}.$$

where

$$\Psi(x) = x\log(r) - \frac{x}{\rho} - x\log\Big(\varphi(x/\sigma_2.\rho)\Big).$$

If we choose ϵ such that $0 < \epsilon < \dfrac{1-\delta}{1+\delta}$ then $\dfrac{\delta(1+\epsilon)}{1-\epsilon} < 1$ and thus

$$M_K(f,r) \leq C_0.r^{s_k} + C. \sup_{k>k_0} e^{\Psi(s_k)}.$$

We note that $\Psi(x) = 0$ is equivalent to

$$\log(r) + \frac{1}{\rho} - \frac{x}{\sigma_2.\rho}.\frac{\varphi'(x/\sigma_2.\rho)}{\varphi(x/\sigma_2.\rho)} - \log\Big(\varphi(x/\sigma_2.\rho)\Big) = 0, \tag{3.13}$$

then the solution x_r of the equation (3.13) verify

$$\log(r) - \frac{\rho}{\epsilon} < \varphi(\frac{x_r}{\sigma_2.\rho}) < \log(r) + \frac{\rho}{\epsilon} \text{ for } r > r_1$$

and thus

$$\begin{cases} \Psi(x) \leq \dfrac{x_r}{\rho} + x_r\Big(\log(r) - \log\big(\varphi(\dfrac{x_r}{\sigma_2.\rho})\big)\Big) \leq (1+\epsilon)\dfrac{x_r}{\rho} \\ \dfrac{x_r}{\sigma_2} \leq (\theta.r)^{\varphi(\theta)} \text{ where } \theta = e^{\epsilon/\rho} \end{cases}$$

Since for every $\theta > 1$ we have $(\theta.r)^{\varphi(\theta)} \leq (\theta.r)^{\rho+\epsilon}.r^{\rho(r)}$ then

$$e^{\Psi(x_r)} \leq e^{(1+\epsilon).\theta^{\rho+\epsilon}}.\sigma_2.r^{\rho(r)} \text{ for } r > r_1$$

and consequently, for $r > r_1$,

$$M_K(f,r) \leq C_0.r^{s_{k_0}} + A.\theta^{\rho+\epsilon}.\sigma_2.r^{\rho(r)}.$$

whence

$$\frac{\log\big(M_K(f,r)\big)}{r^{\rho(r)}} \leq \sigma_1 + o(1),$$

passing to the upper limit we get $\sigma(K,f) \leq \sigma_1$. Which leads a contradiction and this shows the result.

2.2.(K,m)-type of f with respect to the proximate order

For the entire functions infinite order we introduce the notion of m-order defined by:

$$\rho_m = \limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r))}{\log(r)}, \tag{3.14}$$

for $m \geq 2$. The function f is said to be of index-pair $(m,1)$ if $\rho_{m-1} = +\infty$ and $\rho_m < +\infty$. The number ρ_m is called the m-order of f.

Definition 3.3.
If $\rho(r)$ is a proximate order associated to the m-order ρ_m, the (K,m)-type with respect to the proximate order $\rho(r)$ is defined by:

$$\sigma_m(K,f) = \limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \tag{3.15}$$

Let $f = \sum_{k=0}^{+\infty} f_k(f).A_k$ the development of f with respect to the sequence of extremal polynomials.

Theorem 3.3.
The (K,m)-type of f with respect to the proximate order is given by the formula:

$$\sigma_m(K,f) = \limsup_{k\to+\infty} \Big(\varphi(\log^{[m-2]}(s_k))\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k}, \tag{3.16}$$

for $m > 2$.

Proof of theorem 3.3.
Put $\rho_m = \rho$ and $\sigma = \limsup_{k\to+\infty} \Big(\varphi(\log^{[m]}(s_k))\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k}$.
Show that $\sigma_m(K,f) \leq \sigma$.
We have for every $\epsilon > 0$ there exists k_0 such that for every $k > k_0$

$$\varphi(\log^{[m-2]}(s_k))\tau_k.\big|f_k\big|^{1/s_k} \leq \sigma^{1/\rho} + \epsilon, \tag{3.17}$$

thus

$$M_K(f,r) \leq C_0 r^{s_{k(r)}} + \sum_{k=0}^{k_0} |f_k|.\|A_k\|_{\overline{\Omega}_r} + \sum_{k=k_0+1}^{+\infty} \Big(\frac{\sigma^{1/\rho}+\epsilon}{\varphi(\log^{[m-2]}(s_k))}\Big)^{s_k}.r^{s_k}. \tag{3.18}$$

For $\sigma_1 > \sigma$ we have

$$\Big(\frac{\sigma^{1/\rho}+\epsilon}{\varphi(\log^{[m-2]}(s_k))}\Big)^{s_k}.r^{s_k} \leq \Big(\frac{\sigma^{1/\rho}+\epsilon}{\sigma_1^{1/\rho}+\epsilon}\Big)^{s}.\sup_{k>k_0} e^{\Psi(s_k)},$$

where

$$\Psi(x) = x\log(r) + x\log(\sigma^{1/\rho}+\epsilon) + x\log(\varphi(\log^{[m-2]}(x)).$$

The solution x_r of the equation $\Psi'(x) = 0$ verify, for r sufficiently large ($r > r_1$)

$$\Psi(x_r) \leq \epsilon.exp^{[m-2]}((1+\epsilon).\theta^{\rho+\epsilon}.r^{\rho(r)}), \text{ where } \theta = (\sigma^{1/\rho} + \epsilon).e^{\epsilon}$$

therefore

$$M_K(f,r) \leq C_0 r^{s_{k(r)}} + A.e^{\Psi(x_r)}, \text{ where A is a constant.}$$

This gives $\limsup_{r\to+\infty} \dfrac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \leq \sigma_1$ and since this is true for every $\sigma_1 > \sigma$ then

$$\limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \leq \sigma.$$

Show now that $\sigma_m(K,f) \geq \sigma$.

By definition of $\sigma_m(K,f)$ we have for every $\epsilon > 0$ there exists $r_0(\epsilon)$ such that for every $r > r_0(\epsilon)$,

$$M_K(f,r) \leq \exp^{[m-2]}\left[(\sigma_m(K,f)+\epsilon)(r\theta)^{r\theta}\right], \text{ and } \theta > 1,$$

thus

$$\left|f_k\right|.\tau_k^{s_k} \leq C_0'.\sup_{k>k_0} \exp^{\Psi(s_k)}.$$

where

$$\Psi(x) = -s_k \log\left(\frac{r}{1+\epsilon}\right) + \exp^{[m-2]}\left[(\sigma_m(K,f)+\epsilon)(r\theta)^{r\theta}\right].$$

For r sufficiently large the solution of the equation $\Psi'(x) = 0$ verify

$$E^{-1}_{[m-2]}\Big(s_k(\frac{1}{\rho}-1)\Big) \leq (\sigma_m(K,f)+\epsilon)(r_k\theta)^{r_k\theta} \leq E^{-1}_{[m-2]}(s_k(\frac{1}{\rho}+1)). \tag{3.19}$$

Using the relation (3.19) an elementary calculus gives

$$\left|f_k\right|.\tau_k^{s_k}.\varphi\Big(E^{-1}_{[m-2]}(s_k(\frac{1}{\rho}-1))\Big) \leq (\sigma_m(K,f)+\epsilon)^{1/\rho}.\exp^{[m-1]}\Big[E^{-1}_{[m-2]}(s_k(\frac{1}{\rho}+1))\Big]. \tag{3.20}$$

Therefore passing to the upper limit and using the propriety of the function $x \to E_{[m-2]}(x)$ we obtain the result.

4. Best polynomial approximation in terms of L^p-norm.

The object of this section is to study the relationship of the rate of the best polynomial approximation of f in L^p-norm with the -growth with respect to the proximate order of an entire function g such that $g_{/K} = f$.

More precisely we show the following theorem:

Theorem 4.1.

If $\rho(r)$ is a proximate order for $\rho$and f and let $f \in L^p(K,\mu)$ for $p > 0$. Then f is μ-almost-surely the restriction to K of an entire function in $\mathbb{C}^n$, f_1, of finite nonzero order ρ and K-type $\sigma(K, f_1) \in]0, +\infty5$ with respect to the proximate order $\rho(r)$ for ρ if and only if

$$\sigma(K, f_1) = \frac{1}{e.\rho} \limsup_{k\to+\infty} (\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k}, \tag{4.1}$$

where φ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.

We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.

Proof of theorem 4.1.

Suppose that f is μ-almost-surely the restriction to K of an entire function in $\mathbb{C}^n$, f_1, of finite nonzero order ρ and K-type $\sigma(K, f_1) \in]0, +\infty5$ with respect to the proximate order $\rho(r)$ for ρ. We have $f_1 \in L^2(K,\mu)$ and

$$f_1 = \sum_{k=0}^{+\infty} f_k.A_k.$$

Put $\sigma = \dfrac{1}{e.\rho} \limsup_{k\to+\infty} \Big(\varphi(s_k)\tau_k\Big)^{\rho}.\Big|f_k\Big|^{\rho/s_k}$

By the relation (92) for $p \geq 2$ and the relation (96) for $p \in [1, 2[$ of the paper of M. El Kadiri and M. Harfaoui (see (Kadiri & Harfaoui, 2013))

$$\Big(\varphi(s_k)\Big)^{\rho}.(\nu_k.|f_k|)^{\rho/s_k} \leq (A_\epsilon)^{\rho/s_k}\Big(\varphi(s_k)\Big)^{\rho}(1+\epsilon)^{\rho}.(\mathcal{E}_k^p)^{\rho/s_k} \tag{4.2}$$

then

$$\Big(\varphi(s_k)\tau_k\Big)^{\rho}.(|f_k|)^{\rho/s_k} \leq \Big(\varphi(s_k)\Big)^{\rho}(\Big|f_k\Big|.\nu_k)^{\rho/s_k}.\Big(\frac{\tau_k^{s_k}}{\nu_k}\Big)^{\rho/s_k} \tag{4.3}$$

By the relation 3.6 we have

$$\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (A_\epsilon)^{\rho/s_k}.\Big[\Big|f_k\Big|.\nu_k.(1+\epsilon)^{s_k+1} + ...\Big]. \tag{4.4}$$

But

$$\sigma' = \limsup_{k\to+\infty} \Big(\varphi(s_k)\Big)^{\rho}.(\nu_k.\Big|f_k\Big|^{\rho/s_k} = e.\rho.\sigma.$$

Thus, for k sufficiently large

$$\varphi(s_k).(\nu_k.\Big|f_k\Big|)^{1/s_k} \leq (\sigma')^{1/\rho} + \epsilon \Leftrightarrow \nu_k.\Big|f_k\Big| \leq \Big[\frac{(\sigma')^{1/\rho} + \epsilon}{\varphi(s_k)}\Big]^{s_k}.$$

Hence for every $j \in \mathbb{N}$;

$$\nu_{k+j}.\Big|f_{k+j}\Big| \leq \Big[\frac{(\sigma')^{1/\rho} + \epsilon}{\varphi(s_{k+j})}\Big]^{s_{k+j}}.$$

Then, if we put $S = (1+\epsilon)^{s_k}.\nu_k.\big|f_k + (1+\epsilon)^{s_{k+1}}.\nu_{k+1}.\big|f_{k+1}....$, we have

$$S \leq \sum_{j=0}^{+\infty} \epsilon)^{s_{k+j}} [\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_{k+j})}]^{s_{k+j}}$$

which is equivalent to

$$S \leq \Big[\frac{(\sigma)^{1/\rho}+\epsilon}{\varphi(s_{k+j})}\Big]^{s_k} \sum_{j=0}^{+\infty}(1+\epsilon)^{s_{k+j}} \frac{\Big[(\sigma')^{1/\rho}+\epsilon\Big]^{s_k+j}}{\big[(\sigma)^{1/\rho}+\epsilon\big]^{s_k}} \Big[\frac{\big[\varphi(s_k)\big]^{s_k}}{\big[\varphi(s_{k+j})\big]^{s_{k+j}}}\Big]^{s_{k+j}}$$

or

$$S \leq \Big[\frac{(\sigma)^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{s_k} \sum_{j=0}^{+\infty}(1+\epsilon)^{s_{k+j}} \frac{\Big[(\sigma')^{1/\rho}+\epsilon\Big]^{s_k+j}}{\big[(\sigma)^{1/\rho}+\epsilon\big]^{s_k}} \frac{\Big[\varphi(s_k)\Big]^{s_k}}{\Big[\varphi(s_k+j)\Big]^{s_k}+j}$$

Since $\dfrac{\varphi(s_k)}{\varphi(s_k+j)} \leq 1$ we get also

$$S \leq (1+\epsilon)^{s_k}.\frac{\Big((\sigma')^{1/\rho}+\epsilon\Big)^{s_k}}{\varphi(s_k)}.\sum_{j=0}^{+\infty}\Big[(1+\epsilon).\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(j)}\Big]^j.$$

As for k sufficiently large $\dfrac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k+j)} < 1$ the series is convergent to a finite sum L and we will get finally

$$\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (1+\epsilon)^{\rho}(A_\epsilon)^{1/s_k}.L^{\rho/s_k}.\Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{\rho}$$

which equivalent to

$$(\varphi(s_k))^{\rho}.\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (1+\epsilon)^{\rho}(A_\epsilon)^{\rho/s_k}.L^{\rho/s_k}.((\sigma')^{1/\rho}+\epsilon^{\rho}).$$

Passing to the upper limit get

$$\sigma(K,f_1) = \frac{1}{e.\rho}\limsup_{k\to+\infty}(\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k} \leq \sigma.$$

Conversely, suppose now that f satisfies the relation 4.6. We show the result by three steps.

If $f \in L^p(K,\mu)$ with $p \geq 2$ then $f \in L^2(K,\mu)$ and we have $\sum_{k=0}^{+\infty} f_k A_k$ with convergence in $L^2(K,\mu)$, where

$$f_k = \frac{1}{\nu_k^2}\int_K f.\overline{A}_k \quad (k \geq 0).$$

We verify easily by the relations 3.3, 3.6 and the inequality (B.M):

$$\limsup_{k\to+\infty}\Big(\varphi(s_k)\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k} = \limsup_{k\to+\infty}(\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k}. \tag{4.5}$$

By this inequality the series $\sum f_k A_k$ considered in $\mathbb{C}^n$ converges normally on every compact of $\mathbb{C}^n$ to a function denoted f_1 by the inequality (B.M) and the inequality of the coefficient of $|f_k|$. We have obviously $f_1 = f$ μ-a.s on K and the proof is completed by the theorem3.1.

If $p \in [0, 2[$ we take p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, then $p' \geq 2$. Applying the previous arguments of the first step to p' and Hölder and Bernstein inequality we obtain the result.

If $0 < p < 1$, of course, for $0 < p < 1$ the L_p-norm does not satisfy the triangle inequality. But our relations (4.2) and (4.3) are also satisfied for $0 < p < 1$ (see (Harfaoui & Kumar, 2014)), because using Holder's inequality we have, for some $M > 0$ and all $r > p$ (p fixed)

$$\| f \|_{L^p(K,\mu)} \leq M. \| f \|_{L^r(K,\mu)} .$$

Using the inequality

$$\int_K | f |^p \, d\mu \leq \| f \|_K^{p-r} . \int_K | f |^r \, d\mu$$

we get

$$\| f \|_{L^p(K,\mu)} \leq \| f \|_K^{1-(r/p)} . \| f \|_{L^r(K,\mu)}^{r/p} .$$

We deduce that (K, μ) satisfies the Bernstein-Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that, for all (analytic) polynomials P we have

$$\| P \|_K \leq C(1 + \epsilon)_{deg(P)}. \ \| P \|_{L^p(K,\mu)} .$$

Thus if (K, μ) satisfies the Bernstein-Markov inequality for one $p > 0$ then (4.2) and (4.3) are satisfied for all$p > 0$.

The rest of proof is easily deduced using the same reasoning as in step.1 and step.2

Theorem 4.2.

If $\rho(r)$ is a proximate order for $\rho_m \in]0, +\infty[$ ($m > 2$), and f and let $f \in L^p(K, \mu)$ for $p > 0$. Then f is μ-almost-surely (μ-a.s) the restriction to K of an entire function in $\mathbb{C}^n$, f_1, of finite nonzero m-order ρ_m and (K, m)-type $\sigma_m(K, f_1) \in]0, +\infty 5$ with respect to the proximate order $\rho(r)$ for ρ if and only if

$$\sigma_m(K, f_1) = \limsup_{k \to +\infty} \Big(\varphi((\log^{[m-2]}(k))) \Big)^{\rho_m} . (\mathcal{E}_k^p)^{\rho_m / k}, \tag{4.6}$$

where φ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.
We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.

Proof of theorem 4.2.

The theorem can be proved on similar lines as those of the proof of the theorem 4.1 because the relations (4.2) and (4.3) are still valid by iteration of logarithm . Hence we omit the proof.

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