Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 4 (2) (2014) 169–184

Some Results in Connection with the Bounds for the Zeros of Entire Functions in the Light of Slowly Changing Functions

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Abstract

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. In this paper we would like to establish the bounds for the moduli of zeros of entire functions on the basis of slowly changing functions.

Keywords: Zeros of entire functions, proper ring shaped region, slowly changing functions.
2010 MSC: 30C15, 30C10, 26C10.

1. Introduction, Definitions and Notations.

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots\dots + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0$$

be a polynomial of degree n. Datt and Govil (Datt & Govil, 1978); Govil and Rahaman(Govil & Rahaman, 1968); Marden (Marden, 1966); Mohammad (Mohammad, 1967); Chattopadhyay, Das, Jain and Konwer (Chattopadhyay, 2005); Joyal, Labelle and Rahaman (Joyal, Labelle & Rahaman 1967) Jain (Jain, 1976), (Jain, 2006) Sun and Hsieh (Sun & Hsie, 1996); Zilovic, Roytman, Combettes and Swamy (Zilovic, Roytman); Das and Datta (Das & Datta, 2008) etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions on the basis of slowly changing functions.

The following definitions are well known :

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Definition 1.1. (Valiron, 1949)The order ρ and lower order λ of an entire function f are defined as

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker (Singh & Barker, 1977) defined it in the following way:

Definition 1.2. (Singh & Barker, 1977) A positive continuous function $L(r)$ is called a slowly changing function if for ε (> 0),

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \text{ for } r > r(\varepsilon) \quad \text{and}$$

uniformly for $k(\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to $\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0$.

Somasundaram and Thamizharasi (Somasundaram & Thamizharasi, 1988) introduced the notions of L-order and L-lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

Definition 1.3. (Somasundaram & Thamizharasi, 1988) The L-order ρ^L and the L-lower order λ^L of an entire function f are defined as

$$\rho^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}.$$

The more generalised concept for L-order and L-lower order are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 1.4. The L^*-order ρ^{L^*} and the L^*-lower order λ^{L^*} of an entire function f are defined as

$$\rho^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]}.$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *If $f(z)$ is an entire function of L-order ρ^L, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [rL(r)]^{\rho^L + \varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [rL(r)]$.

Proof. Let us suppose that $f(0) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c \cdot \frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m.

We further assume that $f(z)$ has no zeros on $|z| = 2[rL(r)]$ and the zeros z_i's of $f(z)$ in $|z| < [rL(r)]$ are in non decreasing order of their moduli so that $|z_i| \le |z_{i+1}|$. Also let ρ^L suppose to be finite.

Now we shall make use of Jenson's formula as state below

$$\log |f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log \left| f\left(R\, e^{i\phi}\right)\right| d\phi. \tag{2.1}$$

Let us replace R by $2r$ and n by $N(2r)$ in (2.1).

$$\therefore \log |f(0)| = -\sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log \left| f\left(2r\, e^{i\phi}\right)\right| d\phi.$$

Since $f(0) = 1, \therefore \log |f(0)| = \log 1 = 0$.

$$\therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log \left| f\left(2r\, e^{i\phi}\right)\right| d\phi. \tag{2.2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \ge \sum_{i=1}^{N(r)} \log \frac{2r}{|z_i|} \ge N(r)\log 2 \tag{2.3}$$

because for large values of r, $\log \frac{2r}{|z_i|} \ge \log 2$.

$$\text{R.H.S} = \frac{1}{2\pi}\int_0^{2\pi} \log \left| f\left(2r\, e^{i\phi}\right)\right| d\phi \le \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r). \tag{2.4}$$

Again by definition of order ρ^L of $f(z)$ we have fore every $\varepsilon > 0$, and as $L(2r) \sim L(r)$,

$$\log M(2r) \le [2rL(2r)]^{\rho^L + \varepsilon/2} \log M(2r) \le [2rL(r)]^{\rho^L + \varepsilon/2}. \tag{2.5}$$

Hence from (2.2) by the help of (2.3), (2.4) and (2.5) we have

$$N(r)\log 2 \le [2rL(r)]^{\rho^L + \varepsilon/2}$$

$$N(r) \le \frac{2^{\rho^L + \varepsilon/2}}{\log 2} \cdot \frac{[rL(r)]^{\rho^L + \varepsilon}}{[rL(r)]^{\varepsilon/2}} \le [rL(r)]^{\rho^L + \varepsilon}.$$

This proves the lemma. □

In the line of Lemma 2.1, we may state the following lemma:

Lemma 2.2. *If $f(z)$ is an entire function of L^*-order ρ^{L^*}, then for every $\varepsilon > 0$ the inequality*

$$N(r) \le [re^{L(r)}]^{\rho^{L^*}+\varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \le [re^{L(r)}]$.

Proof. With the initial assumptions as laid down in Lemma 1, let us suppose that $f(z)$ has no zeros on $|z| = 2[re^{L(r)}]$ and the zeros z_i's of $f(z)$ in $|z| < [re^{L(r)}]$ are in non decreasing order of their moduli so that $|z_i| \le |z_{i+1}|$. Also let ρ^{L^*} supposed to be finite.

In view of (2.1), (2.2), (2.3) and (2.4), by definition of ρ^{L^*} and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\log M(2r) \le [2re^{L(2r)}]^{\rho^{L^*}+\varepsilon/2}, i.e., \log M(2r) \le [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}.$$

Hence by the help of (2.3), (2.4) and (2) we obtain from (2.2) that

$$N(r)\log 2 \le [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}, N(r) \le \frac{2^{\rho^{L^*}+\varepsilon/2}}{\log 2}\cdot\frac{[re^{L(r)}]^{\rho^{L^*}+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \le [re^{L(r)}]^{\rho^{L^*}+\varepsilon}.$$

Thus the lemma is established. □

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L-order ρ^L. Also for all sufficiently large r in the disc $|z| \le [rL(r)]$, $\left|a_{N(r)}\right| \ne 0$, $|a_0| \ne 0$. and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \le |z| \le t_0$$

where t_0 is the greatest positive root of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$$

and t_0' is the greatest positive root of

$$f(t) \equiv |a_0|\, t^{N(r)+1} - (|a_0| + M')\, t^{N(r)} + M' = 0$$

$$\text{where } M = max\left\{|a_0|, |a_1|, ... \left|a_{N(r)-1}\right|\right\}$$
$$\text{and } M' = max\left\{|a_1|, |a_2|, ... \left|a_{N(r)}\right|\right\}.$$

Proof. Now
$$P(z) \approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}$$
because $N(r)$ exists for $|z| \le [rL(r)]$; r is sufficiently large and $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region given in Theorem 3.1 which we are to prove.

Now
$$\begin{aligned} |P(z)| &\approx \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}\right| \\ &\ge \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right|. \end{aligned}$$
Also
$$\left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \le |a_0| + ... + \left|a_{N(r)-1}\right| |z|^{N(r)-1} \le M\left(1 + |z| + ... + |z|^{N(r)-1}\right)$$
$$= M\frac{|z|^{N(r)} - 1}{|z| - 1} \text{ if } |z| \ne 1. \tag{3.1}$$
Therefore using (3.1) we obtain that
$$|P(z)| \ge \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \ge \left|a_{N(r)}\right| |z|^{N(r)} - M\frac{|z|^{N(r)} - 1}{|z| - 1}.$$
Hence
$$|P(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} - M\frac{|z|^{N(r)} - 1}{|z| - 1} > 0$$
$$\begin{aligned} &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} > M\frac{|z|^{N(r)} - 1}{|z| - 1} \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} > M\left(|z|^{N(r)} - 1\right) \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} - M\ |z|^{N(r)} + M > 0 \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) |z|^{N(r)} + M > 0. \end{aligned}$$
Therefore on $|z| \ne 1$, $|P(z)| > 0$ if $\left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) |z|^{N(r)} + M > 0$. Now let us consider
$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0. \tag{3.2}$$
Clearly the maximum number of changes in sign in (3.2) is two. So the maximum number of positive roots of $g(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of (3.2). So $g(t) = 0$ must have another positive root t_1(say).
Let us take $t_0 = max\{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not, for some $t = t_2 > t_0$, $g(t_2) < 0$.
Now $g(t_2) < 0$ and $g(\infty) > 0$ imply that $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction.
Therefore for $t > t_0$, $g(t) > 0$ and so $t_0 > 1$.
Hence $|P(z)| > 0$ for $|z| > t_0$.
$$\text{Therefore all the zeros of } P(z) \text{ lie in the disc } |z| \le t_0. \tag{3.3}$$

Again let us consider

$$Q(z) = z^{N(r)} P\left(\frac{1}{z}\right) \approx z^{N(r)} \left\{a_0 + \frac{a_1}{z} + ... + \frac{a_{N(r)}}{z^{N(r)}}\right\} = a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}$$

$$\text{i.e., } |Q(z)| \geq |a_0| |z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \text{ for } |z| \neq 1.$$

Now

$$\left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1| |z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \leq M' \left(|z|^{N(r)-1} + ... + 1\right)$$

$$= M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1. \tag{3.4}$$

Using (3.4) we get that

$$|Q(z)| \geq |a_0| |z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \geq |a_0| |z|^{N(r)} - M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1.$$

Therefore for $|z| \neq 1$,

$$|Q(z)| > 0 \text{ if } |a_0| |z|^{N(r)} - M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) > 0$$

$$\text{i.e., if } |a_0| |z|^{N(r)} > M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right)$$

$$\text{i.e., if } |a_0| |z|^{N(r)+1} - |a_0| |z|^{N(r)} - M' |z|^{N(r)} + M' > 0$$

$$\text{i.e., if } |a_0| |z|^{N(r)+1} - (|a_0| + M') |z|^{N(r)} + M' > 0.$$

So for $|z| \neq 1$, $|Q(z)| > 0$ if $|a_0| |z|^{N(r)+1} - (|a_0| + M') |z|^{N(r)} + M' > 0$. Let us consider

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0.$$

Since the maximum number of changes of sign in $f(t)$ is two, the maximum number of positive roots of $f(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of $f(t) = 0$. So $f(t) = 0$ must have another positive root t_2 (say).

Let us take $t'_0 = max\{1, t_2\}$. Clearly for $t > t'_0$, $f(t) > 0$. If not, for some $t_3 > t'_0$, $f(t_3) < 0$. Now $f(t_3) < 0$ and $f(\infty) > 0$ implies that $f(t) = 0$ have another positive root in the interval (t_3, ∞) which is a contradiction.

Therefore for $t > t'_0$, $f(t) > 0$.

Also $t'_0 \geq 1$. So $|Q(z)| > 0$ for $|z| > t'_0$.

Therefore $Q(z)$ does not vanish in $|z| > t'_0$.

Hence all the zeros of $Q(z)$ lie in $|z| \leq t'_0$.

Let $z = z_0$ be a zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that $Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^{N(r)} .P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} .0 = 0$. Therefore $Q\left(\frac{1}{z_0}\right) = 0$. So

$z = \frac{1}{z_0}$ is a root of $Q(z) = 0$. Hence $\left|\frac{1}{z_0}\right| \le t_0'$ implies that $|z_0| \ge \frac{1}{t_0'}$.
As z_0 is an arbitrary root of $P(z) = 0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in } |z| \ge \frac{1}{t_0'}. \tag{3.5}$$

From (3.3) and (3.5) we get that all the zeros of $P(z)$ lie in the proper ring shaped region $\frac{1}{t_0'} \le |z| \le t_0$ where t_0 and t_0' are the greatest positive roots of the equations $g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$ and $f(t) \equiv |a_0|\, t^{N(r)+1} - (|a_0| + M')\, t^{N(r)} + M' = 0$ where M and M' are given in the statement of Theorem 3.1. This proves the theorem. □

In the line of Theorem 3.1, we may state the following theorem in view of Lemma 2.2:

Theorem 3.2. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}. Also for all sufficiently large r in the disc $|z| \le [re^{L(r)}]$, $\left|a_{N(r)}\right| \ne 0$, $|a_0| \ne 0$. and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \le |z| \le t_0$$

where t_0 is the greatest positive root of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$$

and t_0' is the greatest positive root of

$$f(t) \equiv |a_0|\, t^{N(r)+1} - (|a_0| + M')\, t^{N(r)} + M' = 0$$
$$\text{where } M = \max\left\{|a_0|, |a_1|, ... \left|a_{N(r)-1}\right|\right\}$$
$$\text{and } M' = \max\left\{|a_1|, |a_2|, ... \left|a_{N(r)}\right|\right\}.$$

The proof is omitted.

Theorem 3.3. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L-order ρ^L, $a_{N(r)} \ne 0$, $a_0 \ne 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \le [rL(r)]$ when r is sufficiently large. Further, for $\rho^L > 0$,

$$|a_0|\, (\rho^L)^{N(r)} \ge |a_1|\, (\rho^L)^{N(r)-1} \ge ... \ge \left|a_{N(r)-1}\right| \rho^L \ge \left|a_{N(r)}\right|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^L\left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)} < |z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right).$$

Proof. For the given entire function

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with $a_n \to 0$ as $n > N(r)$, where r is sufficiently large, $N(r)$ exists and $N(r) \le [rL(r)]^{\rho^L + \epsilon}$.
Therefore

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}$$

as $a_0 \neq 0, a_{N(r)} \neq 0$ and $a_n \to 0$ for $n > N(r)$.
Let us consider

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) \approx (\rho^L)^{N(r)} \left(a_0 + a_1 \frac{z}{\rho^L} + a_2 \frac{z^2}{(\rho^L)^2} + ... + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}}\right) \\ &= \left(a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)} z^{N(r)}\right). \end{aligned}$$

Therefore

$$|R(z)| \ge \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right|. \tag{3.6}$$

Now by the given condition $|a_0| (\rho^L)^{N(r)} \ge |a_1| (\rho^L)^{N(r)-1} \ge ...$ provided $|z| \neq 0$,we obtain that

$$\begin{aligned} &\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right| \le |a_0| (\rho^L)^{N(r)} + ... + \left|a_{N(r)-1}\right| \rho^L |z|^{N(r)-1} \\ &\le |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right). \end{aligned}$$

Therefore on $|z| \neq 0$,

$$-\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right| \ge -|a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right). \tag{3.7}$$

Therefore using (3.7) we get from (3.6) that

$$\begin{aligned} |R(z)| &\ge \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right) \\ &\ge \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}} + ...\right) \\ &= |z|^{N(r)} \left[\left|a_{N(r)}\right| - |a_0| (\rho^L)^{N(r)} \left\{\sum_{k=1}^{\infty} \frac{1}{|z|^k}\right\}\right]. \end{aligned}$$

Clearly $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is a geometric series which is convergent for $\frac{1}{|z|} < 1$ i.e., for $|z| > 1$ and converges to $\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1}$ if $|z| > 1$. Hence we get from above that for $|z| > 1 |R(z)| >$

$|z|^{N(r)}\left(\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1}\right)$. Now for $|z| > 1$,

$$\begin{aligned} |R(z)| > 0 \text{ if } & |z|^{N(r)}\left(\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1}\right) \geq 0 \\ \text{i.e., if } & \left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1} \geq 0 \\ \text{i.e., if } & \left|a_{N(r)}\right| \geq (\rho^L)^{N(r)}\frac{|a_0|}{|z|-1} \\ \text{i.e., if } & |z| - 1 \geq (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|} \\ \text{i.e., if } & |z| \geq 1 + (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|} > 1. \end{aligned}$$

Therefore $|R(z)| > 0$ if $|z| \geq 1 + (\rho^L)^{N(r)}\frac{|a_0|}{|a_{N(r)}|}$. So all the zeros of $R(z)$ lie in $|z| < 1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^L)^{N(r)}$. Let z_0 be an arbitrary zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $R(z)$ we have $R(\rho^L z_0) = (\rho^L)^{N(r)}.P(z_0) = (\rho^L)^{N(r)}.0 = 0$.

Hence $z = \rho^L z_0$ is a zero of $R(z)$. Therefore $\left|\rho^L z_0\right| < 1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^L)^{N(r)}$ i.e., $|z_0| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^L)^{N(r)}\right)$. Since z_0 is any zero of $P(z)$ therefore all the zeros of $P(z)$ lie in

$$|z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right). \tag{3.8}$$

Again let us consider $F(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right)$. Now $F(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right)$
$\approx (\rho^L)^{N(r)} z^{N(r)}\left\{a_0 + \frac{a_1}{\rho^L z} + ... + \frac{a_{N(r)}}{(\rho^L z)^{N(r)}}\right\} = a_0(\rho^L)^{N(r)} z^{N(r)} + a_1(\rho^L)^{N(r)-1} z^{N(r)-1} + ... + a_{N(r)}$. Therefore $|F(z)| \geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_1(\rho^L)^{N(r)-1} z^{N(r)-1} + ... + a_{N(r)}\right|$. Again

$$\begin{aligned} & \left|a_1(\rho^L)^{N(r)-1} z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \\ & \leq |a_1|(\rho^L)^{N(r)-1}\left(|z|^{N(r)-1} + ... + |z| + 1\right) \end{aligned}$$

provided $|z| \neq 0$. So $\left|a_1(\rho^L)^{N(r)-1} z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)}\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)$. So for $|z| \neq 0$, $|F(z)| \geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)}\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)$
$= (\rho^L)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^L - |a_1|\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)\right]$. Therefore for $|z| \neq 0$,

$$|F(z)| > (\rho^L)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^L - |a_1|\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right]. \tag{3.9}$$

The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$ *i.e.*, for $|z| > 1$ and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ if } |z| > 1. \tag{3.10}$$

Using (3.9) and (3.10) we have for $|z| > 1$, $|F(z)| > (\rho^L)^{N(r)-1} |z|^{N(r)} \left[|a_0| \rho^L - \frac{|a_1|}{|z|-1}\right]$. Hence for $|z| > 1$,

$$|F(z)| > 0 \text{ if } |z|^{N(r)} (\rho^L)^{N(r)-1} \left[|a_0| \rho^L - \frac{|a_1|}{|z|-1}\right] \geq 0$$

$$\begin{aligned}
&\text{i.e., if } |a_0| \rho^L - \frac{|a_1|}{|z|-1} \geq 0 \\
&\text{i.e., if } |a_0| \rho^L \geq \frac{|a_1|}{|z|-1} \\
&\text{i.e., if } |z| \geq 1 + \frac{|a_1|}{|a_0| \rho^L} > 1.
\end{aligned}$$

Therefore $|F(z)| > 0$ for $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^L}$. So $F(z)$ does not vanish in $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^L}$. Equivalently all the zeros of $F(z)$ lie in $|z| < 1 + \frac{|a_1|}{|a_0|\rho^L}$. Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $a_0 \neq 0$ and $z_0 \neq 0$.
Now let us put $z = \frac{1}{\rho^L z_0}$ in $F(z)$. So we have $F\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} .P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} .0 = 0$. Therefore $z = \frac{1}{\rho^L z_0}$ is a root of $F(z)$.
Hence

$$\left|\frac{1}{\rho^L z_0}\right| < 1 + \frac{|a_1|}{|a_0| \rho^L}$$

$$\begin{aligned}
&\text{i.e., } \frac{1}{|z_0|} < \rho^L \left(1 + \frac{|a_1|}{|a_0| \rho^L}\right) \\
&\text{i.e., } |z_0| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)}.
\end{aligned}$$

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie on

$$|z| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)}. \tag{3.11}$$

From (3.8) and (3.11) we get that all the zeros of $P(z)$ lie on the proper ring shaped region $\frac{1}{\rho^L\left(1+\frac{|a_1|}{|a_0|\rho^L}\right)} < |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)}\right)$ where $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq ... \geq \left|a_{N(r)}\right|$ for $\rho^L > 0$.
This proves the theorem. □

In the line of Theorem 3.3, we may state the following theorem in view of Lemma 2.2 :

Theorem 3.4. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}, $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \leq [re^{L(r)}]$ when r is sufficiently large. Further, for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq ..... \geq \left|a_{N(r)-1}\right| \rho^{L^*} \geq \left|a_{N(r)}\right|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^{L^*}\left(1 + \frac{|a_1|}{|a_0|\rho^{L^*}}\right)} < |z| < \frac{1}{\rho^{L^*}}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^{L^*})^{N(r)}\right).$$

The proof is omitted.

Corollary 3.1. *From Theorem 3.3 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

of degree n, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq ... \geq |a_n|$ lie in the proper ring shaped region

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

just on putting $\rho^L = 1$.

Corollary 3.2. *From Theorem 3.4 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

of degree n, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq ... \geq |a_n|$ lie in the proper ring shaped region

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

just on putting $\rho^{L^*} = 1$.

Theorem 3.5. *Let $P(z)$ be an entire function with L-order ρ^L. For sufficiently large values of r in the disk $|z| \leq [rL(r)]$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, a_0 \neq 0$$

be such that $1 \leq p_1 < p_2 ... < p_m \leq N(r) - 1$, p_i's are integers and for $\rho^L > 0$,

$$|a_0| (\rho^L)^{N(r)} \geq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \geq ... \geq \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m}.$$

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^L t_0'} < |z| < \frac{1}{\rho^L} t_0$$

where t_0 and t_0' are the unique positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)-p_m} - \left|a_{N(r)}\right| t^{N(r)-p_m-1} - |a_0| \left(\rho^L\right)^{N(r)} = 0 \text{ and}$$
$$f(t) \equiv |a_0| \left(\rho^L\right)^{p_1} t^{p_1} - |a_0| \left(\rho^L\right)^{p_1} t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively.

Proof. Let

$$P(z) = a_0 + a_{p_1} z^{p_1} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, \left|a_{N(r)}\right| \neq 0. \tag{3.12}$$

Also for $\rho^L > 0$, $|a_0| \left(\rho^L\right)^{N(r)} \geq \left|a_{p_1}\right| \left(\rho^L\right)^{N(r)-p_1} \geq ... \geq \left|a_{N(r)}\right|$. Let us consider

$$\begin{aligned} R(z) = \left(\rho^L\right)^{N(r)} P\left(\frac{z}{\rho^L}\right) &= \left(\rho^L\right)^{N(r)} \left\{a_0 + a_{p_1} \frac{z^{p_1}}{\left(\rho^L\right)^{p_1}} + ... + a_{p_m} \frac{z^{p_m}}{\left(\rho^L\right)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{\left(\rho^L\right)^{N(r)}}\right\} \\ &= a_0 \left(\rho^L\right)^{N(r)} + a_{p_1} \left(\rho^L\right)^{N(r)-p_1} z^{p_1} + ... + a_{p_m} \left(\rho^L\right)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore

$$|R(z)| \geq \left|a_{N(r)} z^{N(r)}\right| - \left|a_0 \left(\rho^L\right)^{N(r)} + a_{p_1} \left(\rho^L\right)^{N(r)-p_1} z^{p_1} + ... + a_{p_m} \left(\rho^L\right)^{N(r)-p_m} z^{p_m}\right|. \tag{3.13}$$

Now for $|z| \neq 0$,

$$\begin{aligned} &\left|a_0 \left(\rho^L\right)^{N(r)} + a_{p_1} \left(\rho^L\right)^{N(r)-p_1} z^{p_1} + ... + a_{p_m} \left(\rho^L\right)^{N(r)-p_m} z^{p_m}\right| \\ &\leq |a_0| \left(\rho^L\right)^{N(r)} + \left|a_{p_1}\right| \left(\rho^L\right)^{N(r)-p_1} |z|^{p_1} + ... + \left|a_{p_m}\right| \left(\rho^L\right)^{N(r)-p_m} |z|^{p_m} \\ &\leq |a_0| \left(\rho^L\right)^{N(r)} \left(1 + |z|^{p_1} + ... + |z|^{p_m}\right) \\ &= |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right). \end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we have for $|z| \neq 0$

$$\begin{aligned} |R(z)| &\geq \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right) \\ &> \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + ...\right) \\ &= \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \sum_{k=1}^{\infty} \frac{1}{|z|^k}. \end{aligned} \tag{3.15}$$

The geometric series $\sum\limits_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$i.e.,for $|z| > 1$ and converges to $\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum\limits_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1}$ for $|z| > 1$. So on $|z| > 1$,

$$
\begin{aligned}
&|R(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} - \frac{|a_0| (\rho^L)^{N(r)} |z|^{p_m+1}}{|z|-1} \geq 0\\
&\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} \geq \frac{|a_0| (\rho^L)^{N(r)} |z|^{p_m+1}}{|z|-1}\\
&\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} \geq |a_0| (\rho^L)^{N(r)} |z|^{p_m+1}\\
&\text{i.e., if } |z|^{p_m+1}\left(\left|a_{N(r)}\right| |z|^{N(r)-p_m} - \left|a_{N(r)}\right| |z|^{N(r)-p_m-1} - |a_0| (\rho^L)^{N(r)}\right) \geq 0.
\end{aligned}
$$

Let us consider $g(t) \equiv \left|a_{N(r)}\right| |t|^{N(r)-p_m} - \left|a_{N(r)}\right| |t|^{N(r)-p_m-1} - |a_0| (\rho^L)^{N(r)} = 0$. Clearly $g(t) = 0$ has one positive root because the maximum number of changes in sign in $g(t)$ is one and $g(0) = -|a_0| \rho^{N(r)}$ is $-ve$, $g(\infty)$ is $+ve$.
Let t_0 be the positive root of $g(t) = 0$ and $t_0 > 1$. Clearly for $t > t_0$, $g(t) > 0$. If not for some $t_1 > t_0$, $g(t_1) < 0$.
Then $g(t_1) < 0$ and $g(\infty) > 0$. Therefore $g(t) = 0$ must have another positive root in (t_1, ∞) which gives a contradiction .
Hence for $t \geq t_0$, $g(t) \geq 0$ and $t_0 > 1$. So $|R(z)| > 0$ for $|z| \geq t_0$.
Thus $R(z)$ does not vanish in $|z| \geq t_0$.

Hence all the zeros of $R(z)$ lie in $|z| < t_0$.

Let $z = z_0$ be any zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $R(z)$ we have $R(\rho^L z_0) = (\rho^L)^{N(r)}.P(z_0) = (\rho^L)^{N(r)}.0 = 0$. Therefore $R(\rho^L z_0) = 0$ and so $z = \rho^L z_0$ is a zero of $R(z)$ and consequently $\left|\rho^L z_0\right| < t_0$ which implies $|z_0| < \frac{t_0}{\rho^L}$. As z_0 is an arbitrary zero of $P(z)$,

$$
\text{all the zeros of } P(z) \text{ lie in } |z| < \frac{t_0}{\rho^L}. \tag{3.16}
$$

Again let us consider $F(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right)$. Now

$$
\begin{aligned}
F(z) &= (\rho^L)^{N(r)} z^{N(r)} . \left\{a_0 + a_{p_1}\frac{1}{(\rho^L)^{p_1} z^{p_1}} + ... + a_{p_m}\frac{1}{(\rho^L)^{p_m} z^{p_m}} + a_{N(r)}\frac{1}{(\rho^L)^{N(r)} z^{N(r)}}\right\}\\
&= a_0 (\rho^L)^{N(r)} z^{N(r)} + a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\left|a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}\right|\\
&\leq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1} + ... + \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m} |z|^{N(r)-p_m} + \left|a_{N(r)}\right|\\
&\leq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + ... + |z|^{N(r)-p_m} + 1\right).
\end{aligned}
$$

So for $|z| \neq 0$,

$$\begin{aligned} |F(z)| &\geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left| a_{p_1} (\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + ... + a_{p_m} (\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)} \right| \\ &\geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left| a_{p_1} \right| (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + ... + |z|^{N(r)-p_m} + 1 \right) \\ &= |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left| a_{p_1} \right| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\frac{1}{|z|} + \frac{1}{|z|^{p_2-p_1+1}} + ... + \frac{1}{|z|^{N(r)-p_1+1}} \right) \end{aligned}$$

i.e., on $|z| \neq 0$, $|F(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left| a_{p_1} \right| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k} \right)$. The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$i.e., for $|z| > 1$ and converges to $\frac{1}{|z|} \frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1}$ for $|z| > 1$. Therefore for $|z| > 1$

$$|F(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left| a_{p_1} \right| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\frac{1}{|z|-1} \right)$$

$$\begin{aligned} &= (\rho^L)^{N(r)-p_1} \left((\rho^L)^{p_1} |a_0| |z|^{N(r)} - \left| a_{p_1} \right| \frac{|z|^{N(r)-p_1+1}}{|z|-1} \right) \\ &= (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(|a_0| \left(\rho^L \right)^{p_1} |z|^{p_1-1} - \frac{\left| a_{p_1} \right|}{|z|-1} \right) \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |F(z)| > 0 \text{ if } & |a_0| (\rho^L)^{p_1} |z|^{p_1-1} - \frac{\left| a_{p_1} \right|}{|z|-1} \geq 0 \\ \text{i.e., if } & |a_0| (\rho^L)^{p_1} |z|^{p_1-1} \geq \frac{\left| a_{p_1} \right|}{|z|-1} \\ \text{i.e., if } & |a_0| (\rho^L)^{p_1} |z|^{p_1} - |a_0| (\rho^L)^{p_1} |z|^{p_1-1} - \left| a_{p_1} \right| \geq 0. \end{aligned} \tag{3.17}$$

Therefore on $|z| > 1, |F(z)| > 0$ if (3.17) holds. Let us consider $f(t) = |a_0| (\rho^L)^{p_1} t^{p_1} - |a_0| (\rho^L)^{p_1} t^{p_1-1} - \left| a_{p_1} \right| = 0$. Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let t'_0 be the positive root of $f(t) = 0$. Therefore $t'_0 > 1$. Obviously if $t \geq t'_0$ then $f(t) \geq 0$. So for $|F(z)| > 0$, $|z| \geq t'_0$. Therefore $F(z)$ does not vanish in $|z| \geq t'_0$.

Hence all the zeros of $F(z)$ lie in $|z| < t'_0$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Now putting $z = \frac{1}{\rho^L z_0}$ in $F(z)$ we obtain that $F\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} .P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} .P(z_0)$ $= \left(\frac{1}{z_0}\right)^{N(r)} .0 = 0$. Therefore $z = \frac{1}{\rho^L z_0}$ is a zero of $F(z)$. Now $\left| \frac{1}{\rho^L z_0} \right| < t'_0$ *i.e.*, $\left| \frac{1}{z_0} \right| < \rho^L t'_0$ *i.e.*, $| z_0| > \frac{1}{\rho^L t'_0}$. As z_0 is an arbitrary zero of $P(z)$ therefore we obtain that

$$\text{all the zeros of } P(z) \text{ lie in } |z| > \frac{1}{\rho^L t'_0}. \tag{3.18}$$

Using (3.16) and (3.18) we get that all the zeros of $P(z)$ lie in the ring shaped region $\frac{1}{\rho^{L}t_0'} < |z| < \frac{t_0}{\rho^{L}}$ where t_0, t_0' are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose forms are given in the statement of Theorem 3.3. This proves the theorem. □

In the line of Theorem 3.5, we may state the following theorem in view of Lemma 2.2 :

Theorem 3.6. *Let $P(z)$ be an entire function with L^*-order ρ^{L^*}. For sufficiently large values of r in the disk $|z| \le [re^{L(r)}]$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1}z^{p_1} + a_{p_2}z^{p_2} + ... + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}, a_0 \neq 0$$

be such that $1 \le p_1 < p_2 ... < p_m \le N(r) - 1$, p_i's are integers and for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \ge \left|a_{p_1}\right| (\rho^{L^*})^{N(r)-p_1} \ge ... \ge \left|a_{p_m}\right| (\rho^{L^*})^{N(r)-p_m}.$$

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^{L^*}t_0'} < |z| < \frac{1}{\rho^{L^*}}t_0$$

where t_0 and t_0' are the unique positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)-p_m} - \left|a_{N(r)}\right| t^{N(r)-p_m-1} - |a_0| (\rho^{L^*})^{N(r)} = 0 \text{ and}$$
$$f(t) \equiv |a_0| (\rho^{L^*})^{p_1} t^{p_1} - |a_0| (\rho^{L^*})^{p_1} t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively.

The proof is omitted.

Corollary 3.3. *In view of Theorem 3.5 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1}z^{p_1} + ... + a_{p_m}z^{p_m} + a_nz^n$ of degree n with $1 \le p_1 < p_2 < ... < p_m \le n-1$, p_i's are integers such that*

$$|a_0| \ge \left|a_{p_1}\right| \ge .... \ge |a_n|$$

lie in ring shaped region

$$\frac{1}{t_0'} < |z| < t_0$$

where t_0, t_0' are the unique positive roots of the equations

$$g(t) \equiv |a_n| t^{n-p_m} - |a_n| t^{n-p_m-1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0| t^{p_1} - |a_0| t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively just substituting $\rho^{L} = 1$.

Corollary 3.4. *In view of Theorem 3.6 we may state that all the zeros of the polynomial* $P(z) = a_0 + a_{p_1}z^{p_1} + ... + a_{p_m}z^{p_m} + a_n z^n$ *of degree n with* $1 \leq p_1 < p_2 < ... < p_m \leq n-1$, p_i*'s are integers such that*

$$|a_0| \geq \left|a_{p_1}\right| \geq .... \geq |a_n|$$

lie in ring shaped region

$$\frac{1}{t_0'} < |z| < t_0$$

where t_0, t_0' *are the unique positive roots of the equations*

$$g(t) \equiv |a_n|\, t^{n-p_m} - |a_n|\, t^{n-p_m-1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0|\, t^{p_1} - |a_0|\, t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively just substituting $\rho^{L^*} = 1$.

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