



Sixth Order Multiple Coarse Grid Computation for Solving 1D Partial Differential Equation

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Abstract

We present a new method using multiple coarse grid computation technique to solve one dimensional (1D) partial differential equation (PDE). Our method is based on a fourth order discretization scheme on two scale grids and the Richardson extrapolation. For a particular implementation, we use multiple coarse grid computation to compute the fourth order solutions on the fine grid and all the coarse grids. Since every fine grid point has a corresponding coarse grid point with fourth order solution, the Richardson extrapolation procedure is applied for every fine grid point to increase the order of solution accuracy from fourth order to sixth order. We compare the maximum absolute error and the order of solution accuracy for our new method, the standard fourth order compact (FOC) scheme and Wang-Zhang's sixth order multiscale multigrid method. Two convection-diffusion problems are solved numerically to validate our proposed method.

Keywords: partial differential equation, multiple coarse grid computation, multigrid method.

2010 MSC: 65N06, 65N55, 65F10.

1. Introduction

Numerical solutions of partial differential equations (PDEs) play a crucial role in many simulation and engineering modeling applications, such as airplane manufacturing (Gamet *et al.*, 1999), auto manufacturing (Gerlinger *et al.*, 1998), medical imaging (Kang *et al.*, 2004), oil exploration and production (Li *et al.*, 2005), semiconductor (Carey *et al.*, 1996), communications (Kim & Kim, 2004), etc. Over the past several decades, computational mathematicians and engineers have developed many efficient fast algorithms to reduce the computation time. However, the increasing

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demand for higher resolution simulations in less computer time has continuously challenged the computational scientists to come up with more efficient, scalable numerical algorithms to solve PDEs.

In many scientific and engineering applications, such as the global ocean modeling and wide area weather forecasting, the computational domains are huge and the grid spaces are not small. In the context of the finite difference methods, the standard second order discretization scheme or the first order upwind difference scheme yield unsatisfactory results because they may need fine mesh griddings to compute approximate solutions of acceptable accuracy. In addition, the second order scheme may also produce numerical solutions with nonphysical oscillations for the convection dominated problems (Spotz, 1995).

Higher order (more than two) discretization methods are considered to be useful to reduce computational cost in very large scale modelings and simulations, which use relatively coarser mesh griddings to yield approximate solutions of comparable accuracy, compared with lower order discretization. Generally, higher order discretization schemes need more complicated procedures and more preprocessing costs to construct the coefficient matrix. However, they usually yield linear systems of much smaller size, compared with those from the lower order methods.

For the development of fourth order compact difference schemes, Gupta *et al.* proposed a fourth order nine-point compact (FOC) scheme to discretize the two dimensional (2D) convection-diffusion equation with variable coefficients (Gupta *et al.*, 1984). There are also some other similar fourth order compact schemes that have been developed for the convection-diffusion equations. Readers are referred to (Li *et al.*, 1995; Spotz, 1995; Spotz & Carey, 1995) and the references therein for more details.

For the sixth order schemes, Chu and Fan (Chu & Fan, 1998, 1999) proposed a three point combined compact difference (CCD) scheme for solving 2D Stommel Ocean model, which is a convection-diffusion equation. Their scheme can achieve sixth order accuracy for the inner grid points and fifth order accuracy for the boundary grid points. CCD scheme is considered as an *implicit* scheme because it does not compute the solution of the variables of interest directly. It also has a stability problem that for certain problems, if a large meshsize is used, the computed solution may be oscillatory (Zhang & Zhao, 2005).

In contrary, the *explicit* compact schemes compute the solutions of the variables directly. In addition, the explicit schemes have an additional advantage that they can avoid the oscillations in computed solutions. However, the higher order explicit compact schemes are more complicated to develop in higher dimensions, compared with the implicit schemes. As far as we know, there is no existing explicit compact scheme on a single scale grid that is higher than the fourth order.

By using the idea of multiscale computation, Sun and Zhang (Sun & Zhang, 2004) first proposed a sixth order explicit finite difference discretization strategy, which is based on the Richardson extrapolation technique and an operator interpolation scheme. Recently, Wang and Zhang developed an efficient and scalable sixth order explicit compact scheme for 2D/3D Poisson and convection-diffusion equations by using multiscale mutigrid method and an operator based interpolation combined with extrapolation technique (Wang & Zhang, 2009, 2011, 2010). The However, for the operator based interpolation, if the coefficient matrix A is not diagonally dominant like the convection-diffusion equation with very large cell Reynolds number, it may take a large number of iterations to converge. In this paper, we present another technique called the multiple

coarse grid computation technique. This approach can be used to compute the fourth order solutions on the fine grid and every coarse grid, which means that we can directly apply Richardson extrapolation for every grid point on the fine grid and no operator based interpolation is needed.

An outline of the paper is as follows. In Section 2, we illustrate our sixth order strategy by using multiple coarse grid computation technique. Numerical results will be provided in Section 3. Section 4 contains the concluding remarks.

2. Sixth Order Multiple Coarse Grid Computation

Our motivation is to build an efficient and scalable method for solving PDEs like the convection-diffusion equations with high order of solution accuracy. In addition, we want the new method to have good potential to be modified to work on parallel computers. In (Wang & Zhang, 2009), Wang and Zhang successfully increase the order of solution accuracy from fourth order to sixth order by using multiscale multigrid method, Richardson extrapolation and an operator based interpolation. Important properties of the Richardson extrapolation has been studied by Zlatev *et al.* Readers are referred to (Zlatev *et al.*, 2010) and the references therein for more details. The interpolation strategy is a mesh-refinement type of iterative method and it is very efficient for some PDEs like the Poisson equation. Since their discretization scheme is based on the standard explicit fourth order compact scheme, so there is no nonphysical oscillation in the computed solutions. The proof and numerical analysis of this property can be found in (Spotz, 1995). However, it is not efficient and scalable for some problems like the convection-diffusion equation with high Reynolds numbers (Wang & Zhang, 2011). For some cases, the interpolation procedure may take thousands of iterations to converge. In addition, this method does not have a good potential for parallel implementation.

The idea of using multiple coarse grid computation is from the parallel superconvergent multigrid method. In addition to splitting the original grid and filtering residual vector to exploit parallelism, one can use the concurrent relaxation method on multiple grids (Zhu, 1993). The multigrid superconvergent method uses multiple coarse grids to generate better correction for the fine grid solution than that from a single coarse grid. The reason is that for standard multigrid method of 1D problem as in figure 1, the residual of the fine grid is projected to only *even* coarse grid. But we can also project the residual to *odd* coarse grids. Therefore, a combination of error correction from all the coarse grids may make the fine grid converge faster than that from a single coarse grid. In general, for a d dimensional problem, the fine grid can be easily coarsened into 2^d coarse grids. If the computation work for each coarse grid can be loaded to a separate processor and computed simultaneously, we can develop a parallel solver for solving PDEs.

2.1. 1D multiple coarse grid computation

Let's consider the multiple coarse grid computation technique for the one dimensional (1D) convection diffusion equation, which can be written as

$$u_x + b(x)u_x + c(x)u = f(x), \quad 0 \leq x \leq l, \quad (2.1)$$

where the known functions $b(x)$, $c(x)$ and $f(x)$ are assumed to have the necessary derivatives up to certain orders. Eq. (2.1) can be discretized by some finite difference scheme to result in a system

of linear equations

$$A^h u^h = f^h, \quad (2.2)$$

where h is the uniform grid spacing of the discretized domain Ω^h .

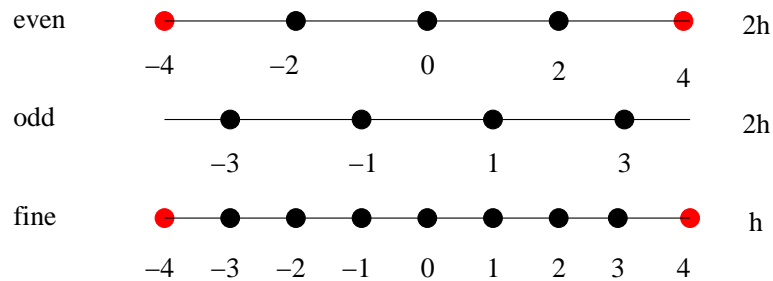


Figure 1. Illustration of the multiple coarse grid for 1D problem.

In order to achieve sixth order solution accuracy, we need to compute the fourth order solutions for the fine grid and two coarse grids like figure 1. Then we can apply the Richardson extrapolation. The fourth order compact (FOC) scheme we use is from (Wang & Zhang, 2011).

From figure 1, we can find out that two coarse grids are generated in such a way that all the even-numbered grid points from Ω_h belong to coarse grid Ω_{even} and all the odd-numbered grid points belong to coarse grid Ω_{odd} . So we have

$$\begin{aligned} \Omega_{even} &= \{x_j | x_j \in \Omega_h \text{ and } (j = \text{even})\}, \\ \Omega_{odd} &= \{x_j | x_j \in \Omega_h \text{ and } (j = \text{odd})\}. \end{aligned}$$

We note that the even indexed coarse grid is easy to be solved by double the mesh size from h to $2h$. However, the coarse grid Ω_{odd} only contains the *black* color grid points from fine grid but no *red* color boundary grid points. It is very difficult to develop the finite difference schemes for coarse grid Ω_{odd} if we only have the inner grid points. One possible approach is to add these red color boundaries to Ω_{odd} and develop special computational stencil for grid point u_{-3} and u_3 as shown in figure 2. For the 1D problem in figure 2, the computational stencil for the grid points near the boundaries are different with other inner grid points. For the inner grid points like u_{-1} and u_1 , their finite difference schemes are based on $2h$ meshsize. However, if we take grid point u_{-3} in Ω_{odd} as an example, its compact finite difference scheme needs the boundary grid point u_{-4} and inner grid point u_{-1} . The meshsize between u_{-4} and u_{-1} are h and $2h$.

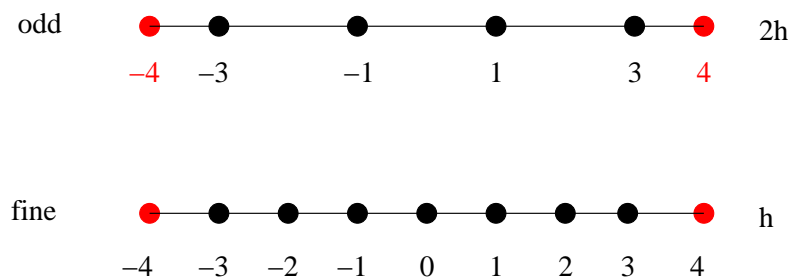


Figure 2. Ω_{odd} with two added red color boundary grid points.

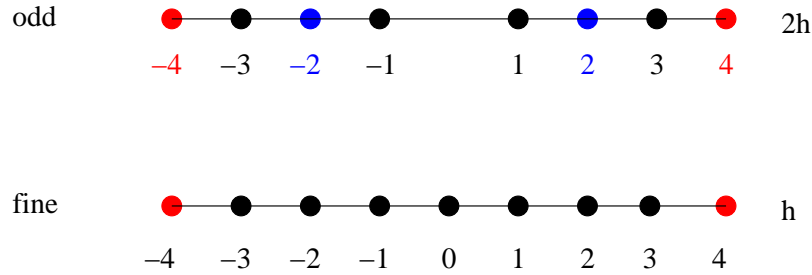


Figure 3. Ω_{odd} with two red color boundary and two blue color inner grid points .

Lemma 2.1. For coarse grid as shown in figure 2, the solution accuracy for the central difference operator becomes first order.

Proof. It can be easily verified by using Taylor series expansion. □

Since the second order central difference operator is degraded to the first order, the FOC scheme which is based on the approximation for the second order terms will be degraded to the second order for these near boundary grid points. In order to compute fourth order solution for every coarse grid point, we add two more grid points to the Ω_{odd} like the blue color grid points in figure 3.

By adding these four grid points, now we can discretize every grid point in Ω_{odd} with fourth order accuracy using FOC scheme. Let's assume the Ω_{odd} contains Nx grid points

$$u_{odd}(0), u_{odd}(1), \dots, u_{odd}(Nx)$$

as in figure 4. Then the Ω_{even} will contains $Nx - 3$ grid points and fine grid will contains $2Nx - 7$ grid points. The grid points on Ω_{odd} are approximated as follows:

- For $j \in \{1, 2, Nx - 2, Nx - 1\}$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+1)$ with meshsize h . The truncation error is $O(h^4)$.
- For $j = 3$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-2)$ and $u_{odd}(j+1)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.
- For $j \in [4, Nx - 4]$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+1)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.
- For $j = Nx - 3$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+2)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.



Figure 4. Representation of modified Ω_{odd} for 1D problem.

By using above discretization strategy, we can approximate the fourth order solution for every grid point on Ω_{odd} . After we get fourth order solutions for the fine grid and two coarse grids, each grid point on the fine grid will have a corresponding grid point on either Ω_{even} or Ω_{odd} . Then we apply Richardson extrapolation (Cheney & Kincard, 1999) for every fine grid point to approximate the sixth order solution like

$$\tilde{u}_j^h = \frac{16u_j^h - u_j^{2h}}{15}, \quad (2.3)$$

where u_j^h is j th grid point from fine grid and u_j^{2h} is the corresponding coarse grid point.

3. Numerical Results

Two 1D convection-diffusion equations are solved using the multiple coarse grid computation strategy discussed in the previous sections. We compared the truncated error and the order of accuracy by using our multiple coarse grid computation technique (MCG), standard fourth order scheme (FOC), and the sixth order operator based interpolation scheme (SOC) in (Wang & Zhang, 2011).

The codes are written in Fortran 77 programming language and run on one node of the Lipscomb HPC Cluster at the University of Kentucky. Each node has 36GB of local memory and runs at 2.66GHz. The initial guess for our test cases is the zero vector. The stopping criteria for the iterative methods we tested and the operator based interpolation procedure is 10^{-10} . The errors reported are the maximum absolute errors over the discrete grid of the finest level.

For the order of solution accuracy, we denote $E(h)$ and $E(H)$ to be the solution error with meshsize h and H , respectively. The order of accuracy m is calculated from the formula

$$\begin{aligned} \frac{E(h)}{E(H)} &= \frac{h^m}{H^m} \\ \implies m &= \log_{(h/H)}(E(h)/E(H)). \end{aligned}$$

The order of accuracy is formally defined when the meshsize approaches zero. Therefore, when the meshsize is relatively large, the discretization scheme may not achieve its formal order of accuracy.

Problem 1. Let's consider the examples from Sun's previous work (Sun & Zhang, 2004), which is a 1D convection-diffusion equation like

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - u = -\cos x - 2 \sin x, \quad 0 \leq x \leq \pi. \quad (3.1)$$

Eq. (3.1) has the Dirichlet boundary conditions as $u(0) = u(\pi) = 0$. The analytic solution for this problem is $u(x) = \sin x$.

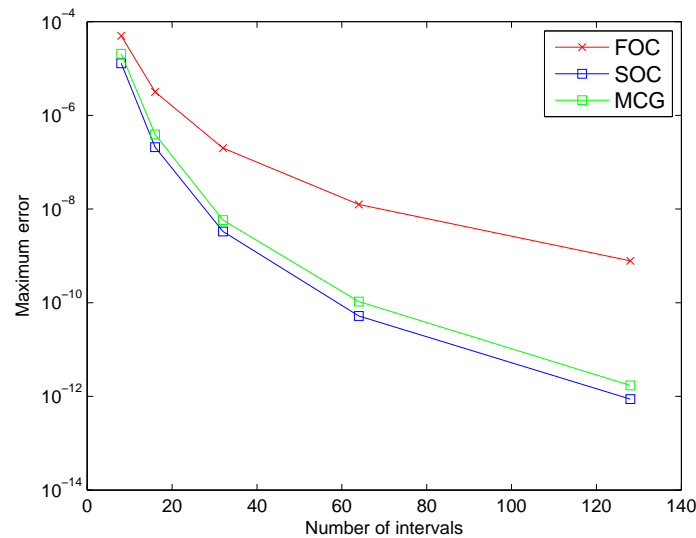


Figure 5. Comparison of maximum errors of FOC, SOC and MCG methods for Problem 1.

The computational results are listed in Table 1 and figure 5. From Table 1, we can see that the multiple coarse grid method (MCG) is more accurate than the fourth order scheme (FOC). Although the MCG method is not as accurate as the SOC but it can achieve the sixth order solution accuracy when the number of intervals is bigger than 8. The reason why MCG is less accurate than SOC is that there are two near boundary grid point using meshsize h to approximate instead of $2h$ in Ω_{odd} .

Table 1. Comparison of maximum errors and the order of accuracy by using FOC, SOC, and MCG methods for Eq. (3.1).

	FOC		SOC		MCG	
h	Error	Order	Error	Order	Error	Order
$\pi/8$	5.02e-5	4.0	1.30e-5	5.9	2.08e-5	5.7
$\pi/16$	3.18e-6	4.0	2.10e-7	6.0	3.94e-7	6.1
$\pi/32$	2.00e-7	4.0	3.32e-9	6.0	5.81e-9	5.8
$\pi/64$	1.25e-8	4.1	5.20e-11	6.0	1.06e-10	6.0
$\pi/128$	7.83e-10	4.1	8.73e-13	6.0	1.71e-12	6.0

Problem 2. We solve another classical 1D convection-diffusion equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = 0, \quad 0 \leq x \leq 1. \quad (3.2)$$

The boundary condition for Eq. (3.2) is $u_0 = 0$ and $u_1 = 1$. The analytic solution is $u(x) = (e^x - 1)/(e - 1)$.

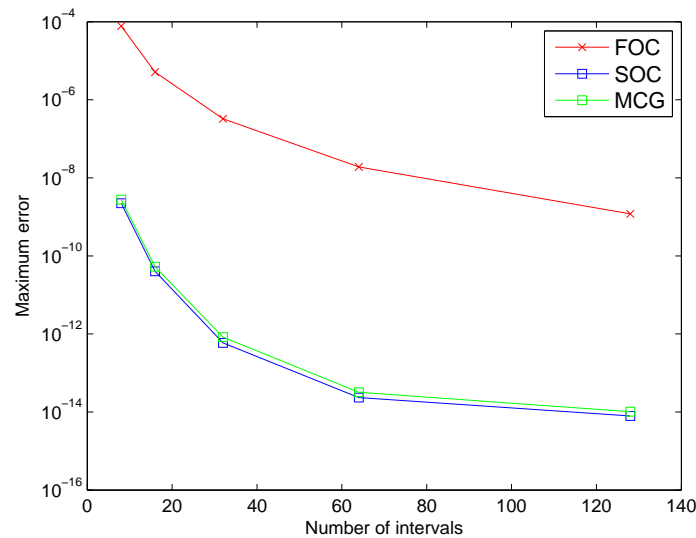


Figure 6. Comparison of maximum errors of FOC, SOC and MCG methods for Problem 2.

The numerical results of Problem 2 are listed in Table 2 and figure 6. We note that when $n > 32$, the order of solution accuracy is not high enough as we hope. The reason is that the computed solutions with $h = 1/64$ and $h = 1/128$ are not as accurate as they should be, due to the stopping criteria we set. Once again, the solutions from our MCG method are more accurate than the FOC method and can achieve the sixth order when $n < 64$.

Table 2. Comparison of maximum errors and the order of accuracy by using FOC, SOC, and MCG methods for Eq. (3.2).

	FOC		SOC		MCG	
h	Error	Order	Error	Order	Error	Order
1/8	7.76 e-5	3.9	2.24e-9	5.9	2.78e-9	5.7
1/16	5.12e-6	4.0	4.01e-11	6.0	5.29e-11	6.0
1/32	3.27e-7	4.0	5.91e-13	6.0	8.27e-13	5.9
1/64	1.91e-8	4.0	2.34e-14	4.9	3.21e-14	4.8
1/128	1.19e-9	4.0	7.93e-15	1.6	1.02e-14	1.6

We want to mention here that the SOC method for both test cases is slightly more accurate than the MCG method, but the MCG method has a very good potential for parallel implementation. The computing tasks for MCG procedure can be divided to three independent processors (one for find grid and two for coarse grids). In addition, since the MCG method does not need the operator based interpolation procedure to approximate the sixth order fine grid solution, it will save a large amount of CPU costs for some high Reynolds number problems (Wang & Zhang, 2011).

4. Concluding Remarks and Future Work

We presented a new sixth order solution method based on the fourth order discretization and multiple coarse grid computation for solving 1D convection-diffusion equation. Our numerical experiments show that the new sixth order strategy is more accurate than the standard fourth order scheme and can achieve the sixth order solution accuracy.

It is worth pointing out that our solution strategy can be applied to solve many other types of PDEs, because it does not require the additional work to redesign the discretization schemes. The advantage of using multiple coarse grids is that we can use it to increase the order of accuracy without using operator based interpolation scheme. However, there is still a lot of work that needs to be done to develop a useful multiple coarse grid computation method that can be applied to real-world problems. In this paper, we just use the standard Gauss-Seidel iterative method for MCG strategy, because our goal is to test whether the MCG method can achieve the sixth order accuracy or not. For some real applications, we should use multigrid method and implement the multiple coarse grid computation in the multigrid cycle.

For the future research work, we will extend our 1D multiple coarse grid computation method to higher dimensional problems. For 2D problems, we will generate four course grids by the index of x and y direction as $(even, even)$, $(even, odd)$, $(odd, even)$ and (odd, odd) . Here, $(even, even)$ is the course grid in standard multigrid method. Like 1D strategy, only the $(even, even)$ course grid has the full boundary conditions. We need to find a way to add artificial boundary grid points for other three course grids. Another possible solution is to use algebraic multigrid method instead of geometric multigrid method, this is also one of our research interest in the future.

For the parallelization, the parallel multiscale multigrid (MCG) method has been discussed in (Xiao, 1994; Zhu, 1993). However, these parallel MCG methods are only used to speed up the convergence. As we mentioned in previous section, the computation of each course grid and the fine grid is independent. If we want to solve a 3D problem, we can use nine processors to solve the fourth order solutions on the fine grid and eight coarse grids. Then an Richardson extrapolation, which can also been parallelized, can increase the order of accuracy to sixth order.

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