On Nonuniform Polynomial Stability for Evolution Operators on the Half-line

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Abstract

The main aim of this paper is to study a concept of nonuniform polynomial stability for evolution operators on the half-line. The obtained results are variants for nonuniform polynomial stability of some well-known theorems due to Barbashin, Datko, Rolewicz and Zabczyk in the case of uniform exponential stability. This paper generalizes well-known results for the nonuniform exponential stability (Lupa & Megan, 2012) and the uniform polynomial stability (Megan & Ceausu, 2012).

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1. Introduction and preliminaries

The notion of exponential stability plays an important role in the theory of differential equations in Banach spaces, particularly in the study of asymptotical behaviors. It has gained prominence since appearance of two fundamental monographs of J. L. Massera, J. J. Schäffer (Massera & Schäffer, 1966) and J. L. Daleckii, M. G. Krein (Daleckii & Krein, 1974). These were followed by the important books of C. Chicone and Yu. Latushkin (Chicone & Latushkin, 1999) and L. Barreira and C. Valls (Barreira & Valls, 2008).

The most important stability concept used in the qualitative theory of differential equations is the uniform exponential stability. In some situations, particulary in the nonautonomous setting, the concept of uniform exponential stability is too restrictive and it is important to look for a more general behavior.

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Two different perspectives can be identified to generalize the concept of uniform exponential stability, on the one hand one can define exponential stabilities that depends on the initial time (and therefore are nonuniform) and, on the other hand, one can consider grow rates that are not necessarily exponential.

The first approach leads to the concepts of nonuniform exponential stabilities and can be found in the works (Barreira & Valls, 2008), (Lupa & Megan, 2012), (Megan, 1995), (Minda & Megan, 2011), (Pinto, 1988) and the second approach is presented in the papers (Barreira & Valls, 2009), (Bento & Silva, 2009), (Bento & Silva, 2012), (Megan & Ramneantu, 2011), (Megan & Minda, 2011).

A natural generalization is to consider stability concepts that are both nonuniform and not necessarily exponential. This was the approach followed by Barreira and Valls in (Barreira & Valls, 2009) and A. Bento and C. Silva in (Bento & Silva, 2009), (Bento & Silva, 2012), who studied a nonuniform polynomial dichotomy concept. A principal motivation for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all variational equations in a finite dimensional space admit a nonuniform exponential dichotomy.

In this paper we consider a concept of nonuniform polynomial stability for evolution operators in Banach spaces. This concept has been considered in the case of invertible evolution operators in the papers (Barreira & Valls, 2009) due to L. Barreira and C. Valls, respectively in (Bento & Silva, 2009), (Bento & Silva, 2012) due to A. Bento and C. Silva.

Some results concerning polynomial stability for evolution operators were published in our papers (Megan & Ramneantu, 2011), (Megan & Ceausu, 2012), (Megan & Minda, 2011). We remark that the results obtained in (Megan & Ramneantu, 2011) are for the case of evolution operators with uniform exponential growth. In this paper we consider the case of evolution operators with nonuniform polynomial growth.

The obtained results in this paper can be considered as variants for nonuniform polynomial stability of some well-known theorems due to Barbashin ((Barbashin, 1967)), Datko ((Datko, 1972)) and Rolewicz ((Rolewicz, 1986)) in the case of uniform exponential stability. We remark that our proofs are not adaptations for polynomial stability of the proofs presented in (Barbashin, 1967), (Datko, 1972) and (Rolewicz, 1986). The case of nonuniform exponential stability has been studied in (Lupa & Megan, 2012), (Minda & Megan, 2011), respectively (Megan & Ramneantu, 2011), (Megan & Ceausu, 2012).

Moreover, we note that we consider evolution operators which are not supposed to be invertible and the polynomial stability concept studied in this paper uses the evolution operators in forward time. Thus the stability results obtained in this paper hold for a much larger class of differential equations than in the classical theory of uniform exponential stability.

Let X be a real or complex Banach space and let I be the identity operator on X. The norm on X and on $\mathcal{B}(X)$, the algebra of all bounded linear operators acting on X, will be denoted by $\|\cdot\|$.

Let

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : \ t \geq s\}.$$

We recall that a mapping $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called an *evolution operator* on X if

(e_1) $\Phi(t, t) = I$, for all $t \geqslant 0$;

(e_2) $\Phi(t, s)\Phi(s, r) = \Phi(t, r)$, for all $(t, s), (s, r) \in \Delta$.

Definition 1.1. An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be

(i) *with polynomial growth* (and denote p.g) if there exist $M \geq 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

$$(s + 1)^\omega \parallel \Phi(t, s) \parallel \leq M(t + 1)^\omega (s + 1)^\varepsilon, \quad for \ all \ (t, s) \in \Delta;$$

(ii) *polynomially stable* (and denote p.s) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t + 1)^\alpha \parallel \Phi(t, s) \parallel \leq N(s + 1)^{\alpha+\beta}, \quad for \ all \ (t, s) \in \Delta;$$

(iii) *exponentially stable* (and denote e.s) if there exist $N_1 \geq 1$, $\alpha_1 > 0$ and $\beta_1 \geq 0$ such that

$$e^{\alpha_1 t} \parallel \Phi(t, s) \parallel \leq N_1 e^{(\alpha_1+\beta_1)s}, \quad for \ all \ (t, s) \in \Delta.$$

Definition 1.2. An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be

(i) *measurable*, if for all $(s, x) \in \mathbb{R}_+ \times X$ the mapping $t \longmapsto \parallel \Phi(t, s)x \parallel$ is measurable on $[s, \infty)$.

(ii) *∗-measurable*, if for all $(s, x^*) \in \mathbb{R}_+ \times X^*$ the mapping $s \longmapsto \parallel \Phi(t, s)^* x^* \parallel$ is measurable on $[0, t]$.

2. Results

Theorem 2.1. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a measurable evolution operator. If Φ is p.s then there exist $D \geq 1$, $d > 0$ and $c \geq 0$ such that*

$$\int_s^\infty (\tau + 1)^{d-1} \|\Phi(\tau, s)x\| d\tau \leq D(s + 1)^{d+c} \|x\|, \tag{2.1}$$

for all $s \geq 0$ and $x \in X$.

Proof. If Φ is p.s, then according to Definition 1.1 (ii) there exist the constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that, for all $d \in (0, \alpha)$ and $c = \beta$ we have

$$\int_s^\infty (\tau + 1)^{d-1} \|\Phi(\tau, s)x\| d\tau \leq N(s + 1)^{\alpha+\beta} \|x\| \int_s^\infty (\tau + 1)^{d-\alpha-1} d\tau \leq D(s + 1)^{d+c} \|x\|,$$

for all $(s, x) \in \mathbb{R}_+ \times X$, where $D = \frac{N+\alpha-d}{\alpha-d}$.

□

Theorem 2.2. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a measurable evolution operator with p.g and with the property that there exist $D \geq 1$, $c \geq 0$ and $d > \varepsilon$ such that (2.1) holds, where ε is given by Definition 1.1(i). Then Φ is p.s.*

Proof. Let $x \in X$ and $t \geq 2s + 1$. Because

$$\int_{\frac{t-1}{2}}^{t} (\tau + 1)^{a-1} d\tau = (t + 1)^a \frac{2^a - 1}{a2^a},$$

for all $t \geq 0$ and $a > 0$ we have

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| = N \int_{\frac{t-1}{2}}^{t} (\tau + 1)^{d-\varepsilon-1} \|\Phi(t, s)x\| d\tau$$

$$= N \int_{\frac{t-1}{2}}^{t} (\tau + 1)^{d-\varepsilon-1} \|\Phi(\tau, s)x\| M \left(\frac{t + 1}{\tau + 1} \right)^{\omega} (\tau + 1)^{\varepsilon} d\tau$$

$$\leq 2^{\omega} NM \int_{s}^{\infty} (\tau + 1)^{d-1} \|\Phi(\tau, s)x\| d\tau \leq 2^{\omega} NMD(s + 1)^{d+c} \|x\|.$$

Hence, we have that

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| \leq 2^{\omega} NMD(s + 1)^{d-\varepsilon+c+\varepsilon} \|x\|,$$

for all $t \geq 2s + 1$ and $x \in X$, where $N = \frac{(d-\varepsilon)2^{d-\varepsilon}}{2^{d-\varepsilon}-1}$.

For $t \in [s, 2s + 1)$ we have

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| \leq 2^{d+\omega-\varepsilon} M(s + 1)^d \|x\|$$

and hence,

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| \leq K(s + 1)^{d-\varepsilon+c+\varepsilon} \|x\|,$$

for all $(t, s, x) \in \Delta \times X$, where $K = \max\{2^{\omega} NMD, 2^{d-\varepsilon+\omega} M\}$.

Finally, we obtain that Φ is p.s.

□

A discrete variant of the Theorem 2.2 is

Theorem 2.3. *Let* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *be an evolution operator with p.g and with the property that there exist the constants* $D \geq 1$, $d > 0$ *and* $c \geq 0$ *such that*

$$\sum_{k=n}^{\infty} (k + 1)^d \|\Phi(k, n)x\| \leq D(n + 1)^{d+c} \|x\|,$$

for all $n \in \mathbb{N}$ *and* $x \in X$. *Then* Φ *is p.s.*

Proof. According the hypothesis, if we consider $k = m$ then we have

$$(m + 1)^d \|\Phi(m, n)x\| \leq D(n + 1)^{d+c} \|x\|,$$

for all $m, n \in \mathbb{N}$, $m \geq n$ and $x \in X$, which proves that Φ is p.s.

□

Remark. Theorem 2.3 can be considered a Zabczyk's (Zabczyk, 1974) type theorem for polynomial stability.

Theorem 2.4. *Let* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *be an evolution operator. Then* Φ *is p.s if and only if there exist the constants* $B \geq 1$ *and* $b > c \geq 0$ *such that*

$$\sum_{k=0}^{n} (k+1)^{-b-1} \|\Phi(n,k)x\| \leq B(n+1)^{c-b}\|x\|,$$

for all $n \in \mathbb{N}$ *and* $x \in X$.

Proof. Necessity. If we consider $b \in (\beta, \alpha + \beta)$, $c = \beta$ and $B = \frac{N+\alpha+\beta-b}{\alpha+\beta-b}$ we have

$$\sum_{k=0}^{n} (k+1)^{-b-1}\|\Phi(n,k)x\| \leq N(n+1)^{-\alpha}\|x\| \sum_{k=0}^{n} (k+1)^{\alpha+\beta-b-1}$$

$$\leq N(n+1)^{-\alpha}\|x\| \left(1 + \int_0^n (\tau+1)^{\alpha+\beta-b-1} d\tau \right) \leq B(n+1)^{c-b}\|x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$.

Sufficiency. Let $n \geq k \geq 0$ with $n, k \in \mathbb{N}$. According to the hypothesis we have that

$$(k+1)^{-b-1}\|\Phi(n,k)x\| \leq B(n+1)^{c-b}\|x\|$$

which implies

$$(n+1)^{b-c}\|\Phi(n,k)x\| \leq B(k+1)^{b-c+1+c}\|x\|,$$

for all $x \in X$. Hence, Φ is p.s.   □

Remark. Theorem 2.4 can be considered a Barbashin's type theorem for polynomial stability (see (Barbashin, 1967)).

We consider the set

$$\mathcal{R} = \{R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ | R \ nondecreasing, \ R(t) > 0, \ \forall \ t > 0\}.$$

Theorem 2.5. *Let* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *be a* *-measurable evolution operator with p.g. Then* Φ *is p.s if and only if there exist* $B \geq 1$, $b > c \geq 0$ *and a function* $R \in \mathcal{R}$ *such that*

$$\int_0^t R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|\right) d\tau \leq BR\left((t+1)^{c-b}\|x^*\|\right),$$

for all $(t, s, x^*) \in \Delta \times X^*$.

Proof. Necessity. Let us consider $R(t) = t$, $t \geq 0$. If Φ is p.s, then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for all $b \in (\beta, \alpha + \beta)$ and $c = \beta$ we have

$$\int_0^t (\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|d\tau \leq N(t+1)^{-\alpha}\|x^*\| \int_0^t (\tau+1)^{\alpha+\beta-b-1}d\tau = B(t+1)^{c-b}\|x^*\|,$$

where $B = \frac{N+\alpha+\beta-b}{\alpha+\beta-b}$.

Sufficiency. Let $x \in X$ with $\|x\| \leq 1$ and $a - 1 > B$. For $t \geq as + a - 1$ we have

$$(a - 1)R\left(M^{-1}a^{-b-\omega-1}(s + 1)^{-b-c-1}|\langle x^*, \Phi(t, s)x\rangle|\right)$$

$$= \int_s^{s+a-1} R\left(M^{-1}a^{-b-\omega-1}(s + 1)^{-b-c-1}|\langle \Phi(t, \tau)^*x^*, \Phi(\tau, s)x\rangle|\right) d\tau$$

$$\leq \int_s^{as+a-1} R\left((\tau + 1)^{-b-1}\|\Phi(t, \tau)^*x^*\|a^{-b-\omega-1}\left(\frac{\tau + 1}{s + 1}\right)^{b+\omega+1}\right) d\tau$$

$$\leq \int_0^t R\left((\tau + 1)^{-b-1}\|\Phi(t, \tau)^*x^*\|\right) d\tau < (a - 1)R\left((t + 1)^{c-b}\|x^*\|\right).$$

Since R is nondecreasing, we obtain that

$$M^{-1}a^{-b-\omega-1}(s + 1)^{-b-c-1}|\langle x^*, \Phi(t, s)x\rangle| \leq (t + 1)^{c-b}\|x^*\|\|x\|.$$

By taking supremum relative to $\|x^*\| \leq 1$, we have that

$$(t + 1)^{b-c}\|\Phi(t, s)\| \leq Ma^{b+\omega+1}(s + 1)^{b+c+1}.$$

If $t \in [s, as + a - 1)$ we have

$$(t + 1)^{b-c}\|\Phi(t, s)\| \leq M\left(\frac{t + 1}{s + 1}\right)^{b-c+\omega}(s + 1)^b \leq Ma^{b+\omega+1}(s + 1)^b,$$

and, further,

$$(t + 1)^{b-c}\|\Phi(t, s)\| \leq Ma^{b+\omega+1}(s + 1)^{b+c+1},$$

for all $(t, s) \in \Delta$, which proves that Φ is p.s. $\qquad\square$

Remark. Theorem 2.5 can be considered a Rolewicz's type theorem for polynomial stability (see (Rolewicz, 1986)).

Corollary 2.6. Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-measurable evolution operator with p.g. Then Φ is p.s if and only if there exist $B \geq 1$ and $b > c \geq 0$ such that

$$\int_0^t (\tau + 1)^{-b-1}\|\Phi(t, \tau)^*x^*\| d\tau \leq B(t + 1)^{c-b}\|x^*\|,$$

for all $(t, s, x^*) \in \Delta \times X^*$.

Proof. It follows from Theorem 2.5 for $R(t) = t$.

Remark. A similar result was obtained by N. Lupa and M. Megan in (Lupa & Megan, 2012) for the case of nonuniform exponential stability.

3. Examples

In this section we will give some examples that illustrate the connection between the exponential stability and the polynomial stability, as well as the connection between polynomial growth and polynomial stability. Furthermore, we will present some examples of evolution operators which are not p.s and the integral from (2.1) is convergent, respectively divergent.

In contrast with uniform case (where uniform exponential stability implies uniform polynomial stability, see (Megan & Ramneantu, 2011)) in the nonuniform case there is no connection between the concepts of exponential stability and polynomial stability, as shown in the following examples.

Example 3.1. We consider the function

$$u : [1, \infty) \longrightarrow \mathbb{R}^*_+, \quad u(t) = (t + 1)^3 + 1$$

and the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t, s) = \frac{(s + 1)^2 u(s)}{(t + 1)^2 u(t)} I.$$

We have that

$$(t + 1)^2 \|\Phi(t, s)\| \le 2(s + 1)^5, \quad for \ all \ (t, s) \in \Delta.$$

It results that Φ is p.s. If we suppose that Φ is e.s, then there exist $N_1 \ge 1$, $\alpha_1 > 0$ and $\beta_1 \ge 0$ such that

$$e^{\alpha_1 t}(s + 1)^2[(s + 1)^3 + 1] \le N_1 e^{(\alpha_1 + \beta_1)s}(t + 1)^2[(t + 1)^3 + 1], \quad for \ all \ (t, s) \in \Delta.$$

For $s = 0$ and $t \longrightarrow \infty$, we obtain a contradiction and hence Φ is not e.s.

Example 3.2. The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t, s) = \frac{e^{(2 - \cos s)s}}{e^{(2 - \cos t)t}} I$$

satisfies the condition

$$e^t \|\Phi(t, s)\| \le e^{3s}, \quad for \ all \ (t, s) \in \Delta.$$

Hence Φ is e.s. If we suppose that Φ is p.s then there exist $N \ge 1$, $\alpha > 0$ and $\beta \ge 0$ such that

$$(t + 1)^\alpha e^{(2 - \cos s)s} \le N(s + 1)^{\alpha + \beta} e^{(2 - \cos t)t}, \quad for \ all \ (t, s) \in \Delta.$$

From here, for $t = 2(n + 1)\pi$ and $s = (2n + 1)\pi$ we obtain

$$(2n\pi + 2\pi + 1)^\alpha \ e^{4n\pi + \pi} \le N (2n\pi + \pi)^{\alpha + \beta},$$

which for $n \longrightarrow \infty$ yields a contradiction.

It is obvious that if an evolution operator is p.s then it has p.g. The next example presents an evolution operator with p.g, which is not p.s and the integral from (2.1) is divergent.

Example 3.3. The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t, s) = \frac{(s + 1)^{1-\cos(s+1)}}{(t + 1)^{1-\cos(t+1)}} I$$

satisfies the relation

$$(s + 1)^{\omega} \|\Phi(t, s)\| \leq (t + 1)^{\omega} (s + 1)^{\varepsilon}, \quad for \ all \ (t, s) \in \Delta.$$

It results that Φ has p.g for all $\omega > 0$ and $\varepsilon = 2$.

If we suppose that Φ is p.s then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t + 1)^{\alpha} \frac{(s + 1)^{1-\cos(s+1)}}{(t + 1)^{1-\cos(t+1)}} \leq N(s + 1)^{\alpha+\beta},$$

for all $(t, s) \in \Delta$. For $s = \frac{\pi}{2} - 1$ and $t = 2\pi + 2n\pi - 1$, we obtain

$$(2\pi + 2n\pi)^{\alpha} \frac{\pi}{2} \leq N \left(\frac{\pi}{2}\right)^{\alpha+\beta},$$

which if $n \longrightarrow \infty$, leads to a contradiction. We obtain thus that Φ is not p.s.

Let $d \geq 2$ and $s \geq 0$. Then we have

$$\int_s^{\infty} (\tau + 1)^{d-1} \|\Phi(\tau, s)x\| d\tau \geq (s + 1)^{1-\cos(s+1)} \|x\| \int_s^{\infty} (\tau + 1)^{d-3} d\tau = \infty.$$

The next evolution operator does is not p.s and the integral from (2.1) is divergent.

Example 3.4. We consider the set

$$A = \{n + \frac{1}{n + 1} : n \in \mathbb{N}\}$$

and a function $u : [0, \infty) \longrightarrow [1, \infty)$

$$u(t) = \begin{cases} e^{t+1}, & t \notin A \\ e^2, & t \in A. \end{cases}$$

and the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t, s) = \frac{u(s)}{u(t)} I$$

Let $d > 0$ and $s \geq 0$. Then we have

$$\int_s^{\infty} (\tau + 1)^{d-1} \|\Phi(\tau, s)x\| d\tau = u(s)\|x\| \int_s^{\infty} (\tau + 1)^{d-1} e^{-(\tau+1)} d\tau$$

$$\leq u(s)\|x\| \int_{s+1}^{\infty} y^{d-1} e^{-y} dy = \|x\| u(s)\Gamma(d) < \infty.$$

If we suppose that Φ has p.g then there exist $M \geq 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

$$(s + 1)^{\omega} u(s) \leq M(t + 1)^{\omega}(s + 1)^{\varepsilon} u(t), \quad for \ all \ (t, s) \in \Delta.$$

For $s = n$ and $t = n + \frac{1}{n+1}$ we obtain

$$e^{n+1}(n + 1)^{2\omega} \leq Me^2(n + 1)^{\varepsilon}(n^2 + 2n + 2)^{\omega},$$

which for $n \longrightarrow \infty$ yields a contradiction. Hence, Φ does not have p.g and so Φ is not p.s.

4. Open Problem

Finally, we put the following open problems:

1 There exist evolution operators which are not p.s and the relation (2.1) is satisfied?

2 There are evolution operators with p.g with $\varepsilon > 0$, which are not p.s and the relation (2.1) is satisfied for $d \in (0, \varepsilon)$?

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