Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 4 (2) (2014) 211–220

Zweier I-convergent Sequence Spaces Defined by a Sequence of Moduli

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Abstract

In this article we introduce the sequence spaces $\mathcal{Z}^I(F)$, $\mathcal{Z}^I_0(F)$ and $\mathcal{Z}^I_\infty(F)$ for a sequence of moduli $F = (f_k)$ and study some of the topological and algebraic properties on these spaces.

Keywords: Ideal, filter, sequence of moduli, Lipschitz function, I-convergence field, I-convergent, monotone and solid spaces.
2010 MSC: 41A10, 41A25, 41A36, 40A30.

1. Introduction

Let $\mathbb{R}$, and $\mathbb{C}$ be the sets of all real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \ or \ \mathbb{C}\ \}$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $||x||_\infty = \sup_k |x_k|$. Each linear subspace of ω, for example $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines

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a matrix mapping from λ to μ and we denote it by writing $A : \lambda \longrightarrow \mu$. If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$. The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen (Altay *et al.*, 2006), Başar and Altay (Altay & Başar, 2003), Malkowsky (Malkowsky, 1997), Ng and Lee (Ng & Lee, 1978) and Wang (Wang, 1978). Şengönül (Şengönül, 2007) defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e, $y_i = px_i + (1-p)x_{i-1}$ where $x_{-1} = 0, p \neq 0, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, \text{otherwise}. \end{cases}$$

Following Basar and Altay (Altay & Başar, 2003), Şengönül (Şengönül, 2007) introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows $\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$, $\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}$. Here we quote below some of the results due to Şengönül (Şengönül, 2007) which we will need in order to establish the results of this article.

Theorem 1.1 ((Şengönül, 2007), Theorem 2.1)**.** *The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm*

$$\|x\|_{\mathcal{Z}} = \|x\|_{\mathcal{Z}_0} = \|Z^p x\|_c.$$

Theorem 1.2 ((Şengönül, 2007), Theorem 2.2)**.** *The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.*

Theorem 1.3 ((Şengönül, 2007), Theorem 2.3)**.** *The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.*

Theorem 1.4 ((Şengönül, 2007), Theorem 2.6)**.** *$\mathcal{Z}_0$ is solid.*

Theorem 1.5 ((Şengönül, 2007), Theorem 3.6)**.** *$\mathcal{Z}$ is not a solid sequence space.*

The concept of statistical convergence was first introduced by Fast (Fast, 1951) and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Connor (Connor, 1988, 1989; Connor & Kline, 1996), Connor, Fridy and Kline (Fridy & Kline, 1994) and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k} |\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

Later on it was studied by Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński (Kostyrko *et al.*, 2000). Later on it was studied by Šalát, Tripathy, Ziman (Šalát *et al.*, 2004; Šalát *et al.*, 2005) and Demirci (Connor *et al.*, 2001). Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. A set I$\subseteq 2^X$ (2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $£(I) \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin £(\mathrm{I})$, for $A, B \in £(I)$ we have $A \cap B \in £(I)$ and for each $A \in £(I)$ and $A \subseteq B$ implies $B \in £(I)$. An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $£(I)$ corresponding to I. i.e $£(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$.

Definition 1.1. A sequence space E is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

Definition 1.2. A sequence space E is said to be monotone if it contains the canonical preimages of all its stepspaces.

Definition 1.3. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.4. A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$ where $\pi(k)$ is a permutation on $\mathbb{N}$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write I-lim $x_k = L$.
The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C}\}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write I-lim $x_k = 0$.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

Definition 1.10. A modulus function f is said to satisfy $\triangle_2$-condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

Definition 1.11. Take for I the class I_f of all finite subsets of $\mathbb{N}$. Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X (see (Khan & Ebadullah, 2011; Kostyrko *et al.*, 2000)).

Definition 1.12. For I= I_δ and $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ-convergence is said to be logarithmic statistical covergence (see (Khan & Ebadullah, 2011; Kostyrko *et al.*, 2000)).

Definition 1.13. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$ (see (Khan & Ebadullah, 2011)).

Definition 1.14. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in \ell_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $F(I) = \ell_\infty \cap c^I$ (see (Khan & Ebadullah, 2011; Tripathy & Hazarika, 2011)).

Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function (see (Khan & Ebadullah, 2011)). The following Lemmas will be used for establishing some results of this article.

Lemma 1.1. *Let E be a sequence space. If E is solid then E is monotone (see (Kamthan & Gupta, 1981), page 53).*

Lemma 1.2. *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I (see (Tripathy & Hazarika, 2009, 2011)).*

Lemma 1.3. *If $I \subset 2^N$ and M⊆N. If M ∉I, then M∩N ∉I (see (Tripathy & Hazarika, 2009, 2011)).*

The idea of modulus was structured in 1953 by Nakano (See (Nakano, 1953)). A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if (1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is nondecreasing, and
(4) f is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967, 1973) used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle (Ruckle, 1973) proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle (Ruckle, 1968, 1967, 1973) proved that, for any modulus f, $X(f) \subset \ell_1$ and $X(f)^\alpha = \ell_\infty$, where $X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$. The space $X(f)$ is a Banach space with respect to the norm $\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty$.(See[31]).

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in (Gramsch, 1971). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling (Garling, 1966, 1968), Köthe (Köthe, 1970) and Ruckle (Ruckle, 1968, 1967, 1973). After then Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space $X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$.(See[22-23]).

(c.f (Dems, 2005; Gurdal, 2004; Khan *et al.*, 2012*b*,*a*, 2013; Šalát, 1980; Tripathy & Hazarika, 2009, 2011)).

Recently Khan and Ebadullah in (Khan *et al.*, 2013) introduced the following classes of sequences $\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\} \in I\}$, $\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\}$, $\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}$.

We also denote by $m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$ and $m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f)$.

In this article we introduce the following class of sequence spaces:

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\} \in I\},$$
$$\mathcal{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$ and $m_{\mathcal{Z}_0}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}_0^I(F)$.

2. Main Results

Theorem 2.1. *For a sequence of moduli* $F = (f_k)$*, the classes of sequences* $\mathcal{Z}^I(F), \mathcal{Z}_0^I(F), m_{\mathcal{Z}}^I(F)$ *and* $m_{\mathcal{Z}_0}^I(F)$ *are linear spaces.*

Proof. We shall prove the result for the space $\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly.
Let $(x_k), (y_k) \in \mathcal{Z}^I(F)$ and let α, β be scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C};$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in N : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{2.1}$$

$$A_2 = \{k \in N : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{2.2}$$

Since f_k is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|) \leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|).$$

Now, by (2.1) and (2.2), $\{k \in \text{N}: f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(F)$ Hence $\mathcal{Z}^I(F)$ is a linear space.

□

We state the following result without proof in view of Theorem 2.1.

Theorem 2.2. *The spaces $m^I_{\mathcal{Z}}(F)$ and $m^I_{\mathcal{Z}_0}(F)$ are normed linear spaces, normed by*

$$\|x_k\|_* = \sup_k f_k(|x_k|). \tag{2.3}$$

Theorem 2.3. *A sequence $x = (x_k) \in m^I_{\mathcal{Z}}(F)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F) \tag{2.4}$$

Proof. Suppose that $L = I - \lim x$. Then $B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I_{\mathcal{Z}}(F)$. For all $\epsilon > 0$. Fix an $N_\epsilon \in B_\epsilon$.Then we have $|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$.

Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$. This implies that $J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{k \in \mathbb{N} : x_k \in J\} \in m^I_{\mathcal{Z}}(F)$ that is $diamJ \leq diamJ_\epsilon$ where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_k \supseteq ...$ with the property that $diamI_k \leq \frac{1}{2} diamI_{k-1}$ for (k=2,3,4.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I_{\mathcal{Z}}(F)$ for $(k = 1, 2, 3, ...)$. Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$. □

Theorem 2.4. *Let (f_k) and (g_k) be modulus functions for some fixed k that satisfy the $\triangle_2$-condition. If X is any of the spaces $\mathcal{Z}^I, \mathcal{Z}^I_0, m^I_{\mathcal{Z}}$ and $m^I_{\mathcal{Z}_0}$ etc, then the following assertions hold.*
(a) $X(g_k) \subseteq X(f_k.g_k)$,
(b) $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$.

Proof. (a) Let $(x_n) \in \mathcal{Z}^I_0(g_k)$. Then

$$I - \lim_n g_k(|x_n|) = 0. \tag{2.5}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for $0 < t < \delta$.
Write $y_n = g_k(|x_n|)$ and consider $\lim_n f_k(y_n) = \lim_n f_k(y_n)_{y_n<\delta} + \lim_n f_k(y_n)_{y_n>\delta}$. We have

$$\lim_n f_k(y_n) \leq f_k(2) \lim_n (y_n). \tag{2.6}$$

For $y_n > \delta$, we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since f_k is non-decreasing, it follows that $f_k(y_n) < f_k(1 + \frac{y_n}{\delta}) < \frac{1}{2} f_k(2) + \frac{1}{2} f_k(\frac{2y_n}{\delta})$ Since f_k satisfies the $\triangle_2$-condition, we have $f_k(y_n) < \frac{1}{2} K \frac{y_n}{\delta} f_k(2) + \frac{1}{2} K \frac{y_n}{\delta} f_k(2) = K \frac{y_n}{\delta} f_k(2)$.

Hence

$$\lim_n f_k(y_n) \leq \max(1,K) \delta^{-1} f_k(2) \lim_n (y_n). \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have $(x_n) \in \mathcal{Z}_0^I(f_k.g_k)$. Thus $\mathcal{Z}_0^I(g_k) \subseteq \mathcal{Z}_0^I(f_k.g_k)$. The other cases can be proved similarly.

(b) Let $(x_n) \in \mathcal{Z}_0^I(f_k) \cap \mathcal{Z}_0^I(g_k)$. Then $I - \lim_n f_k(|x_n|) = 0$ and $I - \lim_n g_k(|x_n|) = 0$.

The rest of the proof follows from the following equality $\lim_n (f_k + g_k)(|x_n|) = \lim_n f_k(|x_n|) + \lim_n g_k(|x_n|)$. □

Corollary 2.1. *$X \subseteq X(f_k)$ for some fixed k and $X = \mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.*

Theorem 2.5. *The spaces $\mathcal{Z}_0^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are solid and monotone.*

Proof. We shall prove the result for $\mathcal{Z}_0^I(F)$. Let $(x_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim_k f_k(|x_k|) = 0. \tag{2.8}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
Then the result follows from [9] and the following inequality $f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|)$ for all $k \in \mathbb{N}$. That the space $\mathcal{Z}_0^I(F)$ is monotone follows from the Lemma 1.20. For $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly. □

Theorem 2.6. *The spaces $\mathcal{Z}^I(F)$ and $m_{\mathcal{Z}}^I(F)$ are neither solid nor monotone in general.*

Proof. Here we give a counter example. Let $I = I_\delta$ and $f_k(x) = x^2$ for some fixed k and for all $x \in [0, \infty)$. Consider the K-step space $X_K(f_k)$ of X defined as follows. Let $(x_n) \in X$ and let $(y_n) \in X_K$ be such that

$$(y_n) = \begin{cases} (x_n), \text{if n is even,} \\ 0, \textit{otherwise.} \end{cases}$$

Consider the sequence (x_n) defined by $(x_n) = 1$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(F)$. Thus $\mathcal{Z}^I(F)$ is not monotone. Hence $\mathcal{Z}^I(F)$ is not solid. □

Theorem 2.7. *The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are sequence algebras.*

Proof. We prove that $\mathcal{Z}_0^I(F)$ is a sequence algebra. Let $(x_k), (y_k) \in \mathcal{Z}_0^I(F)$. Then $I - \lim f_k(|x_k|) = 0$ and $I - \lim f_k(|y_k|) = 0$. Then we have $I - \lim f_k(|(x_k.y_k)|) = 0$. Thus $(x_k.y_k) \in \mathcal{Z}_0^I(F)$ is a sequence algebra. For the space $\mathcal{Z}^I(F)$, the result can be proved similarly. □

Theorem 2.8. *The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free in general.*

Proof. Here we give a counter example. Let $I = I_f$ and $f_k(x) = x^3$ for some fixed k and for all $x \in [0, \infty)$. Consider the sequence (x_n) and (y_n) defined by $x_n = \frac{1}{n}$ *and* $y_n = n$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$, but $(y_n) \notin \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$. Hence the spaces $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free. □

Theorem 2.9. *If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.*

Proof. Let $A \in I$ be infinite and $f_k(x) = x$ for some fixed k and for all $x \in [0, \infty)$. If

$$x_n = \begin{cases} 1, \text{for } n \in A, \\ 0, otherwise, \end{cases}$$

then by lemma 1.22 $(x_n) \in \mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$.
Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(n) = \begin{cases} \phi(n), \text{for } n \in K, \\ \psi(n), otherwise, \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(n)} \notin \mathcal{Z}^I(F)$ and $x_{\pi(n)} \notin \mathcal{Z}_0^I(F)$. Hence $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric. □

Theorem 2.10. $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$.

Proof. Let $(x_k) \in \mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_k(|x_k - L|) = 0$. We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I(F)$. The inclusion $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$ is obvious. □

Theorem 2.11. *The function $\hbar : m_{\mathcal{Z}}^I(F) \to \mathbb{R}$ is the Lipschitz function, where $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$, and hence uniformly continuous.*

Proof. Let $x, y \in m_{\mathcal{Z}}^I(F)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq \|x - y\|_*\} \in I,$$
$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F).$$

Hence also $B = B_x \cap B_y \in m_{\mathcal{Z}}^I(F)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarity. □

Theorem 2.12. *If $x, y \in m_{\mathcal{Z}}^I(F)$, then $(x.y) \in m_{\mathcal{Z}}^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m_{\mathcal{Z}}^I(F),$$
$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m_{\mathcal{Z}}^I(F).$$

Now,

$$|x_k y_k - \hbar(x)\hbar(y)| = |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|. \quad (2.9)$$

As $m_{\mathcal{Z}}^I(F) \subseteq \mathcal{Z}_\infty^I(F)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn[10] we get $|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$ For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m_{\mathcal{Z}}^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarity. □

Acknowledgments: The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

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