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Common Fixed Points of Fuzzy Mappings in Quasi-Pseudo Metric and Quasi-Metric Spaces

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Abstract

In this paper, we prove common fixed point theorems for fuzzy mappings satisfying a new inequality initiated by Constantin (1991) in complete quasi-pseudo metric space and we also obtain some new common fixed point theorems for a pair of fuzzy mappings on complete quasi-metric space under a generalized contractive condition. Our results generalized many recent fixed point theorems.

Keywords: fuzzy sets, fuzzy mappings, common fixed points, quasi-pseudo metric space, quasi-metric space, fuzzy contraction mappings.
2010 MSC: 47H10, 54H25, 54A40.

1. Introduction

It is a well known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the period of last forty years the theory of fixed points has been developed regarding the results which are related to finding the fixed points of self and non-self nonlinear mappings. In 1922, Banach proved a contraction principle which states that for a complete metric space (X, d), the mapping $T: X \to X$ satisfying the following contraction condition

$$d(Tx, Ty) \le \alpha d(x, y) \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad 0 < \alpha < 1$$

has a unique fixed point in X. Banach contraction principle plays a fundamental role in the emergence of modern fixed point theory and it gains more attention because it is based on iteration, so it can be easily applied using computer. Initially Zadeh (1965) introduced the concept of Fuzzy Sets in 1965, has been an attempt to develop a mathematical framework in which two system or phenomena which due to intrinsic indefiniteness-as distinguished from mere statistical variation can't themselves be characterized precisely. The classical work of Zadeh (1965) stimulated a great interest among mathematicians, engineers, biologists, economists, psychologists and experts in other areas who use mathematical method in their research.

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The notion of fixed points for fuzzy mappings was introduced by Weiss (1975) and Butnariu (1982). Fixed point theorems for fuzzy set valued mappings have been studied by Heilpern (1981) who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces which is a fuzzy extension of Banach fixed point theorem and Nadler (1969) theorem for multi-valued mappings. Park & Jeong (1997) proved some common fixed point theorems for fuzzy mappings satisfying in complete metric space which are fuzzy extensions of some theorems in Beg & A. (1992); Park & Jeong (1997).

Motivated and inspired by the works of Arora & V. (2000), Constantin (1991) and Park & Jeong (1997) the purpose of this paper is to prove some common fixed point theorems for fuzzy mappings satisfying new contractive-type condition of Constantin (1991) in complete quasi-pseudo metric space. Our results are the fuzzy extensions of some theorems in Beg & A. (1992); Iseki (1995); Popa (1985); Singh & Whitfield (1982) . Also, our results generalize the results of Arora & V. (2000), Heilpern (1981), and Park & Jeong (1997).

Recently Chen (2011, 2012) considered a new type contraction ψ contractive mapping in complete quasi metric space. The aim of this paper is to introduced a new class of fuzzy contraction mappings, which will be call fuzzy ψ contractive mappings in complete quasi metric space and to prove the existence of common fixed point for these contractions.

2. Basic concepts

For this purpose we need the following definitions and Lemmas.

Definition 2.1. Sahin *et al.* (2005) A quasi-pseudo metric on a non-empty set X is a non-negative real valued function d on $X \times X$ such that, for all $x, y, z \in X$:

(i) $d(x, x) = 0$, and
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X, d) is called a quasi-pseudo metric space, if d is a quasi-pseudo metric on X. A quasi-pseudo metric d such that $x = y$ whenever $d(x, y) = 0$ is a quasi metric so that a quasi pseudo metric space we do not assume that $d(x, y) = d(y, x)$ for every x and y. Each quasi-pseudo metric d on X induces a topology $\tau(d)$ which has base the family of all d balls $B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ If d is a quasi-pseudo metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also quasi-pseudo metric on X. By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote $\min\{d, d^{-1}\}$ and $\max\{d, d^{-1}\}$ respectively.

Definition 2.2. Gregori. & Pastor (1999) Let (X, d) be a quasi-pseudo metric space and let A and B be non-empty subsets of X. Then the Hausdroff distance between subsets of A and B is defined by

$$H(A, B) = max\{sup_{a\in A} d(a, B), sup_{b\in B} d(b, A)\}$$

where $d(a, B) = inf\{d(a, x) : x \in B\}$.

Note that: $H(A, B) \geq 0$ with $H(A, B) = 0$ if and only if closure of A is equal to closure of B, $H(A, B) = H(B, A)$ and $H(A, B) \leq H(A, C) + H(C, B)$ for any non-empty subset A, B and C of X when d is a metric on X, clearly H is the usual Hausdroff distance.

Definition 2.3. Gregori. & Pastor (1999) Let (X, d) be a quasi-pseudo metric space. The families $W^*(X)$ and $W'(X)$ of fuzzy sets on (X, d) are defined by

$$W^*(X) = \{A \text{ in } I^X : A_1 \text{ is non-empty, } d\text{ - closed and } d^{-1}\text{-}compact\},$$

$$W'(X) = \{A \text{ in } I^X : A_1 \text{ is non-empty, } d\text{ - closed and } d\text{-compact}\}.$$

As per Heilpern (1981), the family W(X) of fuzzy sets on metric linear space (X, d) is defined as follows: $A \in W(X)$ if and only if A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $supA(x) = 1$ for $x \in X$.
If (X, d) is a metric linear space, then we have

$$W(X) \subset W^*(X) = W'(X) = \{A \in I^X : A_1 \text{ is non-empty and } d\text{-compact}\} \subset I^X.$$

Definition 2.4. Gregori. & Pastor (1999) Let (X, d) be a quasi-pseudo metric space and let $A, B \in W^*(X)$ or $A, B \in W'(X)$ and $\alpha \in [0, 1]$. Then we define

$$p_\alpha(A, B) = Inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$\delta_\alpha(A, B) = sup\{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where H is the Hausdroff distance deduced from the quasi-pseudo metric d on X, $p(A, B) = Sup\{p_\alpha(A, B) : \alpha \in [0, 1]\}$, $\delta(A, B) = Sup\{\delta_\alpha(A, B) : \alpha \in [0, 1]\}$, $D(A, B) = Sup\{ D_\alpha(A, B) : \alpha \in [0, 1]\}$. It is noted that p_α is non-decreasing function of α.

Definition 2.5. Gregori. & Pastor (1999) Let X be an arbitrary set and Y be any quasi-pseudo metric space. G is said to be a fuzzy mapping if G is a mapping from the set X into $W^*(Y)$ or $W'(Y)$. This definition is more general than the one given in Heilpern (1981). A fuzzy mapping G is a fuzzy subset on $X \times Y$ with membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of y in $G(x)$.

Definition 2.6. Sahin *et al.* (2005) A point x is a fixed point of the mapping $G : X \to I^X$, if $\{x\} \subseteq G(x)$.

Note that : If $A, B \in I^X$, then $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.
The following Lemmas were proved by Gregori. & Pastor (1999).

Lemma 2.1. *Let (X, d) be a quasi-pseudo metric space and let $x \in X$ and $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ iff $p_\alpha(x, A) = 0$, for each $\alpha \in [0, 1]$.*

Lemma 2.2. *Let (X, d) be a quasi-pseudo metric space and let $A \in W^*(X)$. Then $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$ and $\alpha \in [0, 1]$.*

Lemma 2.3. *Let (X, d) be a quasi-pseudo metric space and let $\{x_0\} \subset A$. Then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $A, B \in W^*(X)$ and $\alpha \in [0, 1]$.*

Above Lemmas were proved by Heilpern (1981) for the family $W(X)$ in a metric linear space.

Proposition 1. *Let (X, d) be a complete quasi-pseudo metric space and $G : X \to W^*(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Proposition 2. *Let (X, d) be a quasi-pseudo metric space and $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Now we shall use the notations as in Isufati & Hoxha (2010).
In the following, the letter Γ denotes the set of positive integers.
If A is a subset of a topological space (X, τ), we will denote by $cl_\tau A$ the closure of A in (X, τ).
A quasi-metric on a non-empty set X is a non-negative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X,d) is called a quasi-metric space, if d is a quasi-metric on X.
Each quasi-metric d on X induces a T_0 topology $\mathcal{T}(d)$ on X, which has a base, the family of all $d-$ balls $\{B_d(x, r) : x \in X, r > 0\}$, where, $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

If d is a quasi-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also quasi metric on X. By $d \wedge d^{-1}$ we denote min $\{d, d^{-1}\}$ and also we denote d^s the metric on X by $d^s(x, y) = max\{d(x, y), d(y, x)\}$ for all $x, y \in X$.

A sequence $(x_n)_{n\in\Gamma}$ in a quasi metric space (X, d) is called left $k-$ Cauchy Reilly *et al.* (1982) if for each $\varepsilon > 0$ there is a $n_\varepsilon \in \Gamma$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \Gamma$ with $m \geq n \geq n_\varepsilon$. Let (X, d) be a quasi-metric space and let $\mathcal{K}_0^s(X)$ be the collection of all non-empty compact subset of the metric space (X, d^s). Then the Hausdroff distance H_d on $\mathcal{K}_0^s(X)$ is defined by

$$H_d(A, B) = \max\{\ \sup d(a, B) : a \in A,\ \ \sup d(A, b) : b \in B\}\ \text{ whenever } A, B \in \mathcal{K}_0^s(X).$$

A fuzzy set on X is an element of I^X where $I = [0, 1]$. If A is a fuzzy set in X, then the number $A(x)$ is called the grade of membership of x in A. The α- level set of A, denoted by A_α, and defined by $A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$ and $A_0 = \overline{\{x : A(x) > 0\}}$ where the closure is taken in (X, d^s).

Definition 2.7. Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space. A fuzzy set A in quasi-metric space (X, d) will be called an approximate quantity. The family $\mathcal{A}(X)$ of all fuzzy sets on (X, d) is defined by $\mathcal{A}(X) = \{A \in I^X : A_\alpha$ is d^s -compact for each $\alpha \in [0, 1]$ and $supA(x) = 1 : x \in X\}$.

Definition 2.8. Gregori & Romaguera (2000) Let $A, B \in \mathcal{A}(X)$ then A is said to be more accurate than B, denoted by $A \subset B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.

Definition 2.9. Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A, B \in \mathcal{A}(X)$ and α in $[0, 1]$. Then we define $p_\alpha(A, B) = Inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha)$, $D_\alpha(A, B) = H_d(A_\alpha, B_\alpha)$, $p(A, B) = sup\{P_\alpha(A, B) : \alpha \in [0, 1]\}$, $D(A, B) = sup\{D_\alpha(A, B) : \alpha \in [0, 1]\}$, for $x \in X$, we write $p_\alpha(x, A)$ instead of $p_\alpha(\{x\}, A)$. We denote that p_α is a non-decreasing function of α and D is metric on $\mathcal{A}(x)$.

Definition 2.10. Gregori & Romaguera (2000) A fuzzy mapping on a quasi-metric space (X, d) is a function F defined on X, which satisfies the following two conditions

(i) $F(x) \in \mathcal{A}(X)$ for all $x \in X$,
(ii) If $a, z \in X$ such that $(F(z))(a) = 1$ and $p(a, F(a)) = 0$ then $(F(a))(a) = 1$.

We need the following lemmas for our main result which was given by Gregori & Romaguera (2000).

Lemma 2.4. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A, B \in \mathcal{A}(X)$ and $x \in A_1$. There exist $y \in B_1$ such that $d(x, y) \leq D_1(A, B)$.*

Lemma 2.5. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A \in \mathcal{A}(X)$ and $y \in A_.$ Then $p(x, A) \leq d(x, y)$ for each $x \in X$.*

Lemma 2.6. *Gregori & Romaguera (2000) Let $x \in X$, $A \in \mathcal{A}(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$, then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Lemma 2.7. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and $A \in \mathcal{A}(X)$. Then $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$.*

Lemma 2.8. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A \in \mathcal{A}(X)$ and $x \in A$. Then $p_\alpha(x, B) \leq D_\alpha(A, B)$ for each $B \in \mathcal{A}(X)$ and each $\alpha \in [0, 1]$.*

Lemma 2.9. *Gregori & Romaguera (2000) Let A and B be non-empty compact subset of a quasi-metric space (X, d) if $a \in A$, then there exists $b \in B$, such that $d(a, b) \leq H(A, B)$.*

Lemma 2.10. *Gregori & Romaguera (2000) Let (X, d) be a complete quasi metric space and let F be a fuzzy mapping from X into $\mathcal{A}(X)$ and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

We consider the set of function $\Psi = \{\psi : R^{+5} \to R^+\}$ satisfying the following conditions

(i) ψ strictly increasing, continuous function in each coordinate and

(ii) for all $g \in R^+$ such that $\psi(g, g, g, 0, 2g) < g$, $\psi(g, g, g, 2g, 0) < g$, $\psi(0, 0, g, g, 0) < g$ and $\psi(g, 0, 0, g, g) < g$.

Example 2.11. *Let $\psi : R^{+5} \to R^{+5}$ denote by $\psi(g_1, g_2, g_3, g_4, g_5) = k\, max(g_1, g_2, g_3, \frac{g_4}{2}, \frac{g_5}{2})$ for $k \in (0, 1)$ then ψ satisfies above conditions (i) and (ii).*

3. Main Result

Following Constantin (1991) we consider the set $\mathcal{G}$ of all continuous functions $g : [0, \infty)^5 \to [0, \infty)$ with the following properties:

(1) g is non-decreasing in the $2^{nd}, 3^{th}, 4^{th}$ and 5^{th} variable,

(2) if $u, v \in [0, \infty)$ are such that $u \leq g(v, v, u, u + v, 0)$ or $u \leq g(v, u, v, 0, u + v)$ then $u \leq qv$ where $0 < q < 1$ is a given constant,

(3) if $u \in [0, \infty)$ is such that $u \leq g(u, 0, 0, u, u)$ then $u = 0$.

Now we are ready to prove our main theorems.

Theorem 3.1. *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there is a $g \in \mathcal{G}$ such that for $x, y \in X$*

$$D(G_1(x), G_2(y)) \leq g(d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x)))$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. Let $x_0 \in X$. Then by Proposition 2.1 there exists an $x_1 \in X$ such that $\{x_1\} \subset G_1(X_0)$. From Proposition 2.1 there exists $x_2 \in (G_2(x_1))_1$. Since $(G_1(x_0))_1, (G_2(x_1))_1 \in CP(X)$ then by Proposition 2.2 we obtain,

$$\begin{aligned} d(x_1, x_2) \;\; &\leq \;\; D_1(G_1(x_0), G_2(x_1)) \leq D(G_1(x_0), G_2(x_1)) \leq g(d(x_0, x_1), p(x_0, G_1(x_0)), p(x_1, G_2(x_1)), \\ & \quad\;\; p(x_0, G_2(x_1)), p(x_1, G_1(x_0))) \leq g(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \end{aligned}$$

therefore, $d(x_1, x_2) \leq qd(x_0, x_1)$. Following similar process we obtain, $d(x_2, x_3) \leq qd(x_1, x_2)$. By induction, we produce a sequence (x_n) of points of X such that for each $k \geq 0$ $\{x_{2k+1}\} \subset G_1(x_{2k})$, and $\{x_{2k+2}\} \subset G_2(x_{2k+1})$, $d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \leq \cdots \leq q^n d(x_0, x_1)$. Furthermore, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \{q^n + q^{n+1} + \cdots + q^{m-1}\} d(x_0, x_1) \leq \frac{q^n}{(1 - q)} d(x_0, x_1). \end{aligned}$$

It follows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset G_i(z), i = 1, 2$. Now by Lemma 2.2 $p_0(z, G_2(z)) \leq d(z, x_{2n+1}) + p_0(x_{2n+1}, G_2(z))$. Then by Lemma 2.3,

$$\begin{aligned} p(z, G_2(z)) &\leq d(z, x_{2n+1}) + D(G_1(x_{2n}), G_2(z)) \leq d(z, x_{2n+1}) + f(d(x_{2n}, z), p(x_{2n}, G_1(x_{2n})), \\ &\quad p(z, G_2(z)), p(x_{2n}, G_2(z)), p(z, G_1(x_{2n}))) \\ &\leq d(z, x_{2n+1}) + g(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, G_2(z)), p(x_{2n}, G_2(z)), d(z, x_{2n+1})). \end{aligned}$$

As $n \to \infty$, we obtain from above inequality that $p(z, G_2(z)) \leq g(0, 0, p(z, G_2(z)), p(z, G_2(z)), 0)$, so by properties of g we have $p(z, G_2(z)) = 0$. by (2). So by Lemma 2.1, we get $\{z\} \subset G_2(z)$. Similarly, it can be shown that $\{z\} \subset G_1(z)$. □

As corollaries of Theorem 3.1, we have the following:

Corollary 3.2 (Park & Jeong (1997); Theorem 3.1)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there exists a constant $\alpha, 0 \leq \alpha < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq \alpha \cdot \max\{d(x, y), p(x, G_1(x)), p(y, G_2(y)), \frac{[p(x,G_2(y))+p(y,G_1(x))]}{2}\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha{\cdot}\max\{x_1, x_2, x_3, \frac{(x_4+x_5)}{2}\}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.1. □

Corollary 3.3 (Park & Jeong (1997); Theorem 3.2)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. satisfying $D(G_1(x), G_2(y)) \leq k[p(x, G_1(x)) \cdot p(y, G_2(y))]^{\frac{1}{2}}$, for all $x, y \in X$ and $0 < k < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = k[x_2 \cdot x_3]^{\frac{1}{2}}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.2. □

Corollary 3.4 (Park & Jeong (1997); Theorem 3.4)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$, such that*

$$D(G_1(x), G_2(y)) \leq \alpha \cdot \frac{p(y, G_1(y))[1 + p(x, G_2(x))]}{1 + d(x, y)} + \beta d(x, y)$$

for all $x \neq y, \alpha, \beta > 0$ and $\alpha + \beta < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \frac{x_3(1+x_2)}{(1+x_1)} + \beta x_1$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.3. □

Corollary 3.5 (Arora & V. (2000); Theorem 3.2)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there exists a constant $r, 0 \leq r < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq r{\cdot}\max\{d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x))\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = r{\cdot}\max\{x_1, x_2, x_3, x_4, x_5\}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain corollary 3.4. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Iseki (1995).

Corollary 3.6. *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If for each $x, y \in X$, such that $D(G_1(x), G_2(y)) \leq \alpha[p(x, G_1(x)) + p(y, G_2(y))] + \beta[p(x, G_2(y)) + p(y, G_1(x))] + \gamma d(x, y)$ where α, β, γ are non-negative and $2\alpha + 2\beta + \gamma < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha[x_2 + x_3] + \beta[x_4 + x_5] + \gamma x_1$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain corollary 3.5. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Singh & Whitfield (1982).

Corollary 3.7. *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there exists a constant $\alpha, 0 \leq \alpha < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq \alpha \cdot \max\{d(x, y), \frac{[p(x,G_1(x))+p(y,G_2(y))]}{2}, \frac{[p(x,G_2(y))+p(y,G_1(x))]}{2}\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \max\{x_1, \frac{[x_2+x_3]}{2}, \frac{[x_4+x_5]}{2}\}$ Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.6. □

Remark. If there exists a function $g \in \mathcal{G}$ such that for all $x, y \in X$

$$\delta(G_1(x), G_2(y)) \leq g(d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x))),$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as special case of Theorem 3.1 because (see, Hicks (1997); page 414) $D(G_1(x), G_2(y)) \leq \delta(G_1(x), G_2(y))$. Moreover, this result generalize Theorem 3.3 of Park & Jeong (1997).

The following theorem extends Theorem 3.1 to a sequence of fuzzy mappings:

Theorem 3.8. *Let X be a complete quasi-pseudo metric space and let $\{G_n : n \in Z^+\}$ be fuzzy mappings from X into $W^*(X)$. If there is a $g \in \mathcal{G}$ such that for all $x, y \in X$*

$$D(G_0(x), G_n(y)) \leq g(d(x, y), p(x, G_0(x)), p(y, G_n(y)), p(x, G_n(y)), p(y, G_0(x)))$$

then there exists a common fixed point of the family $\{G_n : n \in Z^+\}$.

Proof. From Theorem 3.1, we get a common fixed point $x_i, i = 1, 2, \ldots,$ for each pair $(G_0, G_i), i = 1, 2, \ldots$. Applying Lemma 2.2, one can have that $p_\alpha(x_i, G_0 x_i) = P_\alpha(x_i, G_i(x_i)) = 0$, for all $i = 1, 2, \ldots$. Thus one can deduce form Lemma 2.3, for $i \neq j$, that

$$\begin{aligned} d(x_i, x_j) &= p_\alpha(x_i, G_j(x_j)) \leq D_\alpha(G_i(x_i), G_j(x_j)) \leq D(G_i(x_i), G_j(x_j)) \\ &\leq g(d(x_i, x_j), p(x_i, G_i(x_i)), p(x_j, G_j(x_j)), p(x_i, G_j(x_j)), p(x_j, G_i(x_i))) \\ &= g(d(x_i, x_j), 0, 0, d(x_i, x_j), d(x_i, x_j)). \end{aligned}$$

Therefore $d(x_i, x_j) = 0$, i.e., $x_i = x_j$ for all $i, j \in \mathbb{N}$. □

Corollary 3.9. *(Arora & V. (2000);Theorem (3.4)) Let X be a complete quasi-pseudo metric space and let $\{G_n : n \in \mathbb{N}^+\}$ be fuzzy mappings from X into $W^*(X)$. If for each $x, y \in X$, and $r \in (0, \frac{1}{2}), n = 1, 2, , ...,$ such that $D(G_0(x), G_i(y)) \leq r\max\{d(x, y), p(x, G_o(x)), p(y, G_i(y)), p(x, G_i(y)), p(y, G_0(x))\}$. Then there exists a common fixed point of the family $\{G_n : n \in \mathbb{N}^+\}$.*

Theorem 3.10. *Let (X, d) be a complete quasi-metric space, let $T_1, T_2 \colon X \to \mathcal{A}(X)$ be fuzzy ψ contractive mappings satisfies $D(T_1x, T_2y) \leq \psi\{(d(x, y), p(x, T_1x), p(y, T_2y), p(x, T_2y), p(y, T_1x)\}$ then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.*

Proof. Let $x_0 \in X$ then by Lemma 2.10 there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$ for $x_1 \in T_2(x_1))_1$ is non-empty compact subset of X. Since $(T_1(x_0))_1, (T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0)_1)$, then by lemma 2.9 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1, x_2) \leq D_1(T_1(x_0), T_2(x_1))$ so, from Lemma 2.6 and properties of ψ function, we have

$$\begin{aligned} d(x_1, x_2) &\leq D_1(T_1(x_0), T_2(x_1)) \leq D(T_1(x_0), T_2(x_1)) \\ &\leq \psi(d(x_0, x_1), p(x_0, T_1x_0), p(x_1, T_2x_1), p(x_0, T_2x_1), p(x_1, T_1x_0)) \\ &\leq \psi(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \end{aligned}$$

and

$$\begin{aligned} d(x_2, x_1) &\leq D_1(T_2(x_1), T_1(x_0)) \leq D(T_2(x_1), T_1(x_0)) \\ &\leq \psi(d(x_1, x_0), p(x_1, T_2x_1), p(x_0, T_1x_0), p(x_1, T_1x_0), p(x_0, T_2x_1)) \\ &\leq \psi(d(x_1, x_0), d(x_1, x_2), d(x_0, x_1), 0, d(x_0, x_1) + d(x_1, x_2)) \end{aligned}$$

by induction, we have a sequence (x_n) of points such that for all $n \in R^+ \cup \{0\}$ we have $\{x_{2n+1}\} \subset T_1(x_{2n})$ and $\{x_{2n+2}\} \subset T_2(x_{2n+1})$ then

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0) \tag{3.1}$$

and

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \tag{3.2}$$

so, by the properties of the ψ function we have that for each $n \in R^+$ $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ and $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$. The sequence $(b_m)_{m \in R^+}$, such that $b_m = d(x_m, x_{m+1})$ is a non-increasing sequence and bounded below. Thus it must converges to some $b \geq 0$. By the inequality 3.1 and 3.2 we have

$$b \leq b_m \leq \psi(b_{m-1}, b_{m-1}, b_m, b_{m-1} + b_m, 0) < b \tag{3.3}$$

passing to the limit, as $m \longrightarrow \infty$, and by properties of the ψ function we have $b \leq b \leq \psi(b, b, b, 2b, 0) < b$ which is contradiction. Hence $b = 0$. Thus, the sequence $(x_n)_{n \in R^+}$ must be a Cauchy sequence.

Similarly, the sequence $(c_n)_{n \in R^+}$ such that $c_n = d(x_{n+1}, x_n)$ is a non-increasing sequence and bounded below. Thus, it must converges to some $c \geq 0$.
By the inequality 3.1 and 3.2 we have

$$c \leq c_n \leq \psi(c_{n-1}, c_{n-1}, c_n, c_{n-1} + c_n, 0) < b \tag{3.4}$$

passing to the limit, as $n \longrightarrow \infty$, and by properties of the ψ function we have $c \leq c \leq \psi(c, c, c, 2c, 0) < c$ which is possible if and only if $c = 0$.

We next claim that to prove that for each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in R^+$, such that for all $m > n > n_0(\varepsilon)$

$$d(x_m, x_n) < \varepsilon. \tag{3.5}$$

Suppose that 3.5 is false then, there exists some $\varepsilon > 0$ such that for all $k \in R^+$, there exists the smallest number m_k, such that $m_k, n_k \in R^+$ with $m_k > n_k \leq k$ satisfying $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ so,

$$\begin{aligned}
\varepsilon &\leq d(x_{m_k}, x_{n_k}) \leq D(Tx_{m_k-1}, Tx_{n_k-1}) \\
&\leq \psi(d(x_{m_k-1}, x_{n_k-1}), p(x_{m_k-1}, Tx_{m_k-1}), p(x_{n_k-1}, Tx_{n_k-1}), p(x_{m_k-1}, Tx_{n_k-1}), p(x_{n_k-1}, Tx_{m_k-1}) \\
&\leq \psi(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k}) \\
&\leq \psi(c_{m_k-1} + d(x_{m_k}, x_{n_k}) + c_{n_k-1}, c_{m_k-1}, c_{n_k-1}, c_{m_k-1} + d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k}) + c_{n_k-1}).
\end{aligned}$$

Letting $k \to \infty$ we have $\varepsilon \leq \psi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \varepsilon$ which is a contradiction. It follows from 3.5 that (x_n) is a Cauchy sequence since (X, d) is a complete quasi-metric space,then there exists $z \in X$ such that $lim_{n\to\infty} x_n = z$.

Next we show that $\{z\} \subset T_2(z)$.

By Lemmas 2.7 and 2.8 we get $p_\alpha(z, T_2z) \leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2z) \leq d(z, x_{2n+1}) + D_\alpha(T_1x_n, T_2z)$ for each $\alpha \in [0, 1]$. Taking supremum on α in the last inequality, we obtain from the properties of ψ that

$$\begin{aligned}
p_\alpha(z, T_2z) &\leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2z) \leq d(z, x_{2n+1}) + D_\alpha(T_1x_{2n}, T_2z) \\
&\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z), p(x_{2n}, T_1x_{2n}), p(z, T_2z), p(x_{2n}, T_2z), d(z, x_{2n+1})) \\
&\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, T_2z), p(x_{2n}, T_2z), d(z, x_{2n+1})).
\end{aligned}$$

As $n \to \infty$, we have $p(z, T_2z) \leq \psi(0, 0, p(z, T_2z), p(z, T_2z), 0) < p(z, T_2z)$. It yields that $p(z, T_2z) = 0$. So, we get from Lemma 2.10 that $\{z\} \subset T_2z$. Similarly we prove that $\{z\} \subset T_1z$. □

Corollary 3.11. *Let (X, d) be a complete quasi metric space and let $T : X \to \mathcal{A}(X)$ be a fuzzy ψ contraction mapping then there exists $z \in X$ such that $\{z\} \subset T(z)$.*

Proof. If put $T_1 = T_2 = T$ in theorem 3.3 we get the conclusion of corollary 3.8. □

Corollary 3.12. *Let (X, d) be a complete quasi metric space and let $T : X \to \mathcal{A}(X)$ be a fuzzy ψ contraction mapping, such that for all $x, y \in X$ $D(T_1x, T_2y) \leq \psi(d(x, y), p(x, T_1x), p(y, T_2y), \frac{p(x,T_2y)}{2}, \frac{p(y,T_1x)}{2})$ then there exists $z \in X$ such that $\{z\} \subset T_1z$ and $\{z\} \subset T_2z$.*

Proof. We consider the function $\psi : R^{+5} \to R^{+5}$ denoted by $\psi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\}$ for $k \in (o, 1)$. Since $\psi \in \Psi$ we can apply theorem 3.3 and obtain Corollary 3.9. □

Remark. As examples of the main results we can taking theorems in which the contractions conditions are compatible with the condition (i) and (ii).

Remark. If there is a $\psi \in \Psi$ such that for all $x, y \in X$

$$\delta(T_1x, T_2y) \leq \psi(d(x, y), p(x, T_1x), p(y, T_2y), p(x, T_2y), p(y, T_1x))$$

then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because $D_1(T_1x, T_2y) \leq \delta(T_1x, T_2y)$ for all $x, y \in X$. The following theorem generalizes Theorem 3.3 to a sequence of fuzzy contractive mappings.

Theorem 3.13. *Let $(T_n : n \in (0, \infty) \cup \{0\})$ be a sequence of fuzzy mappings from a complete quasi metric space X into $\mathcal{A}(X)$. If there is a $\psi \in \Psi$ such that for all $x, y \in X$*

$$D(T_0x, T_ny) \leq \psi(d(x, y), p(x, T_0x), p(y, T_ny), p(x, T_ny), p(y, T_0x))$$

for all $n \in (0, \infty) \cup \{0\}$, then there exists a common fixed point of the family $(T_n : n \in (0, \infty) \cup \{0\})$.

Proof. Putting $T_1 = T_0$ and $T_2 = T_n$ for all $n \in N$ in Theorem 3.3 then there exists a common fixed point of the family $(T_n : n \in (0, \infty) \cup \{0\})$. □

4. Conclusion and future work

Fuzzy sets and mappings play an important role in the fuzzification of systems. In particular, in the recent years the fixed point theory for fuzzy mappings has been developed largely. We generalize, extend and unify several known results of metric spaces, into a weaker and generalize setting of quasi-pseudo metric space and quasi metric space for fuzzy mappings. We use a more generalize contractive condition than the existing ones, also we prove our results in quasi-pseudo metric space, quasi metric space and so as to obtain better results under weaker conditions. We conclude this paper with an open problem: Is it possible to prove the results of this paper in the setting of b-metric and partial metric spaces?

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