



Second Hankel Determinant for Generalized Sakaguchi Type Functions

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Abstract

In this paper, we have obtained sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ belonging to a new subclass of generalized Sakaguchi type functions introduced by (Frasin, 2010).

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1. Introduction

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}) \quad (1.1)$$

and S be the subclass of A consisting of univalent functions. For two functions $f, g \in A$, we say that the function $f(z)$ is subordinate to $g(z)$ in Δ and write $f < g$, or $f(z) < g(z)$; ($z \in \Delta$) if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \Delta$), such that $f(z) = g(w(z))$, ($z \in \Delta$). In particular, if the function g is univalent in Δ , the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Denote by S_S^* the subclass of S consisting of functions given by (1.1) satisfying $\operatorname{Re} \left[\frac{zf'(z)}{f(z)-f(-z)} \right] > 0$ for $z \in \Delta$. These functions introduced by (Sakaguchi, 1959) are called functions starlike with respect to symmetric points.

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Recently (Frasin, 2010) introduced and studied a generalized Sakaguchi type class $S(\alpha, s, t)$ if it satisfies

$$\operatorname{Re} \left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} > \alpha \quad (1.2)$$

for some $0 \leq \alpha < 1$, $s, t \in C$ with $s \neq t$ and for all $z \in \Delta$. Also denote by $T(\alpha, s, t)$ the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, s, t)$. The class $S(\alpha, 1, t)$ was introduced and studied by Owa et al. (Owa et al., 2005, 2007). If $t = -1$, the class $S(\alpha, 1, -1) \equiv S_s(\alpha)$ (Sakaguchi, 1959) is called Sakaguchi function of order α (see (Cho et al., 1993; Owa et al., 2005)), where as $S_s(0) = S_s^*$ (Sakaguchi, 1959).

Note that $S(\alpha, 1, 0) \equiv S^*(\alpha)$ and $T(\alpha, 1, 0) \equiv C(\alpha)$ which are, respectively, the familiar classes of starlike functions of order α ($0 \leq \alpha < 1$) and convex functions of order α ($0 \leq \alpha < 1$).

Mathur & Mathur (Trilok Mathur & Ruchi Mathur, 2012) investigated the classes $S_s^*(\phi, s, t)$ and $T(\phi, s, t)$ defined as follows.

Definition 1.1. Let $\phi(z) = 1 + B_1z + B_2z^2 + \dots$ be univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. The function $f \in A$ is in the class $S_s^*(\phi, s, t)$ if

$$\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} < \phi(z), \quad s \neq t.$$

Remark. $T(\phi, s, t)$ denotes the subclass of A consisting functions $f(z)$ such that $zf'(z) \in S_s^*(\phi, s, t)$. Observe that $S_s^*(\phi, 1, 0) \equiv S_s^*(\phi)$ and $T(\phi, 1, 0) \equiv C(\phi)$, which are the classes introduced and studied by Ma and Minda (Ma & Minda, 1994). Also note that $S_s^*(\phi, 1, -1) \equiv S_s^*(\phi)$, Shanmugam et al. (Shanmugam et al., 2006).

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas (Noonan & Thomas, 1976) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & & \vdots \\ a_{n+q-1} & \cdots & & a_{n+2q-2} \end{vmatrix}$$

This determinant has also been considered by several authors. For example, Noor (Noor, 1983) determined the rate of growth of $H_q(n)$ as $n \rightarrow \infty$ for functions f given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles (Hayami & Owa, 2010; Janteng et al., 2008; Kharudin et al., 2011; Noor, 1983; Selvaraj & Vasanthi, 2010) for different classes of functions.

Easily, one can observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In the present investigation, we see the upper bound for the functional $|a_2a_4 - a_3^2|$ belonging to a new subclass $S_s^*(\phi, s, t)$ of generalized Sakaguchi type functions introduced by (Frasin, 2010).

2. Preliminary Results

Let P denote the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (2.1)$$

which are regular in Δ and satisfy $\operatorname{Re} p(z) > 0$, $z \in \Delta$. Throughout this paper we assume that $p(z)$ is given by (2.1) and $f(z)$ is given by (1.1). To prove the main results we shall require the following lemmas.

Lemma 2.1. (Duren, 1983) Let $p \in P$, then $|c_k| \leq 2$, $k = 1, 2, \dots$ and the inequality is sharp.

Lemma 2.2. (Libera & Zlotkiewicz, 1982, 1983) Let $p \in P$, then

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad (2.2)$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \quad (2.3)$$

for some x, y such that $|x| \leq 1$ and $|y| \leq 1$.

3. Main Results

Theorem 3.1. If $f \in S_S^*(\phi, s, t)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{B_1}{(3 - s^2 - st - t^2)^2}, \text{ provided } s + t \neq 2. \quad (3.1)$$

The result obtained is sharp.

Proof. Let $f \in S_S^*(\phi, s, t)$. Then there exists a Schwarz function $w(z) \in A$ such that

$$\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} = \phi(w(z)), \quad (z \in \Delta, s \neq t) \quad (3.2)$$

If $P_1(z)$ is analytic and has positive real part in Δ and $P_1(0) = 1$, then

$$P_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (3.3)$$

From (3.3) we obtain

$$w(z) = \frac{c_1}{2} z + \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \quad (3.4)$$

Let

$$p(z) = \frac{(s-t)zf'(z)}{f(sz) - f(tz)} = 1 + b_1 z + b_2 z^2 + \cdots; \quad (z \in \Delta) \quad (3.5)$$

which gives

$$\begin{aligned} b_1 &= (2 - s - t)a_2, \\ b_2 &= (3 - s^2 - st - t^2)a_3 + (s + t)(s + t - 2)a_2^2, \\ b_3 &= (4 - s^3 - st^2 - s^2t - t^3)a_4 + 2(s^2 + st + t^2)(s + t - 1)a_2a_3 \\ &\quad + 2a_2^3(s + t)^2 - 3a_2a_3(s + t). \end{aligned}$$

Since $\phi(z)$ is univalent and $P < \phi$, therefore using (3.4) we obtain

$$\begin{aligned} P(z) = \phi(w(z)) &= 1 + \frac{B_1c_1}{2}z + \left\{ \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) B_1 + \frac{1}{4}c_1^2B_2 \right\} z^2 \\ &\quad + \left[\frac{B_1}{2} \left\{ 2c_3 + c_1 \left(\frac{c_1^2}{2} - c_2 \right) - c_1c_2 \right\} \right. \\ &\quad \left. + \frac{B_1c_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_3c_1^3}{8} \right] z^3 + \dots \end{aligned} \quad (3.6)$$

Now from (3.5) and (3.6) we have

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} = 1 + \frac{B_1c_1}{2}z + \left\{ \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) B_1 + \frac{1}{4}c_1^2B_2 \right\} z^2 + \dots \quad (3.7)$$

On equating the coefficient of z , z^2 and z^3 in (3.7) we obtain

$$\begin{aligned} a_2 &= \frac{B_1c_1}{2(2-s-t)}, \\ a_3 &= \frac{1}{(3-s^2-st-t^2)} \left\{ \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) B_1 + \frac{1}{4}c_1^2B_2 + \frac{(s+t)B_1^2c_1^2}{4(2-s-t)} \right\}, \\ a_4 &= \frac{1}{(4-s^3-st^2-s^2t-t^3)} \left[B_1c_3 + c_1^3 \left[\frac{B_1}{4} - \frac{B_2}{4} + \frac{B_3}{8} - \frac{B_1^3(s+t)^2}{4(2-s-t)^3} \right. \right. \\ &\quad \left. \left. + \frac{(s+t+2(s^2+st+t^2))}{16(2-s-t)(3-s^2-st-t^2)} \left\{ B_1B_2 - B_1^2 + \frac{B_1^3}{(2-s-t)} \right\} \right] \right. \\ &\quad \left. + c_1c_2 \left[-\frac{3B_1}{2} + \frac{B_2}{2} \right. \right. \\ &\quad \left. \left. + \frac{B_1^2}{8(2-s-t)(3-s^2-st-t^2)} [3(s+t) - 2(s^2+st+t^2)(s+t-1)] \right] \right]. \end{aligned}$$

Thus we have,

$$|a_2a_4 - a_3^2| = \left| \frac{B_1c_1}{2P_1} \left\{ B_1c_3 + c_1^3 \left[\frac{2(B_1 - B_2) + B_3}{8} - \frac{B_1^3(s+t)^2}{4(2-s-t)^3} \right. \right. \right. \\ \left. \left. + \frac{3(s+t) - 2(s^2 + st + t^2)(s+t-1)}{16(2-s-t)(3-s^2-st-t^2)} \left\{ B_1B_2 - B_1^2 + \frac{B_1^3}{(2-s-t)} \right\} \right] \right\} \\ \left. + c_1c_2 \left[-\frac{3B_1}{2} + \frac{B_2}{2} + \frac{B_1^2}{8(2-s-t)(3-s^2-st-t^2)} \right. \right. \\ \left. \left. [3(s+t) - 2(s^2 + st + t^2)(s+t-1)] \right] \right. \\ \left. - \frac{1}{P_3} \left\{ c_1^4f_1(s,t) + c_1^2g_1(s,t) + \frac{c_1^2B_1^2}{16} + \frac{c_2^2B_1^2}{4} \right\} \right|.$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction $c \in [0, 2]$. Substituting for c_2 and c_3 , from Lemma 2.2 and applying the triangle inequality with $\rho = |x|$, we obtain

$$|a_2a_4 - a_3^2| \leq c^4 \left[\frac{f(s,t)}{P_1} + \frac{B_1g(s,t)}{4P_1} - \frac{f_1(s,t)}{P_3} - \frac{g_1(s,t)}{2P_3} - \frac{B_1}{16P_3} \right] \\ + c^2(4-c^2)\rho \left[\frac{B_1}{4P_1} + \frac{B_1^2}{4P_1} - \frac{B_1^2\rho}{8P_1} - \frac{g_1(s,t)}{2P_3} - \frac{B_1}{8P_3} \right] \\ + \frac{c^2B_1^2}{16P_3} + \frac{B_1^2c(4-c^2)(1-\rho^2)}{4P_1} + \frac{B_1\rho^2(4-c^2)^2}{16P_3} \\ = F(\rho) \quad (3.8)$$

where,

$$P_1 = (2-s-t)(4-s^3-st^2-s^2t-t^3), \\ P_2 = 8(2-s-t)(3-s^2-st-t^2), \\ P_3 = (3-s^2-st-t^2)^2,$$

$$f_1(s,t) = \frac{B_2^2}{16} + \frac{(s+t)^2B_1^4}{16(2-s-t)^2} - \frac{B_1B_2}{8} + \frac{B_1^2B_2(s+t)}{8(2-s-t)} + \frac{(s+t)B_1^3}{8(2-s-t)}, \\ g_1(s,t) = \frac{(s+t)B_1^3}{4(2-s-t)} + \frac{B_1B_2}{4} - \frac{B_1^2}{4}, \\ f(s,t) = \frac{B_1^2}{4} - \frac{B_1B_2}{8} + \frac{B_1B_3}{8} - \frac{B_1^4(s+t)^2}{8(2-s-t)^3} \\ + \frac{[(B_1B_2 - B_1^2)(2-s-t)] + B_1^3[3(s+t) - 2(s^2 + st + t^2)(s+t-1)]}{32(2-s-t)^2(3-s^2-st-t^2)}, \\ g(s,t) = -B_1 + \frac{B_2}{2} - \frac{B_1^2[3(s+t) - 2(s^2 + st + t^2)(s+t-1)]}{2},$$

with $\rho = |x| \leq 1$. Furthermore

$$F'(\rho) = c^2(4 - c^2) \left[\frac{B_1}{4P_1} + \frac{B_1^2}{4P_1} - \frac{B_1^2\rho}{8P_1} - \frac{g_1(s, t)}{2P_3} - \frac{B_1}{8P_3} \right] \\ + \frac{B_1^2c\rho(4 - c^2)(4 - c)}{8P_1} + \frac{B_1^2(4 - c^2)^2\rho}{8P_3}.$$

For a $c \in [0, 2]$, $F(\rho) \leq F(1)$, that is

$$|a_2a_4 - a_3^2| \leq c^4 \left[\frac{f(s, t)}{P_1} + \frac{B_1g(s, t)}{4P_1} - \frac{f_1(s, t)}{P_3} - \frac{g_1(s, t)}{2P_3} - \frac{B_1}{16P_3} \right] \\ + c^2(4 - c^2) \left[\frac{B_1}{4P_1} + \frac{B_1^2}{4P_1} - \frac{B_1^2}{8P_1} - \frac{g_1(s, t)}{2P_3} - \frac{B_1}{8P_3} \right] \\ + \frac{c^2B_1^2}{16P_3} + \frac{B_1(4 - c^2)^2}{16P_3} \\ = G(c).$$

By elementary calculus we have $G''(c) \leq 0$ for $0 \leq c \leq 2$ and $G(c)$ has real critical point at $c = 0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Thus the maximum of $G(c)$ occurs at $c = 0$. Hence,

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{(3 - s^2 - st - t^2)^2}$$

If $p(z) \in P$ with $c_1 = c = 0$, $c_2 = 2$, $c_3 = 1$, then we obtain

$p(z) = (1 - z) + \frac{z}{(1-z)^2} = 1 + 2z^2 + z^3 + \dots \in P$. The result is sharp for the functions defined by

$$\left\{ \frac{(s - t)zf'(z)}{f(sz) - f(tz)} \right\} = \phi(z), \quad s \neq t$$

and

$$\left\{ \frac{(s - t)zf'(z)}{f(sz) - f(tz)} \right\} = \phi(z^2), \quad s \neq t.$$

□

Remark. If $f \in S_S^*(\phi, 1, -1)$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{2}.$$

Since $f(z) \in T(\phi, s, t)$ if and only if $zf'(z) \in S_S^*(\phi, s, t)$, proceeding on similar lines as in Theorem 3.1 we obtain the upper bound for the functional $|a_2a_4 - a_3^2|$ belonging to the class $T(\phi, s, t)$ which is stated below without proof.

Theorem 3.2. If $f \in T(\phi, s, t)$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9(3 - s^2 - st - t^2)^2}, \quad \text{provided } s + t \neq 2. \quad (3.9)$$

The result obtained is sharp.

Remark. If $f \in T(\phi, 1, -1)$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{36}.$$

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