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L_p - Approximation of Analytic Functions on Compact Sets Bounded by Jordan Curves

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Abstract

This paper is concerned with functions analytic on compact sets bounded by Jordan curves having rapidly increasing maximum modulus such that order of function is infinite. To study the precise rates of growth of such functions the concept of index has been used. The $q-$order and lower $q-$order of analytic functions have been obtained in terms of L_p-approximation error. Our results improve and refine the results of Andre Giroux (Giroux, 1980) and Kapoor and Nautiyal (Kapoor & Nautiyal, 1982) for non entire case.

Keywords: L_p-approximation error, index$-q$, transfinite diameter, Faber series.
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1. Introduction

Let D be a compact set containing at least two points such that its complement D' with respect to the extended complex plane is a simply connected domain containing the point at infinity. In view of Riemann mapping theorem, there exists a one-one analytic function $z = \varphi(w)$ which maps $\{w : |w| > 1\}$ conformly onto D' such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. Thus, in a neighborhood of infinity, the function has the expansion

$$z = \varphi(w) = d\left[w + d_0 + \frac{d_{-1}}{w} + \dots\right]$$

where the number $d > 0$ is called the transfinite diameter of D. If we define $\eta(w) = \varphi(w/d)$, then η maps $\{w : |w| > d\}$ onto D' in a one-one conformal manner. If $w = \Omega(z)$ is the inverse function of η then $\Omega(\infty) = \infty$ and $\lim_{z\to\infty} \Omega(z)/z = 1$.

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For $1 \le p < \infty$, let $L_p(D)$ denote the space of analytic functions f in D such that $\|f\|_{D,p} = \left(\frac{1}{A}\int\int_D |f(z)|^p dxdy\right)^{1/p} < \infty$, where A is the area of D.

Let L_r is an analytic Jordan curve for each $r > d$. If D_r denotes the domain bounded by L_r then $D \subset D_r$ for each $r > d$. Let $H(\overline{D}; R)$ denotes the class of all functions that are regular in D_R with a singularity on $L_R(d < R < \infty)$. Since $\overline{D} \subset D_R$ for $R > d$ it follows that every $f \in H(\overline{D}; R)$ is analytic in $\overline{D}$ and so $\int\int_D |f(z)|^p dxdy < \infty$ and $f \in L_p(D)$.

We now prove the following:

Theorem 1.1. *Every $f \in H(\overline{D}; R)$ can be represented by the Faber series*

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z), \quad z \in D_R \tag{1.1}$$

with

$$a_n = \frac{1}{2\pi i}\int_{|\xi|=1} f(\eta(\xi))\xi^{-n-1} d\xi, \;\; d < r < R \tag{1.2}$$

if and only if

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}. \tag{1.3}$$

The series in (1.1) converges absolutely and uniformly on every compact subset of D_R and diverge outside L_R.

Proof. Let $f \in H(\overline{D}; R)$. If $z \in D_R$, then $z \in D_R$ for some r satisfying $d < r < R$. Using Cauchy's integral formula,

$$f(z) \;=\; \frac{1}{2\pi i}\int_{L_r} \frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{f(\eta(\xi)\eta'(\xi))}{n(\xi)-z} d\xi = \frac{1}{2\pi i}\int_{|\xi|=r} \left(\sum_{n=0}^{\infty} \frac{f(\eta(\xi))}{\xi^{n+1}} P_n(z)\right) d\xi.$$

Since the series under the integral sign converges uniformly on $|\xi| = r$, it can be integrated term by term. Thus we have

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z), \;\; z \in D_R.$$

If

$$\overline{M}(r, f) = \max_{|\xi|=r} |f(\eta(\xi))|,$$

then (1.2) gives

$$|a_n| \le \frac{\overline{M}(r, f)}{r^n}, \;\; n = 0, 1, 2, \ldots, \tag{1.4}$$

which are analogous of Cauchy's inequality for Taylor series. From (1.4), we have

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{r}.$$

Since this holds for every r satisfying $d < r < R$, we have

$$\lim_{n\to\infty} \sup |a_n|^{1/n} \leq \frac{1}{R}.$$

We now show that inequality does not holds in the above relation. If

$$\lim_{n\to\infty} \sup |a_n|^{1/n} = \frac{1}{R_0} < \frac{1}{R},$$

let r satisfy $d < r < R_0$; then for every ε such that $0 < \varepsilon < R_0 - r$, we have

$$|a_n| < \frac{1}{(R_0 - \varepsilon/2)^n} < \frac{1}{(r + \varepsilon/2)^n} \quad \texttt{for} \quad n \geq n_0.$$

On the other hand, for every $z \in L_r$, we have $|P_n(z)| < (r + \varepsilon/4)^n$ for $n \geq n_1$.

Thus, for all $z \in L_r$, $|a_n P_n(z)| < \left(\frac{r+\varepsilon/4}{r+\varepsilon/2}\right)^n < 1$ for $n \geq \max(n_0, n_1)$.

The above inequality shows that the series $\sum_{n=0}^{\infty} a_n P_n(z)$ converges uniformly on L_r and hence on D_r. Since this is true for every r satisfying $d < r < R_0$, it follows that the series $\sum_{n=0}^{\infty} a_n P_n(z)$ converges uniformly on every compact subset of D_{R_0} to a function, $F(z)$, say. The function $F(z)$ must be regular in D_{R_0} since each term of the series is a regular function. However, on $D_R \subset D_{R_0}$, the series converges to $f(z)$ so that $F(z)$ is the analytic continuation of $f(z)$ to D_{R_0}. Since this contradicts the hypothesis that f has a singular point on $L_R (R < R_0)$, we must have (1.3).

To show that the series (1.3) diverges outside L_R, let z_0 lie outside L_R. Then we have $|\Omega(z_0)| > R$. In view of $\lim_{n\to\infty} |P_n(z)|^{1/n} = |\Omega(z)|$ and (1.3) we obtain $\lim_{n\to\infty} \sup |a_n P_n(z_0)|^{1/n} > 1$, showing that the series $\sum_{n=0}^{\infty} a_n P_n(z_0)$ diverges.

Conversely, if (1.3) holds, then as above, we can show that the series $\sum_{n=0}^{\infty} a_n P_n(z)$ converges uniformly on compact subsets of D_R to a regular function, $f(z)$, say. If $f(z)$ had no singular point on L_R then it would be possible to extend $f(z)$ analytically to a bigger domain D_{R_0}, say. But then the first part of the theorem would give that

$\lim_{n\to\infty} \sup |a_n|^{1/n} \leq 1/R_0 < 1/R$, a contradiction. Hence the proof is completed. □

In view of above theorem, there exists a sequence of polynomials converging uniformly on compact subsets of D_R to $f(z)$, it follows that this sequence converges in the norm of $L_p(D)$ also to f. If p_{n-1} denotes the collection of all polynomials of degree not exceeding $n - 1$ and we set

$$E_n^p(f) = \inf_{g \in p_{n-1}} \|f - g\|_{D,p}, \quad n = 1, 2, \ldots,$$

then it is clear that $E_n^p(f)$ is a non increasing sequence tending to zero as $n \to \infty$. Our next theorem holds for $1 \leq p < \infty$.

Theorem 1.2. *If $f \in H(\overline{D}; R)$, then*

$$\lim_{n\to\infty} \sup \left[E_n^p(f)\right]^{1/n} = \frac{d}{R}. \tag{1.5}$$

Proof. If $f \in H(\overline{D}; R)$ we have, by (1.5)

$$\begin{aligned} E_n^p(f) &= \inf_{g \in p_{n-1}} \left(\int \int_D |f(z) - g(z)|^p dxdy \right)^{1/p} \leq \left(\int \int_D |f(z) - Q_{n-1}(z)|^p dxdy \right)^{1/p} \\ &\leq A^{1/p} \max_{z \in \overline{D}} |f(z) - Q_{n-1}(z)|, \end{aligned}$$

where $Q_{n-1}(z)$ is the polynomial of degree not exceeding $n-1$ and A is the area of domain D. Using a result of (Markushevich, 1967), p. 114 for $d < r' < r < R$ and $n > n_0$ we get

$$E_n^p(f) \leq A^{1/p} \overline{M}(r, f) \left(\frac{r'}{r - r'} \right) (r'/r)^n \tag{1.6}$$

where $\overline{M}(r, f) = \max_{z \in L_r} |f(z)|$. This leads to $\lim_{n\to\infty} \sup \left[E_n^p(f) \right]^{1/n} \leq r'/r$.

Since the above relation holds for all r', r satisfying $d < r' < r < R$, we must have

$$\lim_{n\to\infty} \sup \left[E_n^p(f) \right]^{1/n} \leq \frac{d}{R}. \tag{1.7}$$

To obtain the reverse inequality in (1.7), we note that, since every $f \in H(\overline{D}; R)$ is in $H_2(D)$, there exists a closed orthonormal system $\{\chi_n(z)\}_{n=0}^{\infty}$ of polynomials in $H_2(D)$ such that f can be represented by its Fourier series with respect to the system $\{\chi_n(z)\}_{n=0}^{\infty}$ that converges uniformly on compact subsets of D_R to f. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n \chi_n(z) \quad z \in D_R, \tag{1.8}$$

where $a_n = \int \int_D f(z) \overline{\chi_n}(z) dxdy$.

If $g \in p_{n-1}$, then

$$|a_n| = \left| \int \int_D (f(z) - g(z)) \overline{\chi_n}(z) dxdy \right| \leq \left(\int \int_D |f(z) - g(z)|^p dxdy \right)^{1/p} \left(\int \int_D |\overline{\chi_n(z)}|^{p/p-1} dxdy \right)^{1-1/p}.$$

Using (1.3) and the fact that the above inequality holds for every $g \in p_{n-1}$, we get, for $r^* > d$, $|a_n| \leq E_n^p(f).\overline{M}(r, f) \left(\frac{r^*}{d} \right)^n A^{1-1/p}$, by (1.7) we get $\lim_{n\to\infty} \sup \left[E_n^p(f) \right]^{1/n} \geq \frac{d}{r^*} \lim_{n\to\infty} \sup |a_n|^{1/n} = \frac{d^2}{r^* R}$.

Since the above inequality is valid for every $r^* > d$, we must have

$$\lim_{n\to\infty} \sup \left[E_n^p(f) \right]^{1/n} \geq \frac{d}{R}. \tag{1.9}$$

Combining (1.7) and (1.9) we get (1.5). □

2. Fast Growth and Approximation Errors

We now obtain relations that indicate how the growth of an $f \in H(\overline{D}; R)$ depends on $E_n^p(f)$ and vice versa.

For function $f \in H(\overline{D}; R)$, set

$$\rho_R(q) = \limsup_{r \to R} \frac{\log^{[q]} \overline{M}(r, f)}{\log(Rr/(R - r))} \tag{2.1}$$

where $\log^{[0]} \overline{M}(r, f) = \overline{M}(r, f)$ and $\log^{[q]} \overline{M}(r, f) = \log\left(\log^{[q-1]} \overline{M}(r, f)\right), q = 1, 2, \ldots.$ To avoid the trivial cases we shall assume throughout that $\overline{M}(r, f) \to \infty$ as $r \to R$.

Definition 2.1. A function $f \in H(\overline{D}; R)$, is said to have the index q if $\rho_R(q) < \infty$ and $\rho_R(q - 1) = \infty, q = 1, 2, \ldots.$ If q is the index of $f(z)$, then $\rho_R(q)$ is called the $q-$order of f.

Definition 2.2. A function $f \in H(\overline{D}; R)$ and having the index$-q$ is called to have lower $q-$order $\lambda_R(q)$ if

$$\lambda_R(q) = \liminf_{r \to R} \frac{\log^{[q]} \overline{M}(r, f)}{\log(Rr/(R - r))}, q = 1, 2, \ldots. \tag{2.2}$$

Definition 2.3. A function $f \in H(\overline{D}; R)$ and having the index q is said to be of regular $q-$growth if $\rho_R(q) = \lambda_R(q), q = 1, 2, \ldots, f(z)$ is said to be of irregular $q-$growth if $\rho_R(q) > \lambda_R(q), q = 1, 2, \ldots.$

In 1980 Andre Giroux (Giroux, 1980) obtained necessary and sufficient conditions, in terms of the rate of decrease of the approximation error $E_n^p(f)$, such that $f \in L_p(D), 2 \leq p \leq \infty$, has an analytic continuation as an entire function having finite growth parameters. In 1982 Kapoor and Nautiyal (Kapoor & Nautiyal, 1982) considered the approximation error $E_n^p(f)$ on a Caratheodory domain and had extended the results of Giroux for the case $1 \leq p < 2$. All these results do not give any specific information about the growth when function is not entire and for the functions having rapidly increasing maximum modulus such that order of function is infinite. Although, Kumar (Kumar, 2004, 2007*b*,*a*, 2010, 2011, 2013) and Kumar and Mathur (Kumar & Amit, 2006) obtained some results in this direction but our results are different from all those of above papers.

In this paper an attempt has been made to study the growth of $f \in H(\overline{D}; R)$ involving $E_n^p(f)$ for $1 \leq p < \infty$ when f is not entire and having $\overline{M}(r, f) \to \infty$ as $r \to R$. To obtain the results in general setting we shall assume that $f \in H(\overline{D}; R)$ is represented by the gap power series $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda n}$ where $\{\lambda_n\}_{n=0}^{\infty}$ is strictly increasing sequence of integers and $a_n \neq 0$ for all n.

Theorem 2.1. *If $f \in H(\overline{D}; R)$ having the index$-q$ and $q-$order $\rho_R(q)$, then*

$$\rho_R(q) + A(q) = \limsup_{n \to \infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+((E_{\lambda_n}^p(f))(R/d)^{\lambda_n})}, q = 2, 3, \ldots, \tag{2.3}$$

where, $A(q) = 1$ if $q = 2$ and $A(q) = 0$ if $q = 3, 4, \ldots.$

Proof. If $f \in H(\overline{D};R)$ is of $q-$order $\rho_R(q)$, given $\varepsilon > 0$, there exists $r_0(\varepsilon)$ such that, for $r_0 < r < R$, we have

$$\log^{[q-1]} \overline{M}(r,f) < \left(\frac{Rr}{R-r}\right)^{\rho_R(q)+\varepsilon}.$$

Using inequality (1.6) for $n > n_0$ and $d < r' < r < R, r_0 < r'$, we obtain

$$\begin{aligned} \log E_{\lambda_n}^{(p)}(f)(R/d)^{\lambda_n} &< \frac{1}{p}\log A + \exp^{[q-2]}\left(\frac{Rr}{R-r}\right)^{\rho_R(q)+\varepsilon} + \log\frac{r'}{r-r'} + \lambda_n \log\frac{r'}{d} + \lambda_n \log\frac{R}{r} \\ &< \frac{1}{p}\log A + \exp^{[q-2]}\left(\frac{Rr}{R-r}\right)^{\rho_R(q)+\varepsilon} + \log\frac{r'}{r-r'} + \lambda_n\left(\frac{r'-d}{d}\right) + \lambda_n\left(\frac{R-r}{r}\right). \end{aligned} \tag{2.4}$$

Let us consider r such that

$$\begin{aligned} &\frac{Rr}{R-r} = \left(\log^{[q-2]}(\lambda_n R/\rho_R(q)+\varepsilon)\right)^{1/\rho_R(q)+A(q)+\varepsilon}, \quad \text{and} \\ &r' = \lambda d + (1-\lambda)(Rd/r), 0 < \lambda < 1. \end{aligned} \tag{2.5}$$

For $q = 2$ the inequality (2.4) with (2.5) gives for $n > n_1$,

$$\log^+ E_{\lambda_n}^p(f)(R/d)^{\lambda_n} < M(\lambda_n R)^{(\rho_R(2)+\varepsilon)/(\rho_R(2)+1+\varepsilon)}$$

where M is a constant. It gives

$$\limsup_{n\to\infty} \frac{\log^+\log^+ E_{\lambda_n}^p(f)(R/d)^{\lambda_n}}{\log\lambda_n} \le \frac{\rho_R(2)}{\rho_R(2)+1}. \tag{2.6}$$

For $q = 3, 4, \ldots,$ the inequality (2.4) gives for $n > n_1$, with (2.5) that

$$\log^+ E_{\lambda_n}^p(f)(R/d)^{\lambda_n} < \exp^{[q-2]}\left(\log^{[q-2]}(\lambda_n R/\rho_R(q)+\varepsilon)\right)[1+0(1)],$$

from which a simple calculation would yield

$$\rho_R(q) \ge \limsup_{n\to\infty} \frac{\log^{[q-1]}\lambda_n}{\log\lambda_n - \log^+\log^+ E_{\lambda_n}^p(f)(R/d)^{\lambda_n}}. \tag{2.7}$$

To prove that reverse inequality, use (1.8) and (1) to get, for $z \in \overline{D}_r$, $d < r^* < r < R$,

$$\begin{aligned} |f(z)| &\le \sum_{n=0}^{\infty} |a_n||\chi_{\lambda_n}(z)| \\ &\le MA^{1-1/p}\sum_{n=0}^{\infty} E_n^p(f)\left(\frac{r^*}{d}\right)^n |\chi_{\lambda_n}(z)|. \end{aligned}$$

It is known that for any $r^* > d$ there exists a constant $M^|$ such that

$$|\chi_n(z)| \le M'(r^*/d)^n n = 0, 1, 2, \ldots, z \in \overline{D}.$$

Now for $d < r^* < r < R$, we get

$$\overline{M}(r,f) \le M^* M^! A^{1-1/p} \sum_{n=0}^{\infty} E_n^p(f)(R/d)^{\lambda_n} \left(\frac{{r^*}^2 r}{d^2 R}\right)^{\lambda_n}.$$

Taking $r^* = \sqrt[d]{\lambda + (1-\lambda)(R/r)}, 0 < \lambda < 1$, the above inequality gives

$$\overline{M}(r,f) \le BM\left(\frac{\lambda r + (1-\lambda)R}{R}, G\right), \tag{2.8}$$

where B is constant, $G(s) = \sum_{n=0}^{\infty} E_n^p(f)(R/d)^{\lambda_n} s^{\lambda_n}$ and $M(t,G) = \max_{|s|=t} |G(s)|$. It can be easily seen that $G(s)$ is analytic in $|s| < 1$. If the $q-$order of $G(s)$ in unit disc is $\rho_0(g)$ then

$$\rho_R(q) = \limsup_{r\to R} \frac{\log^{[q]} \overline{M}(r,f)}{\log(Rr/(R-r))} \le \limsup_{r\to R} \frac{\log^{[q]} M(((\lambda r + (1-\lambda)R)/R), G)}{\log(Rr/(R-r))} = \rho_0(q).$$

Applying Theorem 1 of (Kapoor & Gopal, 1979) for $\rho_0(q)$, we obtain

$$\rho_R(q) + A(q) \le \limsup_{n\to\infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+ E_{\lambda_n}^p(f)(R/d)^{\lambda_n}} q = 2, 3, \ldots, \tag{2.9}$$

combining (2.6), (2.7) and (2.9) we get (2.3) i.e., the proof of theorem is completed. □

Theorem 2.2. *Let $f \in H(\overline{D}; R)$ having the index$-q$, $q-$order $\rho_R(q)$ and lower $q-$order $\beta_R(q)(0 \le \beta_R(q) \le \infty)$, then for any increasing sequence $\{n_k\}$ of natural numbers,*

$$\beta_R(q) + A(q) \ge \liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ E_{\lambda_{n_k}}^p(f)(R/d)^{\lambda_{n_k}}}. \tag{2.10}$$

Proof. Let the right hand side of (2.10) be δ. Without loss of generality we can assume $\delta > 0$. For any ε such that $0 < \varepsilon < \delta$, and for all $k > k_0 = k_0(\varepsilon)$, we get

$$\log^+ E_{\lambda_{n_k}}^p(f)(R/d)^{\lambda_{n_k}} > \lambda_{n_k} (\log^{[q-2]} \lambda_{n_{k-1}})^{-1/(\delta-\varepsilon)}.$$

Choosing a sequence r_k such that $r_k \le r \le r_{k+1}$, where $\frac{R-r_k}{r_k} = \frac{1}{e}(\log^{[q-2]} \lambda_{n_{k-1}})^{1/(\delta-\varepsilon)}$.
Using inequality (1.6) we obtain

$$\begin{aligned}
\log \overline{M}(r,f) &\ge \log E_{\lambda_{n_k}}^p(f)(R/d)^{\lambda_{n_k}} - \frac{1}{p}\log A - \log\frac{r'}{r-r'} - \lambda_{n_k}\log\frac{r'}{d} - \lambda_{n_k}\log R/r \\
&\ge \log E_{\lambda_{n_k}}^p(f)(R/d)^{\lambda_{n_k}} - \frac{1}{p}\log A - \log\frac{r'}{r_k-r'} - \lambda_{n_k}\log\frac{r'}{d} - \lambda_{n_k}\log R/r_k \\
&> \lambda_{n_k}(\log^{[q-2]} \lambda_{n_{k-1}})^{-1/(\delta-\varepsilon)} - \lambda_{n_k}\left(\frac{R-r_k}{r_k}\right) = (1-1/e)\lambda_{n_k}(\log^{[q-2]} \lambda_{n_{k-1}})^{-1/(\delta-\varepsilon)} \\
&> (e-1)\left(\frac{R-r}{r}\right)\exp^{[q-2]}\left(e\left(\frac{R-r}{r}\right)\right)^{-(\delta-\varepsilon)}.
\end{aligned}$$

Now after a simple calculation the above estimate yields

$$\beta_R(q) + A(q) \ge \delta. \tag{2.11}$$

Hence the proof is completed. □

To prove our next theorem we need the following lemma.

Lemma 2.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in unit disc having index q, q−order* $\rho(q) > 0$ *and lower q−order* $\beta(q)$*. Further, let* $\varphi(n) \equiv |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *forms a non-decreasing function of n for* $n > n_0$ *and*

$$\beta(q) + A(q) = \liminf_{r\to 1} \frac{\log^{[q-1]} \nu(r)}{-\log(1-r)}. \tag{2.12}$$

Then

$$\beta(q) + A(q) = \liminf_{n\to\infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+ |a_n|}$$

where for $|z| = r, \mu(r) = \max_{n>0}\{|a_n| r^{\lambda_n}\}, \nu(r) = \max\{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\}, 0 < r < 1.$

Proof. The proof of this lemma follows on the lines of a result in (Kapoor, 1972), so we omit the details. □

A function f, analytic in unit disc is said to be admissible if its lower q−order satisfies (2.12).

Theorem 2.3. *Let* $f \in H(\overline{D}; R)$ *having the index−q, q−order* $\rho_R(q)$ *and lower q−order* $\beta_R(q)$*. Further let* $\varphi(n) = |E_{\lambda_n}/E_{\lambda_{n+1}}|^{1/(\lambda_{n+1}-\lambda_n)}$ *forms a nondecreasing function of n for* $n > n_0$*. Then*

$$\beta_R(q) + A(q) \le \liminf_{n\to\infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+ E^p_{\lambda_n}(f)(R/d)^{\lambda_n}}. \tag{2.13}$$

Proof. Using (2.8), we get

$$\beta_R(q) = \liminf_{r\to R} \frac{\log^{[q]} \overline{M}(r,f)}{\log(Rr/(R-r))} \le \liminf_{r\to R} \frac{\log^{[q]} M(((\lambda r + (1-\lambda)R)/R))}{\log(Rr/(R-r))} = \beta_0(q).$$

□

It can be easily seen that $G(s)$ satisfies the hypothesis of Lemma 4. Applying Lemma 2.1 for $\beta_0(q)$, it gives

$$\beta_R(q) + A(q) \le \liminf_{n\to\infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+ E^p_{\lambda_n}(f)(R/d)^{\lambda_n}} q = 2, 3, \ldots,$$

combining Theorem 2.2 and 2.3 we get the following theorem:

Theorem 2.4. *Let* $f \in H(\overline{D}; R)$ *having the index−q, q−order* $\rho_R(q)$ *and lower q−order* $\beta_R(q)$*. Further, let* $\varphi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *forms a nondecreasing function of n for* $n > n_0$*. Then*

$$\beta_R(q) + A(q) = \liminf_{n\to\infty} \frac{\log^{[q-1]} \lambda_{n-1}}{\log \lambda_n - \log^+ \log^+ E^p_{\lambda_n}(f)(R/d)^{\lambda_n}}. \tag{2.14}$$

Theorem 2.5. *Let $f \in H(\overline{D}; R)$ having the index$-q$, $q-$order $\rho_R(q)$ and lower $q-$order $\beta_R(q)$. Then*

$$\beta_R(q) + A(q) = \max_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ E^p_{\lambda_{n_k}}(f)(R/d)^{\lambda_{n_k}}} \right], \tag{2.15}$$

where maximum in (2.15) is taken overall increasing sequences $\{n_k\}$ of natural numbers.

Proof. Let $S(s) = \sum_{k=0}^{\infty} E^p_{\lambda_{n_k}}(f)(R/d)^{\lambda_{n_k}} s^{\lambda_{n_k}}, |s| < 1$, where $\{\lambda_{n_k}\}_{k=0}^{\infty}$ is the sequence of elements in the range set of $\nu(r)$. It can be easily seen that $G(s)$ and $S(s)$ have the same maximum term. Hence, the $q-$order and lower $q-$order of $S(s)$ are the same as those of $G(s)$. Thus, $S(s)$ is of lower $q-$order $\beta_R(q)$. Further, let $\xi(n_k) = \max\{r : \nu(r) = \lambda_{n_k}\}$. Then, $\xi(n_k) = \varphi(n_k)$, and consequently, $\varphi(n_k)$ is an increasing function of k. Therefore, $S(s)$ satisfies the hypothesis of Theorem 2.4 and so by (2.15) we get

$$\beta_R(q) + A(q) = \liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ E^p_{\lambda_{n_k}}(f)(R/d)^{\lambda_{n_k}}}. \tag{2.16}$$

But from Theorem 2.3, we get

$$\beta_R(q) + A(q) \geq \max_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ E^p_{\lambda_{n_k}}(f)(R/d)^{\lambda_{n_k}}} \right]. \tag{2.17}$$

Combining (2.16) and (2.17) we get (2.15). Hence the proof is complete. □

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