



A Mixed Integer Linear Programming Formulation for Restrained Roman Domination Problem

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Abstract

This paper deals with a subgroup of Roman domination problems (RDP) named Restrained Roman domination problem (RRDP). It introduces a new mixed integer linear programming (MILP) formulation for the RRDP. The presented model uses relatively small number of the variables and constraints and could be of use both in theoretical and practical purposes. Proof of its correctness is given, i.e. it was shown that optimal solution to the RRDP formulation is equal to the optimal solution of the original problem.

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1. Introduction

With contiguous territories throughout Europe, North Africa, and the Middle East, the Roman Empire was one of the largest in history (Kelly, 2006). The idea of building "empire without end" (Nicolet, 1991) expressed the ideology that neither time nor space limited the Empire. During the fourth century A.D., Emperor of Rome, Constantine the Great, intended to accomplish that idea. In order to expand the Roman Empire, he dealt with the next problem: How to organize legions such that entire Empire of Rome stayed defended? Since legions were highly trained, it was assumed that they could move fast from one city to another. City was considered to be defended if at least one legion was stationed in it or it was adjacent to a city with two legions within. The second condition was made because legion could move from a stationed city only if such an act won't leave it undefended.

Inspired by this historical problem, a new subgroup of the domination problems, named Roman domination problem (RDP), was proposed by Stewart (1999). RDP can be described as a problem of finding the minimal number of legions such that entire Empire of Rome is defended.

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More details about the RDP can be found in (ReVelle & Rosing, 2000), (Currò, 2014), (Liedloff *et al.*, 2005) and (Xing *et al.*, 2006).

Restrained Roman domination problem (RRDP), previously introduced by Pushpam & Sam-path (2015), is defined also as a problem of finding the minimal number of legions such that entire Empire of Rome is defended but the conditions are slightly changed. Again, a city is considered to be defended if at least one legion is stationed within. But, a city without legion within is consider to be defended if it is adjacent to at least one city with two legions within and to at least one undefended city.

The Roman domination problem and the Restrained Roman domination problem can be illustrated by a graph such that each city of the Empire of Rome is represented by a vertex and, for two connected cities, the corresponding vertices are set to be adjacent.

Assuming that five cities, marked by numbers 1 - 5, are constructed such that a city marked by 1 is only adjacent to a city marked by 2 and that all other cities are adjacent to each other, a small illustration of the RDP and RRDP solutions are given in the figure below.

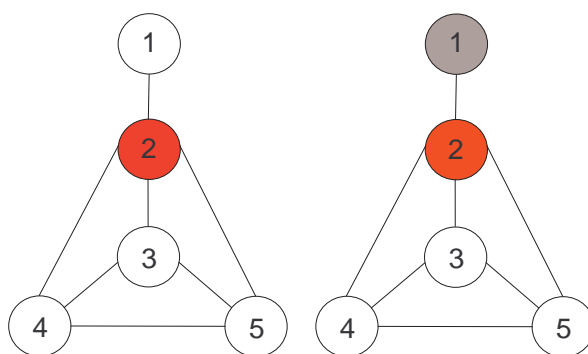


Figure 1. Illustrations of the the RDP (left) and RRDP (right) solutions

Vertex colored in red indicates that corresponding city is defended by two legions, vertex colored in gray indicates that corresponding city is defended by one legion, while vertices colored in white stands for the cities without legions within. For the Roman domination problem (shown on the figure on the left) by using only two legions, all five cities could be defended, i.e. assigning two legions to a city marked by 2, corresponding city and all adjacent cities are consider to be defended. Note that minimal number of legions for the RRDP (shown on the figure on the right) is three, i.e. assigning two legions to a city marked by 2, corresponding city and cities marked by 3, 4 and 5 are set to be defended because they are adjacent to a city with two legions within and adjacent to two cities with no assigned legions; city marked by 1 is set to be defended by one legion, since it can't be adjacent to at least one city without legions within and to at least one city with two legions within, at the same time. Given solution for RRDP is not unique since the same result could be obtained by assigning two legions to a city marked by 3 instead of the city marked by 2.

In the next sections, MILP formulation for the RRDP together with the proof of its validity, are proposed.

2. Problem definition

Let $G = (V, E)$ be an undirected graph with a vertex set V such that each vertex $u \in V$ represents a city of Roman Empire and each edge, $e \in E$, represents an existing road between two adjacent cities. A neighborhood set N_u ($N_u \subset V$), of a vertex $u \in V$, is defined as a set of vertices v adjacent to a vertex u . For a function f

$$f : V \rightarrow \{0, 1, 2\} \quad (2.1)$$

let a number of legions assigned to a city represented by a vertex u to be equal to a value $f(u)$. Additionally, let a function f satisfy the condition that for every vertex $u \in V$ such that $f(u) = 0$ there exists vertices $v, w \in V$ such that $f(v) = 2$ and $f(w) = 0$. In other words, if there is an undefended city u , then there exist at least one city $v, v \in N_u$ with two legions within and at least one undefended city $w, w \in N_u$. Function f is called a restrained Roman domination function.

Mathematically, a proposed problem can be formulated as:

$$\min_f F_1(f) \quad (2.2)$$

subject to:

$$F_1(f) = \sum_{u \in V} f(u) \quad (2.3)$$

$$(\forall u \in V) f(u) = 0 \Rightarrow (\exists v, w \in N_u) (f(v) = 2 \wedge f(w) = 0). \quad (2.4)$$

Now, using a proposed notations, a solution to the illustrated RRDP can be written as: $F_1(f) = 3$ for $f(2) = 2, f(1) = 1$ and $f(3) = f(4) = f(5) = 0$ and it is not unique ($F_1(f) = 3$ for $f(3) = 2, f(1) = 1$ and $f(2) = f(4) = f(5) = 0$).

3. A mixed integer linear programming formulation for the RRDP

For a function f , defined by (2.1), let a continuous decision variable $x_i, x_i \in [0, \infty)$, indicate a number of legions assigned to a corresponding city $i \in V$. Although, $f \in \{0, 1, 2\}$ and $x_i \in [0, \infty)$, x_i and $f(i)$ are with equal values in the optimal solution, and not necessary with equal values for every feasible solution. Let binary decision variables y_i and z_i indicate if there are two or none legions assigned to a corresponding city $i \in V$,

$$y_i = \begin{cases} 1, & f(i) = 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad z_i = \begin{cases} 1, & f(i) = 0 \\ 0, & \text{otherwise} \end{cases}.$$

A mixed integer linear programming (MILP) formulation for the RRDP can now be formulated as follows:

$$\min \sum_{i \in V} x_i \quad (3.1)$$

subject to

$$x_i + \sum_{j \in N_i} y_j \geq 1, \quad i \in V \quad (3.2)$$

$$x_i + \sum_{j \in N_i} z_j \geq 1, \quad i \in V \quad (3.3)$$

$$x_i \geq 2y_i, \quad i \in V \quad (3.4)$$

$$x_i + 2z_i \leq 2, \quad i \in V \quad (3.5)$$

$$x_i \in [0, +\infty); \quad y_i, z_i \in \{0, 1\}, \quad i \in V. \quad (3.6)$$

Further, for vector values $x = [x_i]$, $y = [y_i]$ and $z = [z_i]$, which satisfies constraints (3.2) - (3.6), notation $F_2(x, y, z) = \sum_{i \in V} x_i$ will be used. Now, condition (3.1) which minimizes the number of legions, can be written as $\min_{(x,y,z)} F_2(x, y, z)$. By the constraints (3.2) it is ensured that each undefended vertex i is adjacent to at least one vertex with two legions within. Similarly, by the constraints (3.3) it is ensured that each undefended vertex i is adjacent to at least one vertex which is also undefended. From the inequalities (3.4) and (3.5) it follows that for each city $i \in V$ with at most 1 legion within, corresponding value y_i is set to be equal to zero and that for each city $i \in V$ with at least one legions within, corresponding value z_i is set to be equal to zero. Finally, decision variables x are set to be continuous, while y and z are set to be binary by the constraints (3.6).

A given MILP formulation consists of $2|V|$ variables which are binary and $|V|$ continuous variables. Number of constraints is equal to $4|V|$.

A proof of the validity of the MILP formulation for the RRDP is given in the next proposition.

Proposition 1. *The optimal objective function value $F_1(f)$ of the Restrained Roman domination problem (2.1) - (2.4) is equal to the optimal objective function value $F_2(x, y, z)$ of the MILP formulation (3.1) - (3.6).*

Proof. (\Rightarrow) In this part will be proven that the optimal objective function value of the Restrained Roman domination problem (2.1) - (2.4) is greater or equal to the optimal objective function value of the MILP formulation (3.1) - (3.6), i.e. $F_1(f) \geq F_2(x, y, z)$.

For a fixed city $i \in V$ and a function f given by (2.1), let decision variables x_i , y_i and z_i be defined as

$$x_i = f(i), \quad y_i = \begin{cases} 1, & f(i) = 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad z_i = \begin{cases} 1, & f(i) = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Since $x_i \in [0, +\infty)$, $y_i, z_i \in \{0, 1\}$, conditions (3.4) - (3.6) are satisfied by the definition. For example, a condition (3.5) is satisfied because $z_i = 1$ for $x_i = f(i) = 0$ ($x_i + 2z_i = 2$) and $z_i = 0$ for

$x_i = f(i) = 1$ ($x_i + 2z_i = 1 < 2$). Similarly $z_i = 0$ for $x_i = f(i) = 2$, which again implies that $x_i + 2z_i = 2$.

Assuming that conditions (3.2) and (3.3) holds for a fixed vertex $i \in V$, there are two cases:

Case 1. Let values $f(i)$ are set to be greater or equal to one. Since $x_i = f(i)$, relation $x_i \geq 1$ implies. From the last relation and by the binary notations of the variables y_i and z_i it implies that $x_i + \sum_{j \in N_i} y_j \geq 1$ and $x_i + \sum_{j \in N_i} z_j \geq 1$.

Case 2. Let values $f(i)$ are set to be equal to zero. Satisfying relation (2.4) ($\exists v, w \in N_u$)($f(v) = 2 \wedge f(w) = 0$), it follows that $y_v = 1$ and $z_w = 1$. Therefore, $x_i + \sum_{j \in N_i} y_j = \sum_{j \in N_i} y_j \geq 1$ and $x_i + \sum_{j \in N_i} z_j = \sum_{j \in N_i} z_j \geq 1$.

Finally, since decision variables satisfies the conditions (3.1) - (3.6) for a fixed vertex i , it follows that $F_2(x, y, z) = \sum_{i \in V} x_i = \sum_{i \in V} f(i) = F_1(f)$.

(\Leftarrow) In this part it will be proven that optimal objective function value of the Restrained Roman domination problem (2.1) - (2.4) is less or equal to the optimal objective function value of the MILP formulation (3.1) - (3.6), i.e. $F_1(f) \leq F_2(x, y, z)$.

For a given set of decision variables x_i , y_i and z_i which satisfy conditions (3.1) - (3.6), let a function f be defined as

$$f(i) = \begin{cases} 0, & x_i \in [0, 1) \\ 1, & x_i \in [1, 2) \\ 2, & x_i \in [2, +\infty) \end{cases} \quad (3.7)$$

By the definition of the function f , condition (2.1) holds. Since the condition (2.1) holds, for a fixed vertex $u \in V$ there are two cases:

Case 1. Let $x_u \in [1, +\infty)$. By the definition of the function f , it follows that $f(u) = 1$ or $f(u) = 2$. Now, condition (2.4) holds, since $\perp \Rightarrow p$ is tautology for any logical statement p .

Case 2. Let $x_u \in [0, 1)$. By the definition of the function f , $f(u) = 0$. Because of the condition (3.2), $x_u + \sum_{j \in N_u} y_j \geq 1$, it follows that $\sum_{j \in N_u} y_j \geq 1 - x_u > 0$. Since the decision variables y_j are binary, $\sum_{j \in N_u} y_j$ has to be integer, which implies that $\sum_{j \in N_u} y_j \geq 1$. Therefore, there exists a vertex $v \in N_u$, $y_v = 1$. From the constraints (3.4), and because of the $x_v \geq 2y_v = 2$, it follows that $f(v) = 2$. Similarly, from the constraints (3.3) it follows that $\sum_{j \in N_u} z_j \geq 1 - x_u > 0$. Because of the binary type of the decision variables z_j , $\sum_{j \in N_u} z_j$ has an integer value. Now, since $\sum_{j \in N_u} z_j \geq 1$, there exists a vertex $w \in N_u$ such that $z_w = 1$. Finally, by the constraints (3.5), $x_w \leq 2 - 2z_w = 0$, it follows that $x_w = 0$ and that $f(w) = 0$ which means that condition (2.4) holds also.

By the definition of the function f , it is clear that $f(i) \leq x_i$, for $i \in V$. Therefore, $F_1(f) = \sum_{i \in V} f(i) \leq \sum_{i \in V} x_i = F_2(x, y, z)$.

So, for each feasible solution to the problem (2.1) - (2.4) there exists a feasible solution to the problem (3.1) - (3.6), satisfying the relation $F_2(x, y, z) \leq F_1(f)$, and for each feasible solution to the (3.1) - (3.6) there exists a feasible solution to the (2.1) - (2.4) satisfying the relation $F_1(f) \leq$

$F_2(x, y, z)$. Therefore, it follows that $\min_f F_1(f) = \min_{(x,y,z)} F_2(x, y, z)$. □

Applying the given MILP formulation to the illustrated RRDP, solution $\min_{(x,y,z)} F_2(x, y, z)$ to the proposed problem is equal to 3, and it can be obtained for $x = [1, 0, 2, 0, 0]$, $y = [0, 1, 0, 0, 0]$ and $z = [0, 0, 1, 1, 1]$.

4. Conclusions

This paper is devoted to the Restrained Roman domination problem. A mixed integer linear programming formulation is introduced and the correctness of the corresponding formulation is proved. The presented model uses relatively small number of the variables and constraints, which indicates that presented model can be used both in theoretical and practical considerations. As a future study, it is planned to construct an exact method for solving the corresponding mathematical model. Construction of the metaheuristics for solving the proposed problem can also be a part of a possible future study.

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References

- Currò, Vincenzo (2014). The Roman Domination Problem on Grid Graphs. PhD thesis. Università di Catania.
- Kelly, Christopher (2006). *The Roman Empire: A Very Short Introduction*. Oxford University Press.
- Liedloff, Mathieu, Ton Kloks, Jiping Liu and Sheng-Lung Peng (2005). Roman domination over some graph classes. In: *Graph-Theoretic Concepts in Computer Science*. Springer. pp. 103–114.
- Nicolet, Claude (1991). *Space, Geography, and Politics in the Early Roman Empire*. University of Michigan Press.
- Pushpam, Roushini Leely and Padmapriya Sampath (2015). Restrained roman domination in graphs. *Transactions on Combinatorics* **4**(1), 1–17.
- ReVelle, Charles S and Kenneth E Rosing (2000). Defendens imperium romanum: a classical problem in military strategy. *American Mathematical Monthly* pp. 585–594.
- Stewart, Ian (1999). Defend the roman empire!. *Scientific American* **281**, 136–138.
- Xing, Hua-Ming, Xin Chen and Xue-Gang Chen (2006). A note on roman domination in graphs. *Discrete mathematics* **306**(24), 3338–3340.