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New Čebyšev Type Inequalities for Functions whose Second Derivatives are (s_1, m_1)-(s_2, m_2)-convex on the Co-ordinates

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Abstract

In this paper, we establish some new Čebyšev type inequalities for functions whose second derivatives are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates.

Keywords: Čebyšev type inequalities, co-ordinates (s_1, m_1)-(s_2, m_2)-convex, integral inequality.
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1. Introduction

In 1882, Čebyšev (Chebyshev, 1882) gave the following inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1.1}$$

where $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and $g'$are bounded, where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx - \left(\frac{1}{b-a}\int_a^b f(x)\, dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\, dx\right), \tag{1.2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a, b]$ defined as $\|f\|_\infty = ess \sup_{t\in[a,b]} |f(t)|$.

During the past few years, many researchers established various generalizations, extensions and variants of Čebyšev type inequalities, we can mention the works (Ahmad *et al.*, 2009; Boukerrioua & Guezane-Lakoud, 2007; Guazene-Lakoud & Aissaoui, 2011; Latif & Alomari, 2009;

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Pachpatte & Talkies, 2006; Pachpatte, 2006; Sarikaya *et al.*, 2014). Recently the authors of (Guazene-Lakoud & Aissaoui, 2011), established a new Čebyšev type inequality for functions of two independent variables whose second derivatives are bounded. Also in (Sarikaya *et al.*, 2014), the authors obtained some new Čebyšev type inequalities involving functions whose mixed partial derivatives are s-convex on the co-ordinates. The main purpose of this work is to obtain new Čebyšev type inequalities for functions whose mixed partial derivatives are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates.

This paper is organized as follows: In section 2, we present some preliminaries. In the third section, we prove a new identity for functions of two independent variables then we used it to establish new Čebyšev type inequalities for functions whose mixed partial derivatives are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates.

2. Preliminaries

Throughout this paper we denote by Δ the bidimensional interval in $[0, \infty)^2$, $\Delta =: [a, b] \times [c, d]$ with $a < b$ and $c < d$, $\Delta_0 =: [0, b^*] \times [0, d^*]$ with $b^* > b$, $d^* > d$, $k =: (b-a)(d-c)$ and $\frac{\partial^2 f}{\partial\lambda\partial\alpha}$ by $f_{\lambda\alpha}$.

Definition 2.1. (Dragomir, 2001) A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & \lambda\alpha f(x, y) + \lambda(1-\alpha) f(x, v) \\ & + (1-\lambda)\alpha f(t, y) + (1-\lambda)(1-\alpha) f(t, v), \end{aligned} \tag{2.1}$$

holds for all $\lambda, \alpha \in [0, 1]$ and $(x, y), (x, v), (t, y), (t, v) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, it exists a co-ordinated convex function which is not convex.

Definition 2.2. (Alomari & Darus, 2008) A function $f : \Delta \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & \lambda^s\alpha^s f(x, y) + \lambda^s(1-\alpha)^s f(x, v) \\ & + (1-\lambda)^s \alpha^s f(t, y) + (1-\lambda)^s (1-\alpha)^s f(t, v), \end{aligned} \tag{2.2}$$

holds for all $\lambda, \alpha \in [0, 1]$ and $(x, y), (x, v), (t, y), (t, v) \in \Delta$, for some fixed $s \in (0, 1]$.

s-convexity on the co-ordinates does not imply the s-convexity, it exist a functions which are s-convex on the co-ordinates but are not s-convex.

Definition 2.3. (Bai & Qi, 2013; Chun, 2014) A function $f : \Delta_0 \to \mathbb{R}$ is said (s, m)-convex on Δ, if the following inequality

$$f(\lambda x + m(1-\lambda)t, \lambda y + m(1-\lambda)v) \leq \lambda^s f(x, y) + m(1-\lambda^s) f(t, v), \tag{2.3}$$

holds for all $(x, y), (t, v) \in \Delta$ and $\lambda \in [0, 1]$ and for some fixed $s, m \in (0, 1]$.

Definition 2.4. (Bai & Qi, 2013; Chun, 2014) A function $f : \Delta_0 \to \mathbb{R}$ is said to be (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates on $\Delta_{0,}$ if the following inequality

$$\begin{aligned} f(\lambda x + m_1(1-\lambda)t, \alpha y + m_2(1-\alpha)v) \leq\ & \lambda^{s_1}\alpha^{s_2} f(x,y) + m_2\lambda^{s_1}(1-\alpha^{s_2}) f(x,v) \\ & + m_1(1-\lambda^{s_1})\alpha^{s_2} f(t,y) \\ & + m_1 m_2 (1-\lambda^{s_1})(1-\alpha^{s_2}) f(t,v), \qquad (2.4) \end{aligned}$$

holds for all $(x, y), (x, v), (t, y), (t, v) \in \Delta$ with $\lambda, \alpha \in [0, 1]$ and $s_1, m_1, s_2, m_2 \in (0, 1]$.

3. Main result

Lemma 3.1. *Let $f : \Delta \to \mathbb{R}$ be partially differentiable function on Δ in $\mathbb{R}^2$. If $f_{\lambda\alpha} \in L_1(\Delta)$, then for any $(x, y) \in \Delta \subset \Delta_0$, we have the following identity*

$$\begin{aligned} f(x,y) =\ & \frac{1}{(b-a)}\int_a^b f(m_1 t, y)dt + \frac{1}{(d-c)}\int_c^d f(x, m_2 z)dz \\ & -\frac{1}{k}\int_a^b\int_c^d f(m_1 t, m_2 z)dzdt + \frac{1}{k}\int_a^b\int_c^d (x - m_1 t)(y - m_2 z) \\ & \times \left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z)\, d\alpha d\lambda\right) dzdt, \qquad (3.1) \end{aligned}$$

where $k = (b-a)(d-c)$.

Proof. For any $x, t \in [m_1 a, m_1 b]$ and $y, z \in [m_2 c, m_2 d]$ such that $t \neq x, y \neq z$, we have

$$\begin{aligned} \int_{m_1 t}^{x}\int_{m_2 z}^{y} f_{\sigma\tau}(\sigma,\tau)\, d\tau d\sigma &= \int_{m_1 t}^{x} (f_\sigma(\sigma, y) - f_\sigma(\sigma, m_2 z))\, d\sigma \\ &= f(x,y) - f(x, m_2 z) - f(m_1 t, y) + f(m_1 t, m_2 z), \qquad (3.2) \end{aligned}$$

which implies

$$f(x,y) = f(x, m_2 z) + f(m_1 t, y) - f(m_1 t, m_2 z) + \int_{m_1 t}^{x}\int_{m_2 z}^{y} f_{\sigma\tau}(\sigma,\tau)\, d\tau d\sigma. \qquad (3.3)$$

For $\sigma = \lambda x + m_1(1-\lambda)t$ and $\tau = \alpha y - m_2(1-\alpha)z$, (3.3) becomes

$$\begin{aligned} f(x,y) =\ & f(x, m_2 z) + f(m_1 t, y) - f(m_1 t, m_2 z) \\ & + (x - m_1 t)(y - m_2 z)\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z)\, d\tau d\sigma. \qquad (3.4) \end{aligned}$$

Integrating (3.4) over $[a, b] \times [c, d] \subset \Delta_0$, with respect to t, z, multiplying the resultant equality by $\frac{1}{k}$, we obtain the desired equality. □

Theorem 3.1. *Let $f, g : \Delta_0 \to \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ_0, if $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates, then we have*

$$\begin{aligned} |T(f,g)| \leq\; & \frac{(1+m_1s_1)(1+m_2s_2)}{8(m_1m_2k)^2(1+s_1)(1+s_2)} \int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} \left[M|g(x,y)| + N|f(x,y)|\right] \\ & \times\left[(x-m_1a)^2 + (m_1b-x)^2\right]\left[(y-m_2c)^2 + (m_2d-y)^2\right] dydx, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} T(f,g) =\; & \frac{1}{m_1m_2k}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)g(x,y)dydx - \frac{(d-c)}{m_1^2m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_{m_1a}^{m_1b} f(t,y)dt\right)dydx \\ & -\frac{(b-a)}{m_1m_2^2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_{m_2c}^{m_2d} f(x,z)dz\right)dydx \\ & +\frac{1}{m_1^2m_2^2k^2}\left(\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)dydx\right)\left(\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(t,z)dzdt\right), \end{aligned} \tag{3.6}$$

$$M = \underset{x,t\in[a,b],y,z\in[c,d]}{ess\sup} \left[|f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,z)| + |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,z)|\right],$$
$$N = \underset{x,t\in[a,b],y,z\in[c,d]}{ess\sup} \left[|g_{\lambda\alpha}(x,y)| + |g_{\lambda\alpha}(x,z)| + |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,z)|\right],$$

$(s_1, m_1), (s_2, m_2) \in (0, 1]^2$ *and* $k = (b-a)(d-c)$.

Proof. By Lemma 3.1, we have

$$\begin{aligned} & f(x,y) - \frac{1}{(b-a)}\int_a^b f(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d f(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d f(m_1t,m_2z)dzdt \\ & = \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z)\, d\alpha d\lambda\right)dzdt, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & g(x,y) - \frac{1}{(b-a)}\int_a^b g(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d g(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d g(m_1t,m_2z)dzdt \\ & = \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z)\, d\alpha d\lambda\right)dzdt. \end{aligned} \tag{3.8}$$

Multiplying (3.7) by $\frac{1}{2m_1m_2k}\, g(x,y)$ and (3.8) by $\frac{1}{2m_1m_2k} f(x,y)$, summing the resultant equalities, then integrating on $[m_1a, m_1b] \times [m_2c, m_2d]$ with respect to x, y, we get

$$\begin{aligned}
& \frac{1}{m_1m_2k}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)f(x,y)dydx - \frac{(d-c)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y) \\
& \times\left(\int_a^b f(m_1t,y)dt\right)dydx - \frac{(b-a)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_c^d f(x,m_2z)dz\right)dydx \\
& -\frac{(d-c)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left(\int_a^b g(m_1t,y)dt\right)dydx \\
& -\frac{(b-a)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left(\int_c^d g(x,m_2z)dz\right)dydx \\
& +\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_a^b\int_c^d f(m_1t,m_2z)dzdt\right)dydx \\
& +\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left(\int_a^b\int_c^d g(m_1t,m_2z)dzdt\right)dydx \\
= \; & \frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\Bigg[\int_a^b\int_c^d (x-m_1t)(y-m_2z) \\
& \times\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+m_1(1-\lambda)t, \alpha y-m_2(1-\alpha)z\right)d\alpha d\lambda\right)dzdt\Bigg]dydx \\
& +\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\Bigg[\int_a^b\int_c^d (x-m_1t)(y-m_2z) \\
& \times\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+m_1(1-\lambda)t, \alpha y-m_2(1-\alpha)z\right)d\alpha d\lambda\right)dzdt\Bigg]dydx. \qquad (3.9)
\end{aligned}$$

By Fubini's Theorem, we obtain

$$
\begin{aligned}
T(f,g) &= \frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\Bigg[\int_a^b\int_c^d (x-m_1t)(y-m_2z) \\
&\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z\right)d\alpha d\lambda\right)dzdt\Bigg]dydx \\
&+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\Bigg[\int_a^b\int_c^d (x-m_1t)(y-m_2z) \\
&\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z\right)d\alpha d\lambda\right)dzdt\Bigg]dydx. \qquad (3.10)
\end{aligned}
$$

Using the (s_1, m_1)-(s_2, m_2)-convexity and modulus, (3.10) gives

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |g(x,y)|\Bigg[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z| \\
&\times\Bigg[\int_0^1\int_0^1 \Big(\lambda^{s_1}\alpha^{s_2}\left|f_{\lambda\alpha}(x,y)\right| + m_2\lambda^{s_1}(1-\alpha^{s_2})\left|f_{\lambda\alpha}(x,z)\right| + m_1(1-\lambda^{s_1})\alpha^{s_2}\left|f_{\lambda\alpha}(t,y)\right| \\
&+m_1m_2(1-\lambda^{s_1})(1-\alpha^{s_2})\left|f_{\lambda\alpha}(t,z)\right|\Big)d\alpha d\lambda\Bigg]dzdt\Bigg]dydx \\
&+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |f(x,y)|\Bigg[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z| \\
&\times\Bigg[\int_0^1\int_0^1 \Big(\lambda^{s_1}\alpha^{s_2}\left|g_{\lambda\alpha}(x,y)\right| + m_2\lambda^{s_1}(1-\alpha^{s_2})\left|g_{\lambda\alpha}(x,z)\right| + m_1(1-\lambda^{s_1})\alpha^{s_2}\left|g_{\lambda\alpha}(t,y)\right| \\
&+m_1m_2(1-\lambda^{s_1})(1-\alpha^{s_2})\left|g_{\lambda\alpha}(t,z)\right|\Big)d\alpha d\lambda\Bigg]dzdt\Bigg]dydx. \qquad (3.11)
\end{aligned}
$$

By a simple calculation, we have

$$
\begin{aligned}
|T(f,g)| &\leq \frac{(1+m_1s_1)(1+m_2s_2)M}{2m_1m_2k^2(1+s_1)(1+s_2)}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |g(x,y)|\times\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\,dzdt\right]dydx \\
&+\frac{(1+m_1s_1)(1+m_2s_2)N}{2m_1m_2k^2(1+s_1)(1+s_2)}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |f(x,y)|\times\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\,dzdt\right]dydx. \qquad (3.12)
\end{aligned}
$$

Noting that

$$\int_a^b |x - m_1 t| \; dt = \frac{1}{2m_1}\left[(x - m_1 a)^2 + (m_1 b - x)^2\right], \tag{3.13}$$

$$\int_c^d |y - m_2 z| \, dz = \frac{1}{2m_2}\left[(y - m_2 c)^2 + (m_2 d - y)^2\right]. \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we obtain the required inequality. □

Corollary 3.1. *Let $f, g : \Delta_0 \to \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$, are integrable on Δ_0. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, s_2)-convex on the co-ordinates, then we have*

$$|T(f,g)| \le \frac{1}{8k^2}\int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|]\left[(x-a)^2+(b-x)^2\right]\times\left[(y-c)^2+(d-y)^2\right]dydx, \tag{3.15}$$

where

$$\begin{aligned} T(f,g) &= \frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx - \frac{(d-c)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)dt\right)dydx \\ &\quad -\frac{(b-a)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,z)dz\right)dydx \\ &\quad +\frac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)dydx\right)\left(\int_a^b\int_c^d g(t,z)dzdt\right), \end{aligned} \tag{3.16}$$

M, N, k are defined as in Theorem 3.1 and $(s_1, s_2) \in (0,1]^2$.

Proof. Applying Theorem 3.1, for $m_1 = m_2 = 1$, we obtain the desired inequality. □

Corollary 3.2. *Under the same hypothesis of Theorem 3.1, if $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are m_1-m_2-convex on the co-ordinates, then we have the following inequality*

$$\begin{aligned} |T(f,g)| &\le \frac{(1+m_1)(1+m_2)}{32\,(m_1 m_2 k)^2}\int_{m_1 a}^{m_1 b}\int_{m_2 c}^{m_2 d} [M\,|g(x,y)| + N\,|f(x,y)|] \\ &\quad \times\left[(x - m_1 a)^2 + (m_1 b - x)^2\right]\left[(y - m_2 c)^2 + (m_2 d - y)^2\right]dydx, \end{aligned} \tag{3.17}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $(m_1, m_2) \in (0,1]^2$.

Proof. Using Theorem 3.1, for $s_1 = s_2 = 1$, we obtain the desired inequality. □

Theorem 3.2. *Under the same hypothesis of Theorem 3.1, we have the following inequality*

$$|T(f,g)| \leq \frac{49}{3600}\left[\frac{(1+m_1s_1)(1+m_2s_2)}{(1+s_1)(1+s_2)}\right]^2 MNk^2m_1^2m_2^2\,, \tag{3.18}$$

where $T(f,g)$, M, N, $(s_1,m_1),(s_2,m_2)$ *and* k *are defined as in Theorem 3.1.*

Proof. Let $F, G, \widetilde{F}$ and $\widetilde{G}$ be defined as follows

$$F = f(x,y) - \frac{1}{(b-a)}\int_a^b f(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d f(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d f(m_1t,m_2z)dzdt,$$

$$G = g(x,y) - \frac{1}{(b-a)}\int_a^b g(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d g(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d g(m_1t,m_2z)dzdt,$$

$$\widetilde{F} = \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt,$$

$$\widetilde{G} = \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt.$$

By Lemma 3.1, we have

$$F = \widetilde{F} \text{ and } G = \widetilde{G}, \tag{3.19}$$

which implies

$$F\times G = \widetilde{F}\times\widetilde{G}. \tag{3.20}$$

Integrating both sides of (3.20) over $[m_1a,m_1b]\times[m_2c,m_2d]$ with respect to x,y, multiplying the resultant equality by $\frac{1}{m_1m_2k}$, using Fubini's Theorem and modulus, we get

$$\begin{aligned}|T(f,g)| \leq\ & \frac{1}{m_1m_2k^3}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d}\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\right.\\ &\times\left(\int_0^1\int_0^1 \left|f_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\right| d\alpha d\lambda\right)dzdt\times\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\\ &\left.\times\left(\int_0^1\int_0^1 \left|g_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\right| d\alpha d\lambda\right)dzdt\right]dydx.\end{aligned} \tag{3.21}$$

Using the (s_1, m_1)-(s_2, m_2)-convexity, we obtain

$$\begin{aligned}
|T(f,g)| \leq\ & \frac{1}{m_1 m_2 k^3} \int_{m_1 a}^{m_1 b} \int_{m_2 c}^{m_2 d} \left[\int_a^b \int_c^d |x - m_1 t| \, |y - m_2 z| \left[\int_0^1 \int_0^1 \left[\lambda^{s_1} \alpha^{s_2} \left| f_{\lambda\alpha}(x,y) \right| \right.\right.\right. \\
& + m_2 \lambda^{s_1} (1 - \alpha^{s_2}) \left| f_{\lambda\alpha}(x,z) \right| + m_1 (1 - \lambda^{s_1}) \alpha^{s_2} \left| f_{\lambda\alpha}(t,y) \right| \\
& \left.\left.\left. + m_1 m_2 (1 - \lambda^{s_1})(1 - \alpha^{s_2}) \left| f_{\lambda\alpha}(t,z) \right| \right] d\alpha d\lambda \right] dz dt \right] \\
& \times \left[\int_a^b \int_c^d |x - m_1 t| \, |y - m_2 z| \left[\int_0^1 \int_0^1 \left[\lambda^{s_1} \alpha^{s_2} \left| g_{\lambda\alpha}(x,y) \right| \right.\right.\right. \\
& + m_2 \lambda^{s_1} (1 - \alpha^{s_2}) \left| g_{\lambda\alpha}(x,z) \right| + m_1 (1 - \lambda^{s_1}) \alpha^{s_2} \left| g_{\lambda\alpha}(t,y) \right| \\
& \left.\left.\left. + m_1 m_2 (1 - \lambda^{s_1})(1 - \alpha^{s_2}) \left| g_{\lambda\alpha}(t,z) \right| \right] d\alpha d\lambda \right] dz dt \right] dy dx \\
\leq\ & \left[\frac{(1 + m_1 s_1)(1 + m_2 s_2)}{(1 + s_1)(1 + s_2)} \right]^2 \frac{M \times N}{m_1 m_2 k^3} \\
& \times \left[\int_{m_1 a}^{m_1 b} \left[\int_a^b |x - m_1 t| \, dt \right]^2 dx \right] \left[\int_{m_2 c}^{m_2 d} \left[\int_c^d |y - m_2 z| \, dz \right]^2 dy \right]. \qquad (3.22)
\end{aligned}$$

Taking into account that

$$\left[\int_{m_1 a}^{m_1 b} \left[\int_a^b |x - m_1 t| \, dt \right]^2 dx \right] = \frac{7}{60} m_1^3 (b - a)^5 \qquad (3.23)$$

and

$$\left[\int_{m_2 c}^{m_2 d} \left[\int_c^d |y - m_2 z| \, dz \right]^2 dy \right] = \frac{7}{60} m_2^3 (d - c)^5 . \qquad (3.24)$$

The desired inequality, will be obtained by combining (3.22), (3.23) and (3.24). □

Corollary 3.3. *Let $f, g : \Delta_0 \to \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$, are integrable on Δ_0. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, s_2)-convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{49}{3600} M \times N \times k^2, \qquad (3.25)$$

where $T(f,g)$, M, N and k are defined as in Theorem 3.1.

Proof. Applying Theorem 3.2, for $m_1 = m_2 = 1$, we obtain the desired inequality. □

Corollary 3.4. *Under the same hypothesis of Theorem 3.1, if $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are m_1-m_2-convex on the co-ordinates, we have the following inequality*

$$|T(f,g)| \leq \frac{49}{57600} [(1 + m_1)(1 + m_2)]^2 M \times N \times k^2 m_1^2 \times m_2^2 , \qquad (3.26)$$

where $T(f,g)$, M, N and k are defined as in Theorem 3.1 and $m_1, m_2 \in (0,1]$.

Proof. Applying Theorem 3.2, for $s_1 = s_2 = 1$, we obtain the desired inequality. □

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References

Ahmad, F., N. S. Barnett and S. S. Dragomir (2009). New weighted Ostrowski and Čebyšev type inequalities. *Nonlinear Analysis: Theory, Methods & Applications* **71**(12), e1408–e1412.

Alomari, M. and M. Darus (2008). The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates. *International Journal of Math. Analysis* **2**(13), 629–638.

Bai, S. P. and F. Qi (2013). Some inequalities for (s_1, m_1)-(s_2, m_2)-convex functions on the co-ordinates. *Global Journal of Mathematical Analysis* **1**(1), 22–28.

Boukerrioua, K. and A Guezane-Lakoud (2007). On generalization of Čebyšev type inequalities. *J. Inequal. Pure Appl. Math* **8**(2), Paper No. 55, 4 p.

Chebyshev, P. L. (1882). Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. *InProc.Math.Soc.Charkov(Vol.* **2**, 93–98.

Chun, L. (2014). Some new inequalities of Hermite-Hadamard type for (α_1, m_1)-(α_2, m_2)-convex functions on coordinates. *Journal of Function Spaces* **5950**, Article ID 975950, 7.

Dragomir, S. S. (2001). On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese J Math.* **4**, 775–788.

Guazene-Lakoud, A. and F. Aissaoui (2011). New Čebyšev type inequalities for double integrals. *J. Math. Inequal.* **5**(4), 453–462.

Latif, M. A. and M. Alomari (2009). On Hadamard-type inequalities for h-convex functions on the co-ordinates. *International Journal of Math. Analysis* **3**(33), 1645–1656.

Pachpatte, B. G. (2006). On Čebyšev-Grüss type inequalities via Pečarić's extension of the Montgomery identity. *JIPAM. Journal of Inequalities in Pure & Applied Mathematics [electronic only]* **7**(1), 1–4.

Pachpatte, B. G. and N. A. Talkies (2006). On Čebyšev type inequalities involving functions whose derivatives belong to Lp spaces. *J. Inequal. Pure and Appl. Math* **7**(2), 1–6.

Sarikaya, M. Z., H. Budak and H. Yaldiz (2014). Čebysev type inequalities for co-ordinated convex functions. *Pure and Applied Mathematics Letters* **2**, 244–48.