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Hadamard Product of Certain Harmonic Univalent Meromorphic Functions

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Abstract

In this paper the authors extended certain results concerning the Hadamard product for two classes related to starlike and convex harmonic univalent meromorphic functions, results for each class are obtained. Relevant connections of the results obtained here with various known results for meromorphic univalent functions are indicated.

Keywords: Harmonic functions, meromorphic functions, univalent functions, sense-preserving.
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1. Introduction and definitions

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) that such harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic of f. Also, a necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|$. There are numerous papers on univalent harmonic functions defined in a domain $U = \{z \in \mathbb{C} : |z| < 1\}$ (see (Jahangiri, 1998, 1999), and (Silverman & Silvia, 1999; Silverman, 1998)). Hengartner and Schober (Henartner & Schober, 1987) investigated functions harmonic in the exterior of the unit disc, that is $U^* = \{z \in \mathbb{C} : |z| > 1\}$. They showed that a complex valued, harmonic, sense-preserving univalent function f, defined on U^* and satisfying $f(\infty) = \infty$ must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \quad (A \in \mathbb{C}), \tag{1.1}$$

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where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \quad (0 \le |\beta| < |\alpha|), \tag{1.2}$$

and $a = \overline{f_{\bar{z}}}/f_z$ is analytic and satisfy $|a(z)| < 1$ for $z \in U^*$.

After this work, Jahangiri and Silverman (Jahangiri & Silverman, 1999) defined the class H_0^* of harmonic sense-preserving functions $f(z)$ that are starlike with respect to the origin in U^* given by (1.1) and (1.2) and proved that

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) < |\alpha| - |\beta| - |A| .$$

Denote by Σ_H the class of meromophic functions f that are harmonic univalent and sense-preserving in the exterior of open unit disc U in the form

$$f(z) = h(z) + \overline{g(z)} \tag{1.3}$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \; g(z) = \sum_{n=1}^{\infty} b_n z^{-n}. \tag{1.4}$$

Also, Jahangiri (Jahangiri, 2002) proved that if $f(z)$ given by (1.3) and (1.4) belongs to $\Sigma_H^*(\gamma)$, then

$$\sum_{n=1}^{\infty} \left(\frac{n+\gamma}{1-\gamma} |a_n| + \frac{n-\gamma}{1-\gamma} |b_n| \right) < 1.$$

Several authors have studies the classes of harmonic univalent meromorphic functions (see (Ahuja & Jahangiri, 2002; El-Ashwah *et al.*, 2014) and (Janteng & Halim, 2007)).

Now, we introduce the subclasses $\Sigma_H^*(c_n, d_n, \delta)$, $\Sigma_H^c(c_n, d_n, \delta)$ and $\Sigma_H^k(c_n, d_n, \delta)$ consisting of functions of the form (1.3) and (1.4) which satisfies the inequalities, respectively

$$\sum_{n=1}^{\infty} (c_n |a_n| + d_n |b_n|) < \delta \quad (c_n \ge d_n \ge c_2 > 0; \delta > 0), \tag{1.5}$$

$$\sum_{n=1}^{\infty} n (c_n |a_n| + d_n |b_n|) < \delta \quad (c_n \ge d_n \ge c_2 > 0; \delta > 0), \tag{1.6}$$

and

$$\sum_{n=1}^{\infty} n^k (c_n |a_n| + d_n |b_n|) < \delta \quad (c_n \ge d_n \ge c_2 > 0; \delta > 0). \tag{1.7}$$

It is easy to see that various subclasses of Σ_H consisting of functions $f(z)$ of the form (1.3) and (1.4) can be represented as $\Sigma_H^k(c_n, d_n, \delta)$ for suitable choices of c_n, d_n, δ *and* k studies earlier by various authors.

(i) $\Sigma_H^0(n, n, 1) = H_0^*$ (see Jahangiri and Silverman. ((Jahangiri & Silverman, 1999), with $\alpha = 1$ and $\beta = A = 0$));

(ii) $\Sigma_H^0(n + \gamma, n - \gamma, 1 - \gamma) = \Sigma_H^*(\gamma)(0 \leq \gamma < 1, n \geq 1)$ (see Jahangiri (Jahangiri, 2002));

(iii)$\Sigma_H^0(n(n+2)^m, n(n-2)^m, 1) = MH^*(m)(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}, n \geq 1)$ (see Bostanci and Ozturk (Bostanci & Ozturk, 2010));

(vi) $\Sigma_H^0((n + \gamma)(n + 2)^m, (n - \gamma)(n - 2)^m, 1 - \gamma) = MH^*(m, \gamma)(0 \leq \gamma < 1, m \in \mathbb{N}_0, n \geq 1)$ (see Bostanci and Ozturk (Bostanci & Ozturk, 2011)).

Evidently, $\Sigma_H^0(c_n, d_n, \delta) = \Sigma_H^*(c_n, d_n, \delta)$, and $\Sigma_H^1(c_n, d_n, \delta) = \Sigma_H^c(c_n, d_n, \delta)$. Further $\Sigma_H^{k_1}(c_n, d_n, \delta) \subset \Sigma_H^{k_2}(c_n, d_n, \delta)$ if $k_1 > k_2 \geq 0$, the containment being proper. Moreover, for any positive integer k, we have the following inclusion relation

$\Sigma_H^k(c_n, d_n, \delta) \subset \Sigma_H^{k-1}(c_n, d_n, \delta) \subset ... \subset \Sigma_H^2(c_n, d_n, \delta) \subset \Sigma_H^c(c_n, d_n, \delta) \subset \Sigma_H^*(c_n, d_n, \delta)$.

We also note that for any nonnegative real number k, the class $\Sigma_H^k(c_n, d_n, \delta)$ is nonempty as the function

$$f(z) = z + \sum_{n=1}^{\infty} n^{-k} \frac{\delta}{c_n} \lambda_n z^{-n} + \sum_{n=1}^{\infty} n^{-k} \frac{\delta}{d_n} \lambda_n \overline{z^{-n}}$$

where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n \leq 1$, satisfy the inequality (1.7).

For harmonic meromorphic functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} |a_n| z^{-n} + \sum_{n=1}^{\infty} |b_n| \overline{z^{-n}}$$

and

$$G(z) = z + \sum_{n=1}^{\infty} A_n z^{-n} + \sum_{n=1}^{\infty} B_n \overline{z^{-n}} \quad (A_n, B_n \geq 0),$$

we define the convolution of two harmonic functions f and G as

$$(f * G)(z) = f(z) * G(z)$$

$$= z + \sum_{n=1}^{\infty} |a_n| A_n z^{-n} + \sum_{n=1}^{\infty} |b_n| B_n \overline{z^{-n}},$$

and similarly, we can define the convolution of more than two meromorphic functions.

Several authors such as Mogra (Mogra, 1994, 1991), Aouf and Darwish (Aouf & Darwish, 2006), El-Ashwah and Aouf (El-Ashwah & Aouf, 2009, 2011) studied the convolution properties of meromorphic functions only.

The object of this paper is to establish a results concerning the Hadamard product of functions in the classes $\Sigma_H^k(c_n, d_n, \delta), \Sigma_H^c(c_n, d_n, \delta)$ and $\Sigma_H^*(c_n, d_n, \delta)$.

Throughout this paper, we assume $f(z)$, $g(z)$, $f_i(z)$, and $g_j(z)$ having following form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} \overline{b_n z^{-n}}, \tag{1.8}$$

$$g(z) = z + \sum_{n=1}^{\infty} u_n z^{-n} + \sum_{n=1}^{\infty} \overline{v_n z^{-n}}, \tag{1.9}$$

$$f_i(z) = z + \sum_{n=1}^{\infty} a_{n,i} z^{-n} + \sum_{n=1}^{\infty} \overline{b_{n,i} z^{-n}} \quad (i = 1, 2, ..., s), \tag{1.10}$$

$$g_j(z) = z + \sum_{n=1}^{\infty} u_{n,j} z^{-n} + \sum_{n=1}^{\infty} \overline{v_{n,j} z^{-n}} \quad (j = 1, 2, ..., q). \tag{1.11}$$

Since to a certain extent the work in the harmonic univalent meromorphic functions case has paralleled that of the harmonic analytic univalent case, one is tempted to search analogous results to those of Porwal and Dixt (Porwal & Dixit, 2015) for meromorphic harmonic univalent functions in U^*.

2. Main Results

Theorem 1. *Let the functions* $f_i(z)$ *defined by (1.10) belong to the class* $\Sigma_H^c(c_n, d_n, \delta)$ *for every* $i = 1, 2, ..., s$*; and let the functions* $g_j(z)$ *defined by (1.11) belong to the class* $\Sigma_H^*(c_n, d_n, \delta)$ *for every* $j = 1, 2, ..., q$*. If* $c_n, d_n \geq n\delta, (n \geq 1)$*, then the Hadamard product* $f_1 * f_2 * .... * f_m * g_1 * g_2 * .... * g_q(z)$ *belongs to the class* $\Sigma_H^{2s+q-1}(c_n, d_n, \delta)$*.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty} n^{2s+q-1} \left[c_n \left(\prod_{i=1}^{s} |a_{n,i}| \prod_{j=1}^{q} |u_{n,j}| \right) + d_n \left(\prod_{i=1}^{s} |b_{n,i}| \prod_{j=1}^{q} |v_{n,j}| \right) \right] \leq \delta$$

Since $f_i(z) \in \Sigma_H^c(c_n, d_n, \delta)$, we have

$$\sum_{n=1}^{\infty} n \left(c_n |a_{n,i}| + d_n |b_{n,i}| \right) \leq \delta, \tag{2.1}$$

for every $i = 1, 2, ..., s$, and therefore,

$$nc_n |a_{n,i}| \leq \delta \quad or \quad |a_{n,i}| \leq \left(\frac{\delta}{nc_n} \right)$$

and hence

$$|a_{n,i}| \leq n^{-2}, \tag{2.2}$$

for every $i = 1, 2, ..., s$. Also,

$$nd_n |b_{n,i}| \leq \delta \quad or \quad |b_{n,i}| \leq \left(\frac{\delta}{nc_n} \right)$$

and hence

$$|b_{n,i}| \leq n^{-2}, \tag{2.3}$$

for every $i = 1, 2, ..., s$. Further, since $g_j(z) \in \Sigma_H^*(c_n, d_n, \delta)$, we have

$$\sum_{n=1}^{\infty} \left(c_n \left| u_{n,j} \right| + d_n \left| v_{n,j} \right| \right) \leq \delta, \tag{2.4}$$

for every $j = 1, 2, ..., q$. Hence we obtain

$$\left| u_{n,j} \right| \leq n^{-1} \; and \; \left| v_{n,j} \right| \leq n^{-1} \tag{2.5}$$

for every $j = 1, 2, ..., q$.

Using (2.2) and (2.3) for $i = 1, 2, ..., s$; (2.5) for $j = 1, 2, ..., q-1$ and (2.4) for $j = q$, we have

$$\sum_{n=1}^{\infty} n^{2s+q-1} \left[c_n \left(\prod_{i=1}^{s} \left| a_{n,i} \right| \prod_{j=1}^{q-1} \left| u_{n,j} \right| \right) \left| u_{n,q} \right| + d_n \left(\prod_{i=1}^{s} \left| b_{n,i} \right| \prod_{j=1}^{q-1} \left| v_{n,j} \right| \right) \left| v_{n,q} \right| \right]$$

$$\leq \sum_{n=1}^{\infty} n^{2s+q-1} \left[c_n n^{-2s} n^{-(q-1)} \left| u_{n,q} \right| + d_n n^{-2s} n^{-(q-1)} \left| v_{n,q} \right| \right]$$

$$= \sum_{n=1}^{\infty} c_n \left| u_{n,q} \right| + d_n \left| v_{n,q} \right| \leq \delta.$$

Hence $f_1 * f_2 * .... * f_m * g_1 * g_2 * .... * g_q \in \Sigma_H^{2s+q-1}(c_n, d_n, \delta)$.

We note that the required estimate can also be obtained by using (2.2) and (2.3) for $i = 1, 2, ..., s-1$; (2.5) for $j = 1, 2, ..., q$; and (2.1) for $i = s$. □

Taking into account the convolution of the functions $f_i(z)$ defined by (1.10) for every $i = 1, 2, ..., s$; only in the proof of the above theorem and using (2.2) and (2.3) for $i = 1, 2, ..., s-1$, and the relation (2.1) for $i = s$, we have the following corollary:

Corollary 1. Let the functions $f_i(z)$ defined by (1.10) belong to the class $\Sigma_H^c(c_n, d_n, \delta)$ for every $i = 1, 2, ..., s$. If $c_n, d_n \geq n\delta$ $(n \geq 1)$, then the Hadamard product $f_1 * f_2 * ... * f_s(z)$ belongs to the class $\Sigma_H^{2s-1}(c_n, d_n, \delta)$.

Taking into account the convolution of the functions $g_j(z)$ defined by (1.11) for every $j = 1, 2, ..., q$; only in the proof of the above theorem and using (2.5) for $j = 1, 2, ..., q-1$; and the relation (2.4) for $j = q$, we have the following corollary:

Corollary 2. Let the functions $g_j(z)$ defined by (1.11) belong to the class $\Sigma_H^*(c_n, d_n, \delta)$ for every $j = 1, 2, ..., q$. If $c_n, d_n \geq \delta, (n \geq 1)$, then the Hadamard product $g_1 * g_2 * .... * g_q$ belongs to the class $\Sigma_H^{q-1}(c_n, d_n, \delta)$.

Remarks (i) If the co-analytic parts of $f_i(z)$ and $g_j(z)$ are zero for every $i = 1, 2, ..., s$ and $j = 1, 2, ..., q$, then we obtain the results obtained by El-Ashwah and Aouf (El-Ashwah & Aouf, 2011), with $a_{0,i} = 1, i = 1, 2, ..., s$ and $b_{0,j} = 1, j = 1, 2, ..., q$);

(ii) Taking $c_n = n + 1 + \beta(n + 2\alpha - 1)$ and $\delta = 2\beta(1 - \alpha)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ with the co-analytic parts zero in the above results, we obtain the results obtained by Mogra (Mogra, 1994);

(iii) Taking $c_n = n + \alpha$ and $\delta = 1 - \alpha (0 \leq \alpha < 1)$ with co-analytic parts are zero in the above results, we obtain the result obtained by Mogra (Mogra, 1991);

(iv) Taking $c_n = n^m[(n+1) + \beta(n + 2\alpha - 1)]$ and $\delta = 2\beta(1-\alpha)(0 \le \alpha < 1, 0 < \beta \le 1, m \in \mathbb{N}_0)$ with co-analytic parts are zero in the above results, we obtain the results obtained by El-Ashwah and Aouf (El-Ashwah & Aouf, 2009), with $a_{0,i} = 1, i = 1, 2, ..., s$ and $b_{0,j} = 1, j = 1, 2, ..., q$);

(vi) By specializing the coefficients c_n, d_n and the parameter δ we obtain corresponding results for various subclasses such as H_0^*, $\Sigma_H^*(\gamma)$, $MH^*(m)$, $MH^*(m, \gamma)$.

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