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Katsaras's Type Fuzzy Norm under Triangular Norms

Sorin Nădăban[a,*], Tudor Bînzar[b], Flavius Pater[b], Carmen Ţerei[a], Sorin Hoară[a]

[a] *Aurel Vlaicu University of Arad, Department of Mathematics and Computer Science, Elena Drăgoi 2, RO-310330 Arad, Romania.*

[b] *"Politehnica" University of Timişoara, Department of Mathematics, Piaţa Victoriei 2, RO-300006 Timişoara, Romania.*

Abstract

The aim of this paper is to redefine Katsaras's fuzzy norm using the notion of t-norm.

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1. Introduction and preliminaries

The concept of fuzzy set was introduced by L.A. Zadeh in his famous paper (Zadeh, 1965). A fuzzy set in X is a function $\mu : X \to [0,1]$. We will denote by $\mathcal{F}(X)$ the family of all fuzzy sets in X. The classical union and intersection of ordinary subsets of X can be extended by the following formulas, proposed by L. Zadeh:

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x) \ : \ i \in I\} \ , \ \left(\bigwedge_{i\in I} \mu_i\right)(x) = \inf\{\mu_i(x) \ : \ i \in I\} \ .$$

If $\mu_1, \mu_2 \in \mathcal{F}(X)$, then the inclusion $\mu_1 \subseteq \mu_2$ is defined by $\mu_1(x) \leq \mu_2(x)$.

Definition 1.1. (Chang, 1968) Let X, Y be arbitrary sets and $f : X \to Y$. If μ is a fuzzy set in Y, then $f^{-1}(\mu)$ is a fuzzy set in X defined by

$$f^{-1}(\mu)(x) = \mu(f(x)), (\forall)x \in X \ .$$

*Corresponding author

Email addresses: sorin.nadaban@uav.ro (Sorin Nădăban), tudor.binzar@upt.ro (Tudor Bînzar), flavius.pater@upt.ro (Flavius Pater), carmen.terei@uav.ro (Carmen Ţerei), sorin.hoara@uav.ro (Sorin Hoară)

If μ is a fuzzy set in X then $f(\mu)$ is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \end{cases} .$$

Remark. Previous definition is a special case of Zadeh's extension principle.

Since then many authors have tried to investigate fuzzy sets and their applications from different points of view. An important problem was finding an adequate definition of a fuzzy normed linear space. In the studying of the fuzzy topological vector spaces, Katsaras (1984) introduced for the first time the notion of fuzzy norm on a linear space.

Definition 1.2. (Katsaras & Liu, 1977) A fuzzy set ρ in X is said to be:

1. convex if $t\rho + (1-t)\rho \subseteq \rho, (\forall) t \in [0, 1]$;
2. balanced if $\lambda\rho \subseteq \rho, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$;
3. absorbing if $\bigvee\limits_{t>0} t\rho = 1$;
4. absolutely convex if it is both convex and balanced.

Proposition 1. *(Katsaras & Liu, 1977) Let ρ be a fuzzy set in X. Then:*

1. *ρ is convex if and only if*

$$\rho(tx + (1-t)y) \geq \rho(x) \wedge \rho(y), (\forall)x, y \in X, (\forall)t \in [0, 1];$$

2. *ρ is balanced if and only if $\rho(\lambda x) \geq \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$.*

Definition 1.3. (Katsaras, 1984) A Katsaras fuzzy semi-norm on X is a fuzzy set ρ in X which is absolutely convex and absorbing.

Definition 1.4. (Nădăban & Dzitac, 2014) A fuzzy semi-norm ρ on X will be called Katsaras fuzzy norm if

$$\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \;\Rightarrow\; x = 0 \,.$$

Remark. a) It is easy to see that

$$\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right] \Leftrightarrow \left[\inf_{t>0} \rho\left(\frac{x}{t}\right) < 1, \text{for } x \neq 0\right] .$$

b) The condition $\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right]$ is much weaker than that imposed by Katsaras (1984),

$$\left[\inf_{t>0} \rho\left(\frac{x}{t}\right) = 0, \text{for } x \neq 0\right] .$$

In 1992, Felbin (1992) introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of linear space. Following Cheng & Mordeson (1994), Bag & Samanta (2003) introduced another concept of fuzzy norm. In paper (Bag & Samanta, 2008) it is shown that the fuzzy norm defined by Bag and Samanta is similar to that of Katsaras. As the notion of fuzzy norm as defined by Cheng & Mordeson (1994) and Bag & Samanta (2003) can be generalized for arbitrary t-norms (see (Goleţ, 2010), (Alegre & Romaguera, 2010), (Nădăban & Dzitac, 2014)) motivates us to investigate the extension of Katsaras's fuzzy norm under triangular norm.

Definition 1.5. (Schweizer & Sklar, 1960) A binary operation

$$* : [0,1] \times [0,1] \to [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall) a, b \in [0,1]$;
2. $a * 1 = a, (\forall) a \in [0,1]$;
3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0,1]$;
4. If $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$, then $a * b \leq c * d$.

Example 1.1. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0,1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Remark. $a * 0 = 0, (\forall) a \in [0,1]$.

Definition 1.6. (Nădăban & Dzitac, 2014) Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a Bag-Samanta's type fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, (\forall) x \in X$;

(N2) $[N(x,t) = 1, (\forall) t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall) x \in X, (\forall) t \geq 0, (\forall) \lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x,t) * N(y,s), (\forall) x, y \in X, (\forall) t, s \geq 0$;

(N5) $(\forall) x \in X$, $N(x, \cdot)$ is left continuous and $\lim\limits_{t \to \infty} N(x,t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNL-space).

Remark. Bag & Samanta (2003) gave this definition for $* = \wedge$ and Goleţ (2010), Alegre & Romaguera (2010) gave also this definition in the context of real vector spaces.

2. Fuzzy vector spaces under triangular norms

In paper (Das, 1988) the sum of fuzzy sets, fuzzy subspaces and convex fuzzy sets were re-defined using the notion of a t-norm. In this way, several results are obtained, some of which are generalisation of the results of Katsaras & Liu (1977).

Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm.

Definition 2.1. Let μ_1, μ_2 be fuzzy sets in X. The sum of fuzzy sets μ_1, μ_2 is denoted by $\mu_1 + \mu_2$ and it is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x_1 + x_2 = x} [\mu_1(x_1) * \mu_2(x_2)] \; .$$

If μ is a fuzzy set in X and $\lambda \in \mathbb{K}$, then the fuzzy set $\lambda\mu$ is defined by

$$(\lambda\mu)(x) = \begin{cases} \mu\left(\frac{x}{\lambda}\right) & \text{if} \quad \lambda \neq 0 \\ 0 & \text{if} \quad \lambda = 0, x \neq 0 \\ \vee\{\mu(y) \; : \; y \in X\} & \text{if} \quad \lambda = 0, x = 0 \end{cases} \; .$$

Remark. In the particular case in which $* = \wedge$ we obtain the definition introduced by Katsaras & Liu (1977).

Proposition 2. *If $\alpha, \beta \in \mathbb{K}$ and $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$, then*

1. $\alpha(\beta\mu) = \beta(\alpha\mu) = (\alpha\beta)\mu$;
2. $\mu_1 \leq \mu_2 \Rightarrow \alpha\mu_1 \leq \alpha\mu_2$.

Proof. 1) *Case 1.* $\alpha \neq 0, \beta \neq 0$.

$$(\alpha(\beta\mu))(x) = (\beta\mu)\left(\frac{x}{\alpha}\right) = \mu\left(\frac{x}{\alpha\beta}\right) = ((\alpha\beta)\mu)(x) \; .$$

Similarly,

$$(\beta(\alpha\mu))(x) = (\alpha\mu)\left(\frac{x}{\beta}\right) = \mu\left(\frac{x}{\alpha\beta}\right) = ((\alpha\beta)\mu)(x) \; .$$

Case 2. $\alpha = 0, \beta \neq 0$.
Let $x \neq 0$. Then

$$(\alpha(\beta\mu))(x) = 0; ((\alpha\beta)\mu)(x) = 0; (\beta(\alpha\mu))(x) = (\alpha\mu)\left(\frac{x}{\beta}\right) = 0 \; .$$

For $x = 0$ we have

$$(\alpha(\beta\mu))(x) = \sup_{y \in X}(\beta\mu)(y) = \sup_{y \in X} \mu\left(\frac{y}{\beta}\right) = \sup_{y \in X} \mu(y) \; ;$$

$$((\alpha\beta)\mu)(x) = \sup_{y \in X} \mu(y) \; ; \; (\beta(\alpha\mu))(x) = (\alpha\mu)\left(\frac{x}{\beta}\right) = \sup_{y \in X} \mu(y) \; .$$

Case 3. $\alpha \neq 0, \beta = 0$ is similar.
Case 4. $\alpha = 0, \beta = 0$.
For $x \neq 0$ we have

$$(\alpha(\beta\mu))(x) = 0; ((\alpha\beta)\mu)(x) = 0; (\beta(\alpha\mu))(x) = 0 .$$

If $x = 0$, then

$$(\alpha(\beta\mu))(x) = \sup_{y\in X}(\beta\mu)(y) = \sup_{y\in X}\mu(y) .$$

$$((\alpha\beta)\mu)(x) = \sup_{y\in X}\mu(y) \ ; \ (\beta(\alpha\mu))(x) = \sup_{y\in X}(\alpha\mu)(y) = \sup_{y\in X}\mu(y) .$$

2) Let $x \in X$.
Case 1. $\lambda \neq 0$.

$$(\lambda\mu_1)(x) = \mu_1\left(\frac{x}{\lambda}\right) \leq \mu_2\left(\frac{x}{\lambda}\right) = (\lambda\mu_2)(x) .$$

Case 2. $\lambda = 0$. If $x \neq 0$, then $(\lambda\mu_1)(x) = 0 = (\lambda\mu_2)(x)$. For $x = 0$, we have

$$(\lambda\mu_1)(x) = \sup_{y\in X}\mu_1(y) \leq \sup_{y\in X}\mu_2(y) = (\lambda\mu_2)(x)$$

□

Proposition 3. *Let X, Y be vector spaces over $\mathbb{K}$, $f : X \to Y$ be a linear mapping, $\lambda \in \mathbb{K}$ and $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$. Then*

1. $f(\mu_1 + \mu_2) = f(\mu_1) + f(\mu_2)$;
2. $f(\lambda\mu) = \lambda f(\mu)$.

Proof. The proof is exactly the same as in (Katsaras & Liu, 1977). □

Proposition 4. *Let $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$ and $\alpha, \beta \in \mathbb{K}$. The following sentences are equivalent:*

1. $\alpha\mu_1 + \beta\mu_2 \leq \mu$;
2. *For all $x, y \in X$ we have $\mu(\alpha x + \beta y) \geq \mu_1(x) * \mu_2(y)$.*

Proof. The proof is exactly the same as in (Katsaras & Liu, 1977). □

Definition 2.2. A fuzzy set μ in X is called *-fuzzy linear subspace of X if

1. $\mu + \mu \subseteq \mu$;
2. $\lambda\mu \subseteq \mu, (\forall)\lambda \in \mathbb{K}$.

Proposition 5. *Let $\mu \in \mathcal{F}(X)$. Then μ is a *-fuzzy linear subspace of X if and only in*

$$\mu(\alpha x + \beta y) \geq \mu(x) * \mu(y), (\forall)x, y \in X, (\forall)\alpha, \beta \in \mathbb{K} .$$

Proof. The proof is exactly the same as in (Katsaras & Liu, 1977). □

Definition 2.3. A fuzzy set μ in X is called *-convex if

$$\mu(tx + (1 - t)y) \geq \mu(x) * \mu(y), (\forall)x, y \in X, (\forall)t \in [0, 1] .$$

Remark. If $\mu \in \mathcal{F}(X)$ is *-convex and crisp, then $\mu = \phi_A$ (ϕ_A is the characteristic function of the subset A of X) and

$$\mu(tx + (1-t)y) \geq \mu(x) * \mu(y), (\forall)x, y \in X, (\forall)t \in [0,1] \,.$$

Thus

$$\mu(tx + (1-t)y) = 1, (\forall)x, y \in A, (\forall)t \in [0,1] \,.$$

Hence

$$tx + (1-t)y \in A, (\forall)x, y \in A, (\forall)t \in [0,1] \,.$$

So A is convex in the classical sence.

Proposition 6. *A fuzzy set μ in X is *-convex if and only if*

$$t\mu + (1-t)\mu \subseteq \mu, (\forall)t \in [0,1] \,.$$

Proof. " $\Rightarrow$ " *Case 1.* $t = 0$.

$$(0\mu + 1\mu)(x) = \sup_{x_1+x_2=x} [(0\mu)(x_1) * (1\mu)(x_2)] = \sup_{y \in X} \mu(y) * \mu(x) \leq 1 * \mu(x) = \mu(x) \,.$$

Case 2. $t = 1$ is similar.
Case 3. $t \in (0,1)$.

$$(t\mu + (1-t)\mu)(x) = \sup_{x_1+x_2=x} [(t\mu)(x_1) * ((1-t)\mu)(x_2)] =$$

$$= \sup_{x_1+x_2=x} \left[\mu\left(\frac{x_1}{t}\right) * \mu\left(\frac{x_2}{1-t}\right)\right] \leq$$

$$\leq \sup_{x_1+x_2=x} \mu\left[t \cdot \frac{x_1}{t} + (1-t) \cdot \frac{x_2}{1-t}\right] = \sup_{x_1+x_2=x} \mu(x_1 + x_2) = \mu(x) \,.$$

" $\Leftarrow$ " *Case 1.* $t \in (0,1)$.

$$\mu(tx + (1-t)y) \geq (t\mu + (1-t)\mu)(tx + (1-t)y) \geq (t\mu)(tx) * ((1-t)\mu)((1-t)y) = \mu(x) * \mu(y) \,.$$

Case 2. $t = 0$.

$$\mu(0x + 1y) = \mu(y) = 1 * \mu(y) \geq \mu(x) * \mu(y) \,.$$

Case 3. $t = 1$ is similar. □

Remark. Some *-convexity properties of fuzzy sets, where * is a triangular norm on [0, 1], were investigated in papers (Yandong, 1984; Yuan & Lee, 2004; Nourouzi & Aghajani, 2008) etc.

3. Katsaras's type fuzzy norm

Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm.

Definition 3.1. A fuzzy set ρ in X which is *-convex, balanced and absorbing will be called Katsaras's type fuzzy semi-norm. If in addition

$$\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \;\Rightarrow\; x = 0\,,$$

then ρ will be called Katsaras's type fuzzy norm.

Lemma 1. *If ρ is balanced and absorbing, then $\rho(0) = 1$.*

Proof. As ρ is balanced, we have that $\rho(0) = \rho(0 \cdot x) \geq \rho(x)$. Thus $\rho(0) = \bigvee\limits_{x \in X} \rho(x)$. As ρ is absorbing, we have that $\bigvee\limits_{t>0} \rho\left(\frac{x}{t}\right) = 1$. Hence $\bigvee\limits_{x \in X} \rho(x) = 1$. Thus $\rho(0) = 1$. □

Lemma 2. *If ρ is balanced, then $\rho(\alpha x) = \rho(|\alpha| x), (\forall)x \in X, (\forall)\alpha \in \mathbb{K}$.*

Proof. If ρ is balanced, then $\rho(\lambda x) \geq \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$. Particularly, for $\lambda \in \mathbb{K}$: $|\lambda| = 1$, we have

$$\rho\left(\frac{1}{\lambda} x\right) \geq \rho(x), (\forall)x \in X\,.$$

Replacing x with λx, we obtain $\rho(x) \geq \rho(\lambda x), (\forall)x \in X$. Thus

$$\rho(\lambda x) = \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| = 1\,.$$

Take $\alpha \in \mathbb{K}, \alpha \neq 0$ (if $\alpha = 0$ it is obvious that $\rho(\alpha x) = \rho(|\alpha| x), (\forall)x \in X$). We put in previous equality $\lambda = \frac{\alpha}{|\alpha|}$. It results

$$\rho\left(\frac{\alpha}{|\alpha|} x\right) = \rho(x), (\forall)x \in X \Leftrightarrow \rho(\alpha x) = \rho(|\alpha| x), (\forall)x \in X\,.$$

□

Remark. The following theorems extend some results obtained in (Bag & Samanta, 2008).

Theorem 1. *If ρ is a Katsaras's type fuzzy norm, then*

$$N(x,t) := \begin{cases} \rho\left(\frac{x}{t}\right) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \end{cases}$$

is a Bag-Samanta's type fuzzy norm.

Proof. $(N1)$ $N(x,0)=0,(\forall)x\in X$ is obvious.
$(N2)$ $[N(x,t)=1,(\forall)t>0]\Rightarrow\rho\left(\frac{x}{t}\right)=1,(\forall)t>0\Rightarrow x=0$. Conversely, if $x=0$, then $N(0,t)=\rho(0)=1,(\forall)t>0$.
$(N3)$ We suppose that $t>0$ (if $t=0$ $(N3)$ is obvious). Using previous lemma we have:

$$N(\lambda x,t)=\rho\left(\frac{\lambda x}{t}\right)=\rho\left(\frac{|\lambda|x}{t}\right)=\rho\left(\frac{x}{t/|\lambda|}\right)=N\left(x,\frac{t}{|\lambda|}\right).$$

$(N4)$ If $t=0$, then $N(x,t)=0$ and $N(x,t)*N(y,s)=0*N(y,s)=0$ and the inequality $N(x+y,t+s)\geq N(x,t)*N(y,s)$ is obvious. A similar situation occurs when $s=0$. If $t>0,s>0$, then

$$N(x+y,t+s)=\rho\left(\frac{x+y}{t+s}\right)=\rho\left(\frac{t}{t+s}\cdot\frac{x}{t}+\frac{s}{t+s}\cdot\frac{y}{s}\right)\geq\rho\left(\frac{x}{t}\right)*\rho\left(\frac{y}{s}\right)=N(x,t)*N(y,s).$$

$(N5)$ First, we note that $N(x,\cdot)$ is non-decreasing. Indeed, for $s>t$, we have

$$N(x,s)=N(x+0,t+s-t)\geq N(x,t)*N(0,s-t)=N(x,t)*1=N(x,t).$$

We prove now that $N(x,\cdot)$ is left continuous in $t>0$.
Case 1. $N(x,t)=0$. Thus, for all $s\leq t$, as $N(x,s)\leq N(x,t)$, we have that $N(x,s)=0$. Therefore $\lim\limits_{s\to t,s<t}N(x,s)=0=N(x,t)$.
Case 2. $N(x,t)>0$. Let α be arbitrary such that $0<\alpha<N(x,t)$. Let (t_n) be a sequence such that $t_n\to t,t_n<t$. We prove that there exists $n_0\in\mathbb{N}$ such that $N(x,t_n)\geq\alpha,(\forall)n\geq n_0$. As $\alpha\in(0,N(x,t))$ is arbitrary, we will obtain that $\lim\limits_{n\to\infty}N(x,t_n)=N(x,t)$.

As $N(x,t)=t\rho(x)>0$, we have that $\rho(x)>0$. Let $s=\frac{\alpha}{\rho(x)}$. We note that $s<t$. Indeed,

$$s<t\Leftrightarrow\frac{\alpha}{\rho(x)}<t\Leftrightarrow\alpha<t\rho(x)=N(x,t).$$

As $t_n\to t,t_n<t$ and $s<t$, there exists $n_0\in\mathbb{N}$ such that $t_n>s,(\forall)n\geq n_0$. Then

$$N(x,t_n)=\rho\left(\frac{x}{t_n}\right)\geq\rho\left(\frac{x}{s}\right)=s\rho(x)=\alpha.$$

Since $\bigvee\limits_{t>0}t\rho(x)=1$, we obtain that $\bigvee\limits_{t>0}N(x,t)=1$. Thus $\lim\limits_{t\to\infty}N(x,t)=1$. □

Theorem 2. *If N is a Bag-Samanta's type fuzzy norm, then $\rho:X\to[0,1]$ defined by*

$$\rho(x)=N(x,1),(\forall)x\in X$$

is a Katsaras's type fuzzy norm.

Proof. First, we note that, by $(N2)$, we have $\rho(0)=N(0,1)=1$.
(1) ρ is *-convex.
Let $t\in(0,1)$. Then

$$\rho(tx+(1-t)y)=N(tx+(1-t)y,1)=N(tx+(1-t)y,t+1-t)\geq$$

$$\geq N(tx,t) * N((1-t)y, 1-t) = N(x,1) * N(y,1) = \rho(x) * \rho(y) \ .$$

If $t = 0$, then $\rho(tx + (1-t)y) = \rho(y) = 1 * \rho(y) \geq \rho(x) * \rho(y)$. The case $t = 1$ is similar.
(2) ρ is balanced.
Let $x \in X, \lambda \in \mathbb{K}^*, |\lambda| \leq 1$. As $N(x, \cdot)$ is non-decreasing, we have that

$$\rho(\lambda x) = N(\lambda x, 1) = N\left(x, \frac{1}{|\lambda|}\right) \geq N(x,1) = \rho(x) \ .$$

If $x \in X, \lambda = 0$, then $\rho(\lambda x) = \rho(0) = 1 \geq \rho(x)$.
(3) ρ is absorbing.
Using $(N5)$, we have that

$$\bigvee_{t>0} (t\rho)(x) = \bigvee_{t>0} \rho\left(\frac{x}{t}\right) = \bigvee_{t>0} N\left(\frac{x}{t}, 1\right) = \bigvee_{t>0} N(x,t) = 1 \ .$$

Finally,

$$\rho\left(\frac{x}{t}\right) = 1, (\forall) t > 0 \Rightarrow N\left(\frac{x}{t}, 1\right) = 1, (\forall) t > 0 \Rightarrow N(x,t) = 1, (\forall) t > 0 \Rightarrow x = 0 \ .$$

□

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