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Initial Maclaurin Coefficients Bounds for New Subclasses of Bi-univalent Functions

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Abstract

In this work we introduce the subclasses $\mathcal{L}_{\Sigma}(\theta,\alpha)$ and $\mathcal{L}_{\Sigma}(\theta,\gamma)$ of bi-univalent functions. Furthermore, we obtain coefficient bounds involving the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes.The results presented in this paper would generalize those in related works of several earlier authors.

Keywords: Analytic and univalent functions, Bi-univalent functions, Starlike and convex functions, Coefficients bounds.
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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ with the conditions $f(0) = 0$ and $f'(0) = 1$ and having form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (z \in \mathcal{U}). \tag{1.1}$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

For each θ, $-\pi < \theta \leq \pi$, Silverman and Silvia (Silverman & Silvia, 1999) introduced the class

$$\mathcal{L}(\theta) = \left\{ f \in \mathcal{A} \colon \mathrm{Re}\left(f'(z) + \frac{1+e^{i\theta}}{2} z f''(z) \right) > 0, \qquad z \in \mathcal{U} \right\}$$

and they proved that $\mathcal{L}(\theta) \subset \mathcal{L}(\pi)$, where $\mathcal{L}(\pi)$ is the well known class $\mathcal{R}$ that consists of univalent functions in whose derivatives have positive real part in $\mathcal{U}$ (Alexander, 1915). The class $\mathcal{L}(0)$

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was studied by Singh and Singh (Singh & Singh, 1989), Lewandowski *et al.* (Lewandowski *et al.*, 1976), Chichra (Chichra, 1977), and Silverman (Silverman, 1994).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathcal{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). For a brief history and interesting examples in the class Σ, see (Srivastava *et al.*, 2010).

Brannan and Taha (Brannan & Taha, 1988) (see also (Taha, 1981)) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, respectively (see (Brannan & Taha, 1988)). Thus, following Brannan and Taha (Brannan & Taha, 1988) (see also (Taha, 1981)), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ w \in \mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see (Brannan & Taha, 1988; Taha, 1981)).

Recently, Srivastava *et al.* (Srivastava *et al.*, 2010), Frasin (Frasin, 2014), Frasin and Aouf (Frasin & Aouf, 2011), Goyal and Goswami (Goyal & P.Goswami, 2012), Li and Wang (Li & Wang, 2012), Siregar and Raman (Siregar & Raman, 2012) and Caglar *et al.*(Caglar *et al.*, 2012) introduced new subclasses of bi-univalent functions and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these classes.

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

In order to establish our main results, we shall require the following lemma:

Lemma 1. *(Pommerenke, 1975) If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathcal{U}$ for which*

$$Re\,(p(z)) > 0,\ \ p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}).$$

2. Coefficient bounds for the function class $\mathcal{L}_\Sigma(\theta, \alpha)$

We now introduce the subclass $\mathcal{L}_\Sigma(\theta, \alpha)$ of the functions in the class $\mathcal{A}$ as follows.

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{L}_\Sigma(\theta, \alpha)$ *where* $0 < \alpha \leq 1$ and $\theta \in (-\pi, \pi]$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}) \tag{2.1}$$

and

$$\left|\arg\left(g'(w) + \frac{1 + e^{i\theta}}{2} w g''(w)\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathcal{U}), \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2.3}$$

We first state and prove the following result.

Theorem 1. *Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(\theta, \alpha)$ where $0 < \alpha \leq 1$ and $\theta \in (-\pi, \pi]$. Then*

$$|a_2| \leq \frac{2\alpha}{[(3\alpha + 9 + (1 - \alpha)\cos 2\theta + 6\cos\theta)^2 + ((1 - \alpha)\sin 2\theta + 6\sin\theta)^2]^{1/4}} \tag{2.4}$$

and

$$|a_3| \leq \frac{2\alpha^2}{5 + 3\cos\theta} + \frac{2\alpha}{3\sqrt{5 + 4\cos\theta}}. \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$f'(z) + \left(\frac{1 + e^{i\theta}}{2}\right) z f''(z) = [p(z)]^\alpha \tag{2.6}$$

and

$$g'(w) + \left(\frac{1 + e^{i\theta}}{2}\right) w g''(w) = [q(w)]^\alpha \tag{2.7}$$

where $p(z)$ and $q(w)$ are in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$(3 + e^{i\theta})a_2 = \alpha p_1, \tag{2.10}$$

$$3(2 + e^{i\theta})a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \tag{2.11}$$

$$-(3 + e^{i\theta})a_2 = \alpha q_1 \tag{2.12}$$

and

$$3(2 + e^{i\theta})(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$2(3 + e^{i\theta})^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$\begin{aligned} 6(2 + e^{i\theta})a_2^2 &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) \\ &= \alpha(p_2 + q_2) + \frac{(\alpha - 1)(3 + e^{i\theta})^2}{\alpha} a_2^2. \end{aligned}$$

Thus

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{6\alpha(2 + e^{i\theta}) - (\alpha - 1)(3 + e^{i\theta})^2}$$

that is

$$\left|a_2^2\right| = \frac{\alpha^2 \left|p_2 + q_2\right|}{\left|6\alpha(2 + e^{i\theta}) - (\alpha - 1)(3 + e^{i\theta})^2\right|}$$

Applying Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2| \leq \frac{2\alpha}{[(3\alpha + 9 + (1 - \alpha)\cos 2\theta + 6\cos\theta)^2 + ((1 - \alpha)\sin 2\theta + 6\sin\theta)^2]^{1/4}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$6(2 + e^{i\theta})a_3 - 6(2 + e^{i\theta})a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2\right). \tag{2.16}$$

Upon substituting the value of a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$, it follows that

$$a_3 = \frac{\alpha^2 p_1^2}{(3 + e^{i\theta})^2} + \frac{\alpha(p_2 - q_2)}{6(2 + e^{i\theta})}.$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{2\alpha^2}{5+3\cos\theta} + \frac{2\alpha}{3\sqrt{5+4\cos\theta}},$$

which completes the proof of Theorem 1. □

Choosing $\theta = \pi$ in Theorem 1, we obtain the following particular case due to Srivastava *et al.*(Srivastava *et al.*, 2010):

Corollary 2.1. *(Srivastava et al., 2010) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(\pi,\alpha)$; $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+1}} \tag{2.17}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.18}$$

Putting $\theta = 0$ in Theorem 1, we obtain the following particular case due to Frasin (Frasin, 2014):

Corollary 2.2. *(Frasin, 2014) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0,\alpha)$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+8}} \tag{2.19}$$

and

$$|a_3| \leq \frac{9\alpha^2+8\alpha}{36}. \tag{2.20}$$

3. Coefficient bounds for the function class $\mathcal{L}_\Sigma(\theta,\gamma)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{L}_\Sigma(\theta,\gamma)$ *where* $0 \leq \gamma < 1$, $\theta \in (-\pi,\pi]$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \operatorname{Re}\left(f'(z) + \frac{1+e^{i\theta}}{2} zf''(z)\right) > \gamma \qquad (z \in \mathcal{U}) \tag{3.1}$$

and

$$\operatorname{Re}\left(g'(w) + \frac{1+e^{i\theta}}{2} wg''(w)\right) > \gamma \qquad (w \in \mathcal{U}), \tag{3.2}$$

where the function g is given by (2.3).

Theorem 2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}_\Sigma(\theta,\gamma)$, where $0 \le \gamma < 1, \theta \in (-\pi,\pi]$. Then*

$$|a_2| \le \sqrt{\frac{4(1-\gamma)}{6\sqrt{5+4\cos\theta}}} \tag{3.3}$$

and

$$|a_3| \le \frac{2(1-\gamma)^2}{5+3\cos\theta} + \frac{2(1-\gamma)}{3\sqrt{5+4\cos\theta}}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist p and $q \in \mathcal{P}$ such that

$$f'(z) + \left(\frac{1+e^{i\theta}}{2}\right) zf''(z) = \gamma + (1-\gamma)p(z) \tag{3.5}$$

and

$$g'(w) + \left(\frac{1+e^{i\theta}}{2}\right) wg''(w) = \gamma + (1-\gamma)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(3+e^{i\theta})a_2 = (1-\gamma)p_1, \tag{3.7}$$

$$3(2+e^{i\theta})a_3 = (1-\gamma)p_2, \tag{3.8}$$

$$-(3+e^{i\theta})a_2 = (1-\gamma)q_1 \tag{3.9}$$

and

$$3(2+e^{i\theta})(2a_2^2 - a_3) = (1-\gamma)q_2 \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(3+e^{i\theta})^2 a_2^2 = (1-\gamma)^2(p_1^2+q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$6(2+e^{i\theta})a_2^2 = (1-\gamma)(p_2+q_2).$$

Thus, we have

$$\begin{aligned}\left|a_2^2\right| &\le \frac{(1-\gamma)}{6\left|2+e^{i\theta}\right|}(|p_2|+|q_2|)\\ &\le \frac{4(1-\gamma)}{6\sqrt{5+4\cos\theta}}\end{aligned}$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$6(2+e^{i\theta})a_3 - 6(2+e^{i\theta})a_2^2 = (1-\gamma)(p_2-q_2)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\gamma)(p_2-q_2)}{6(2+e^{i\theta})}.$$

Upon substituting the value of a_2^2 from (3.12), we obtain

$$a_3 = \frac{(1-\gamma)^2(p_1^2+q_1^2)}{2(3+e^{i\theta})^2} + \frac{(1-\gamma)(p_2-q_2)}{6(2+e^{i\theta})}.$$

Applying Lemma 1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \le \frac{2(1-\gamma)^2}{5+3\cos\theta} + \frac{2(1-\gamma)}{3\sqrt{5+4\cos\theta}}$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Choosing $\theta = \pi$ in Theorem 2, we obtain the following particular case due to Srivastava *et al.*(Srivastava *et al.*, 2010):

Corollary 3.1. *(Srivastava* et al.*, 2010) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0,\gamma)$, $0 \le \gamma < 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\gamma)}{3}} \tag{3.13}$$

and

$$|a_3| \le \frac{(1-\gamma)(5-3\gamma)}{3}. \tag{3.14}$$

Putting $\theta = 0$ in Theorem 2, we obtain the following particular case due to Frasin (Frasin, 2014):

Corollary 3.2. *(Frasin, 2014) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0,\gamma)$, $0 \le \gamma < 1$. Then*

$$|a_2| \le \frac{1}{3}\sqrt{2(1-\gamma)} \tag{3.15}$$

and

$$|a_3| \le \frac{(1-\gamma)(9(1-\gamma)+8)}{36}. \tag{3.16}$$

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