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Zweier I-Convergent Double Sequence Spaces Defined by a Sequence of Modulii

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Abstract

In the present article we have introduced the double sequence spaces ${}_2\mathcal{Z}^I(F)$, ${}_2\mathcal{Z}_0^I(F)$ and ${}_2\mathcal{Z}_\infty^I(F)$ for a sequence of modulii $F = (f_{ij})$. We have also studied their topological as well as algebraic properties.

Keywords: Ideal, filter, double sequence, sequence of modulii, Lipschitz function, I-convergence, monotone and solid spaces.
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1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \; or \; \mathbb{C}\}$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $\|x\|_\infty = \sup_k |x_k|$.

The concept of statistical convergence was first introduced by Fast (Fast, 1951) in 1951 and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Connor (Connor, 1998, 1989; Connor & Kline, 1996), Connor, Fridy and Kline (Connor *et al.*, 1994) and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k} |\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

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Later on it was studied by Fridy (Fridy, 1985, 1993) from the sequence space point of view and he linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński (Kostyrko *et al.*, 2000). Later on it was studied by Šalát, Tripathy, Ziman (Šalát *et al.*, 2004) and Demirci (Demirci, 2001).

Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. A set I $\subseteq 2^X$(2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin \mathcal{F}$, for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.

An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I, i.e $\mathcal{F}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$.

Each linear subspace of ω, for example, $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ and we denote it by writing $A : \lambda \longrightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$.

Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$. The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen (Altay *et al.*, 2006), Başar and Altay (Başar & Altay, 2003), Malkowsky (Malkowsky, 1997), Ng and Lee (Ng & Lee, 1978) and Wang (Wang, 1978). Şengönül (Şengönül, 2007) defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1 - p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, \text{otherwise}. \end{cases}$$

Following Basar and Altay (Başar & Altay, 2003), Şengönül Şengönül (2007) introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}$$

Definition 1.1. (Khan & Khan, 2013) A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}), where the two subscripts run through the sequence of natural numbers independent of each other. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$|(x_{ij}) - a| < \epsilon, \qquad for\ all\ i, j \geq N$$

Definition 1.2. A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij} x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.3. E is said to be monotone if it contains the canonical preimages of all its stepspaces.

Definition 1.4. E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$.

Definition 1.5. E is said to be a sequence algebra if $(x_{ij} y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I$. In this case we write I-lim $x_{ij} = L$. The space ${}_2c^I$ of all I-convergent double sequences to L is given by

$${}_2c^I = \{(x_k) \in \omega : \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C}\}.$$

Definition 1.7. A sequence $(x_{ij}) \in \omega$ is said to be I-null if $L = 0$. In this case we write I-lim $x_k = 0$.

Definition 1.8. A sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m, n dependent on ϵ such that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I$.

Definition 1.9. A sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\} \in I$.

Definition 1.10. A modulus function f is said to satisfy $\triangle_2$ condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

Definition 1.11. Take for I the class I_f of all finite subsets of $\mathbb{N}$.Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X.

Definition 1.12. For I= I_δ and $A \subset \mathbb{N} \times \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ-convergence is said to be logarithmic statistical convergence.

Definition 1.13. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.

Definition 1.14. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_{ij}) \in {}_2\ell_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of $_2\ell_\infty$ with respect to the supremum norm, $F(I) = {}_2\ell_\infty \cap {}_2c^I$. Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function. The following Lemmas will be used for establishing some results of this article.

Lemma 1. *Let E be a sequence space. If E is solid then E is monotone.*

Lemma 2. *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I. (Tripathy & Hazarika, 2011).*

Lemma 3. *If $I \subset 2^N$ and M⊆N. If M ∉I, then M∩N ∉I. (Tripathy & Hazarika, 2011).*

The idea of modulus was structured in 1953 by Nakano (Nakano, 1953).
A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if
(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is nondecreasing, and
(4) f is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967)used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle (Ruckle, 1973) proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle (Ruckle, 1973) proved that, for any modulus f:

$$X(f) \subset \ell_1 \text{ and } X(f)^\alpha = \ell_\infty$$

where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling (Garling, 1966), Köthe (Köthe, 1970) and many more. After then Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

Recently Khan et al (Khan *et al.*, 2013), introduced the following classes of sequences

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\},$$
$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f).$$

In this article we introduce the following sequence spaces.

$${}_2\mathcal{Z}^I(F) = \{(x_{ij}) \in\ {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\},$$
$${}_2\mathcal{Z}_0^I(F) = \{(x_{ij}) \in\ {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij}|) \geq \varepsilon\} \in I\},$$
$${}_2\mathcal{Z}_\infty^I(F) = \{(x_{ij}) \in\ {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij}|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$${}_2m_{\mathcal{Z}}^I(F) =\ {}_2\mathcal{Z}_\infty(F) \cap\ {}_2\mathcal{Z}^I(F)$$

and

$${}_2m_{\mathcal{Z}_0}^I(F) =\ {}_2\mathcal{Z}_\infty(F) \cap\ {}_2\mathcal{Z}_0^I(F).$$

2. Main Results

Theorem 1. *For a sequence of modulii $F = (f_{ij})$, the classes of sequences ${}_2\mathcal{Z}^I(F)$, ${}_2\mathcal{Z}_0^I(F)$, ${}_2m_{\mathcal{Z}}^I(F)$ and ${}_2m_{\mathcal{Z}_0}^I(F)$ are linear spaces.*

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in\ {}_2\mathcal{Z}^I(F)$ and let α, β be scalars. Then

$$I - \lim f_{ij}(|x_{ij} - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C};$$
$$I - \lim f_{ij}(|y_{ij} - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{2.1}$$

$$A_2 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|y_{ij} - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{2.2}$$

Since f_{ij} is a modulus function, we have

$$\begin{aligned} f_{ij}(|(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2) &\leq f_{ij}(|\alpha||x_{ij} - L_1|) + f_{ij}(|\beta||y_{ij} - L_2|) \\ &\leq f_{ij}(|x_{ij} - L_1|) + f_{ij}(|y_{ij} - L_2|) \end{aligned}$$

Now, by (2.1) and (2.2), $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_{ij} + \beta y_{ij}) \in\ {}_2\mathcal{Z}^I(F)$. Hence ${}_2\mathcal{Z}^I(F)$ is a linear space. □

We state the following result without proof in view of Theorem 2.1.

Theorem 2. *The spaces* ${}_2m^I_{\mathcal{Z}}(F)$ *and* ${}_2m^I_{\mathcal{Z}_0}(F)$ *are normed linear spaces, normed by*

$$||x_{ij}||_* = \sup_{i,j} f_{ij}(|x_{ij}|). \tag{2.3}$$

Theorem 3. *A sequence* $x = (x_{ij}) \in {}_2m^I_{\mathcal{Z}}(F)$ *I-converges if and only if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}\times\mathbb{N}$ *such that*

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I). \tag{2.4}$$

Proof. Suppose that $L = I - \lim x$. Then $B_\epsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N} : |x_{ij} - L| < \frac{\epsilon}{2}\} \in \mathcal{F}(I)$. For all $\epsilon > 0$, fix an $N_\epsilon \in B_\epsilon$ such that we have $|x_{N_\epsilon} - x_{ij}| \leq |x_{N_\epsilon} - L| + |L - x_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $(i,j) \in B_\epsilon$. Hence $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I)$.

Conversely, suppose that $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I)$. That is $\{(i,j) \in \mathbb{N}\times\mathbb{N} : (|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in \mathcal{F}(I)$ for all $\epsilon > 0$.

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in \mathcal{F}(I)$ as well as $C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$.

Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}(F)$. This implies that $J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in J\} \in \mathcal{F}(I)$ that is $diamJ \leq diamJ_\epsilon$ where the diam of J denotes the length of interval J.

In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_{ij} \supseteq ...$ with the property that $diamI_{ij} \leq \frac{1}{2} diamI_{(i-1)(j-1)}$ for (i,j=2,3,4, ...) and $\{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in {}_2m^I_{\mathcal{Z}}(F)\} \in I_{ij}$ for (i,j=1,2,3,4, ...). Then there exists a $\xi \in \cap I_{ij}$ where $(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $\xi = I - \lim x$. So that $f_{ij}(\xi) = I - \lim f_{ij}(x)$, that is $L = I - \lim f_{ij}(x)$. □

Theorem 4. *Let* (f_{ij}) *and* (g_{ij}) *be modulus functions for some fixed k that satisfy the* $\triangle_2$*-condition. If X is any of the spaces* ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}^I_0$, ${}_2m^I_{\mathcal{Z}}$ *and* ${}_2m^I_{\mathcal{Z}_0}$ *etc., then the following assertions hold.*
(a)$X(g_{ij}) \subseteq X(f_{ij}.g_{ij})$,
(b)$X(f_{ij}) \cap X(g_{ij}) \subseteq X(f_{ij} + g_{ij})$

Proof. (a) Let $(x_{mn}) \in {}_2\mathcal{Z}^I_0(g_{ij})$. Then

$$I - \lim_{m,n} g_{ij}(|x_{mn}|) = 0. \tag{2.5}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_{ij}(t) < \epsilon$ for $0 < t < \delta$. Write $y_{mn} = g_{ij}(|x_{mn}|)$ and consider

$$\lim_{m,n} f_{ij}(y_{mn}) = \lim_{m,n} f_{ij}(y_{mn})_{y_{mn}<\delta} + \lim_{m,n} f_{ij}(y_{mn})_{y_{mn}\geq\delta} \tag{2.6}$$

Now for $y_{mn} < \delta$, we already have $\lim_{m,n} f_{ij}(y_{mn}) < \epsilon$. For $y_{mn} \geq \delta$, we have $y_{mn} < \frac{y_{mn}}{\delta} < 1 + \frac{y_{mn}}{\delta}$. Since f_{ij} is non-decreasing, it follows that $f_{ij}(y_{mn}) < f_{ij}(1 + \frac{y_{mn}}{\delta}) < \frac{1}{2} f_{ij}(2) + \frac{1}{2} f_{ij}(\frac{2y_{mn}}{\delta})$

Since f_{ij} satisfies the $\triangle_2$-condition, therefore for $y_{mn} \geq \delta > 0$ we can choose some $K > 0$ such that $f_{ij}(y_{mn}) < \frac{1}{2} K \frac{y_{mn}}{\delta} f_{ij}(2) + \frac{1}{2} K \frac{y_{mn}}{\delta} f_{ij}(2) = K \frac{y_{mn}}{\delta} f_{ij}(2)$

Hence $\lim_{m,n} f_{ij}(y_{mn}) \leq \max(1,K)\delta^{-1} f_{ij}(2) \lim_{m,n}(y_{mn}) = \epsilon'(say)$. Substituting in equation (2.6), we get

$$\lim_{m,n} f_{ij}(y_{mn}) = \epsilon + \epsilon'. \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have $I-\lim\limits_{m,n} f_{ij}(g_{ij}(|x_{mn}|)) = 0$.

Hence $(x_{mn}) \in {}_2\mathcal{Z}_0^I(f_{ij}.g_{ij})$. Thus ${}_2\mathcal{Z}_0^I(g_{ij}) \subseteq {}_2\mathcal{Z}_0^I(f_{ij}.g_{ij})$. The other cases can be proved similarly.

(b) Let $(x_{mn}) \in {}_2\mathcal{Z}_0^I(f_{ij}) \cap {}_2\mathcal{Z}_0^I(g_{ij})$. Then $I-\lim\limits_{m,n} f_{ij}(|x_{mn}|) = 0$ and $I-\lim\limits_{m,n} g_{ij}(|x_{mn}|) = 0$

The rest of the proof follows from the following equality $\lim\limits_{m,n}(f_{ij}+g_{ij})(|x_{mn}|) = \lim\limits_{m,n} f_{ij}(|x_{mn}|) + \lim\limits_{m,n} g_{ij}(|x_{mn}|)$. □

Corollary 2.1. *$X \subseteq X(f_{ij})$ for some fixed (i,j) and $X = {}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}_0^I$, ${}_2m_{\mathcal{Z}}^I$ and ${}_2m_{\mathcal{Z}_0}^I$.*

Theorem 5. *The spaces ${}_2\mathcal{Z}_0^I(F)$ and ${}_2m_{\mathcal{Z}_0}^I(F)$ are solid and monotone .*

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I(F)$.

Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(F)$. Then

$$I-\lim_{(i,j)} f_{ij}(|x_{ij}|) = 0. \tag{2.8}$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \le 1$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$. Then the result follows from (2.8) and inequality $f_{ij}(|\alpha_{ij}x_{ij}|) \le |\alpha_{ij}|f_{ij}(|x_{ij}|) \le f_{ij}(|x_{ij}|)$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$. The space ${}_2\mathcal{Z}_0^I(F)$ is monotone follows from Lemma 1. For ${}_2m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly. □

Theorem 6. *The spaces ${}_2\mathcal{Z}^I(F)$ and ${}_2\mathcal{Z}_0^I(F)$ are sequence algebras.*

Proof. We prove the result for ${}_2\mathcal{Z}_0^I(F)$. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I(F)$. Then

$$I-\lim f_{ij}(|x_{ij}|) = 0$$

and

$$I-\lim f_{ij}(|y_{ij}|) = 0$$

Therefore, we have

$$I-\lim f_{ij}(|(x_{ij}.y_{ij})|) = 0.$$

Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I(F)$ and hence ${}_2\mathcal{Z}_0^I(F)$ is a sequence algebra. In a similar way we can prove the result for the space ${}_2\mathcal{Z}^I(F)$. □

Theorem 7. *The spaces ${}_2\mathcal{Z}^I(F)$ and ${}_2\mathcal{Z}_0^I(F)$ are not convergence free in general.*

Proof. Here we give a counter example. Let $I = I_f$ and $f_{ij}(x) = x^3$ for some fixed (i,j) and for all $x = (x_{mn}) \in [0,\infty)$. Consider the sequence (x_{mn}) and (y_{mn}) defined by

$$x_{mn} = \frac{1}{m+n} \quad \textit{and} \;\; y_{mn} = m+n \;\text{ for all } (m,n) \in \mathbb{N}\times\mathbb{N}.$$

Then $(x_{mn}) \in {}_2\mathcal{Z}^I(F) \cap {}_2\mathcal{Z}_0^I(F)$, but $(y_{mn}) \notin {}_2\mathcal{Z}^I(F) \cap {}_2\mathcal{Z}_0^I(F)$. Hence the spaces ${}_2\mathcal{Z}_0^I(F)$ and ${}_2\mathcal{Z}_0^I(F)$ are not convergence free. □

Theorem 8. ${}_2\mathcal{Z}_0^I(F) \subset {}_2\mathcal{Z}^I(F) \subset {}_2\mathcal{Z}_\infty^I(F)$.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_{ij}(|x_{ij} - L|) = 0$. We have

$$f_{ij}(|x_{ij}|) \leq \frac{1}{2} f_{ij}(|x_{ij} - L|) + \frac{1}{2} f_{ij}(|L|).$$

Taking the supremum over (i, j) on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}^I_\infty(F)$. The inclusion ${}_2\mathcal{Z}^I_0(F) \subset {}_2\mathcal{Z}^I(F)$ is obvious. □

Theorem 9. *The function $\hbar : {}_2m^I_{\mathcal{Z}}(F) \to \mathbb{R}$ is the Lipschitz function, where ${}_2m^I_{\mathcal{Z}}(F) = {}_2\mathcal{Z}^I_\infty(F) \cap {}_2\mathcal{Z}^I(F)$, and hence uniformly continuous.*

Proof. Let $x = (x_{ij}), y = (_{ij}) \in {}_2m^I_{\mathcal{Z}}(F)$, $x \neq y$.
Then the sets

$$A_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| \geq \|x - y\|_*\} \in I,$$

$$A_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < \|x - y\|_*\} \in \mathcal{F}(I),$$

$$B_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < \|x - y\|_*\} \in \mathcal{F}(I).$$

Hence also $B = B_x \cap B_y \in m^I{}_{{}_2\mathcal{Z}}(F)$, so that $B \neq \phi$.
Now taking $(i, j) \in$ we have, B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y_{ij} - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus $\hbar$ is a Lipschitz function.
For the space ${}_2m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly. □

Theorem 10. *If $x, y \in {}_2m^I_{\mathcal{Z}}(F)$, then $(x.y) \in {}_2m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < \epsilon\} \in \mathcal{F}(I),$$

$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < \epsilon\} \in \mathcal{F}(I).$$

Now,

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)| = |x_{ij}y_{ij} - x_{ij}\hbar(y) + x_{ij}\hbar(y) - \hbar(x)\hbar(y)| \leq |x_{ij}||y_{ij} - \hbar(y)| + |\hbar(y)||x_{ij} - \hbar(x)| \quad (2.9)$$

As ${}_2m^I_{\mathcal{Z}}(F) \subseteq {}_2\mathcal{Z}^I_\infty(F)$, there exists an $M \in \mathbb{R}$ such that $|x_{ij}| < M$ and $|\hbar(y)| < M$.
Using equation (2.9), we get

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(i, j) \in B_x \cap B_y \in {}_2m^I(F)$.
Hence $(x.y) \in {}_2m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.
For the space ${}_2m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly. □

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