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Some Families of q-Series Identities and Associated Continued Fractions

H. M. Srivastava[a,*], S. N. Singh[b], S. P. Singh[b]

[a]*Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada and China Medical University, Taichung 40402, Taiwan, Republic of China*
[b]*Department of Mathematics, Tilak Dhari Post-Graduate College, Jaunpur 222002, Uttar Pradesh, India*

Abstract

In this paper, by using some known q-identities, the authors derive several results involving q-series and associated continued fractions. Some other closely-related q-identities are also considered.

Keywords: q-Series, q-Identities, q-Series identities, Rogers-Ramanujan identities, Continued fractions.
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1. Introduction, Definitions and Notations

For $q, \lambda, \mu \in \mathbb{C}$ $(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, (Slater, 1966); see also the recent works (Cao & Srivastava, 2013), (Choi & Srivastava, 2014), (Srivastava, 2011), (Srivastava & Choi, 2012) and (Srivastava & Karlsson, 1985) dealing with the q-analysis)

$$(\lambda; q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \qquad (|q| < 1;\ \lambda, \mu \in \mathbb{C}), \tag{1.1}$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{1.2}$$

*Corresponding author
Email addresses: `harimsri@math.uvic.ca` (H. M. Srivastava), `snsp39@gmail.com` (S. N. Singh), `snsp39@yahoo.com` (S. P. Singh)

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} \left(1 - \lambda\, q^j\right) \qquad (|q| < 1;\ \lambda \in \mathbb{C}), \tag{1.3}$$

where, as usual, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}$ denotes the set of positive integers (*with* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). For convenience, we write

$$(a_1, \cdots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n \tag{1.4}$$

and

$$(a_1, \cdots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty. \tag{1.5}$$

In the literature on q-series, there usually are two types of identities as follows:

Type 1. Series = Product

and

Type 2. Series = Series.

The most famous identities of Type 1 are the following Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}. \tag{1.7}$$

The identities (1.6) and (1.7) have a remarkably fascinating history. They were first proved in 1894 by Rogers (Rogers, 1894), but his paper was completely overlooked. They were rediscovered (*without any published proof*) by Ramanujan sometime before 1913. These identities were discovered again in 1917 and proved independently by Schur (Schur, 1973).

There are numerous q-identities that are similar to the Rogers-Ramanujan identities (1.6) and (1.7). These include (for example) the q-identities due to Jackson (Jackson, 1928), Rogers (see (Rogers, 1894) and (Rogers, 1917)), Bailey (see (Bailey, 1947) and (Bailey, 1949)), and Slater (Slater, 1952) (see also (McLaughlin & Sills, 2009)). In particular, Slater's paper (Slater, 1952) contains a list of 130 q-identities of the Rogers-Ramanujan type. On the other hand, in terms of continued fractions, Ramanujan stated for $|q| < 1$ that

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n}} = 1 + \frac{q}{1+}\, \frac{q^2}{1+}\, \frac{q^3}{1+} \cdots. \tag{1.8}$$

There are numerous q-identities of Type 2 in the 'Lost' Notebook of Ramanujan (see (Ramanujan, 1988)) and also in other places in the literature on q-series. Our aim in this paper is to consider various q-identities of Type 2 in order to establish a number of results involving continued fractions of the form involved in (1.8).

2. A Set of Main Results

In this section, we propose to derive continued-fraction expressions for the quotients of the series involved in some known q-identities.

First of all, we consider the following identity (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.10)) *and* (McLaughlin *et al.*, 2008, p. 41, Eq. (6.1.7))):

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}. \tag{2.1}$$

and its companion identity given by (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.11)) *and* (McLaughlin *et al.*, 2008, p. 41, Eq. (6.1.8)))

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(\gamma/q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma/q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}. \tag{2.2}$$

I. We now investigate the quotient of the right-hand sides of (2.1) and (2.2) as follows:

$$\frac{(\gamma/q;q^2)_\infty}{(\gamma q;q^2)_\infty} \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}} = \cfrac{1-\frac{\gamma}{q}}{\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}}$$

$$= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n} - \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}}$$

$$= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{(-\gamma)^n q^{n(n-2)}\,(1-q^n)}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-\gamma)^n q^{n(n-1)}}{(q;q)_n}}}$$

$$= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{\sum_{n=1}^{\infty}\frac{(-\gamma)^n q^{n(n-2)}}{(q;q)_{n-1}}}{\sum_{n=0}^{\infty}\frac{(-\gamma)^n q^{n(n-1)}}{(q;q)_n}}}$$

$$= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{(-\gamma/q)}{\cfrac{\sum_{n=0}^{\infty}\frac{(-\gamma)^n q^{n(n-1)}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{(-\gamma)^n q^{n^2}}{(q;q)_n}}}}. \tag{2.3}$$

Proceeding in the same way, we find that

$$\frac{(\gamma/q;q^2)_\infty}{(\gamma q;q^2)_\infty}\frac{\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}} = \frac{1-\frac{\gamma}{q}}{1-}\,\frac{(\gamma/q)}{1-}\,\frac{\gamma}{1-}\,\frac{\gamma q}{1-}\,\frac{\gamma q^2}{1-}\,\frac{\gamma q^3}{1-}\cdots. \tag{2.4}$$

From (2.1), (2.2) and (2.4), we have

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(\gamma/q;q^2)_n(q^2;q^2)_n}} = \frac{1-\frac{\gamma}{q}}{1-}\,\frac{(\gamma/q)}{1-}\,\frac{\gamma}{1-}\,\frac{\gamma q}{1-}\,\frac{\gamma q^2}{1-}\,\frac{\gamma q^3}{1-}\cdots. \tag{2.5}$$

The following special cases and consequences of (2.5) are worthy of note. Firstly, upon setting $\gamma = -q$ in (2.5), we get

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q^2)_n(q^2;q^2)_n}} = \frac{2}{1+}\,\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\cdots, \tag{2.6}$$

which, in light of the known result (Andrews & Berndt, 2005, p. 87, Entry (3.2.3)), yields

$$\frac{2(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q^2)_n(q^2;q^2)_n} = 1+\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\cdots. \tag{2.7}$$

If we use another known result (Andrews & Berndt, 2005, p. 153, Corollary (6.2.6)) in (2.7), we obtain

$$2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q^2)_n(q^2;q^2)_n}=\frac{(q^2;q^4)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}+\frac{(q^2;q^4)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.8}$$

We next set $\gamma=-q^3$ in (2.5) and obtain

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(-q^4;q^2)_n(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q^4;q^4)_n}}=\frac{1+q^2}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^4}{1+}\;\cdots, \tag{2.9}$$

which, in conjunction with a known result (Andrews & Berndt, 2005, p. 87, Entry (3.2.3)), yields the following consequence of (2.5):

$$\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(-q^2;q^2)_{n+1}(q^2;q^2)_n}=\frac{1}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^4}{1+}\;\cdots. \tag{2.10}$$

II. Let us consider the following q-identity of Type 2 (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.7)) *and* (McLaughlin *et al.*, 2008, p. 40, Eq. (6.1.4))):

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n-1)/2}=(-\gamma;q)_\infty\sum_{n=0}^{\infty}\frac{(-a\gamma)^n\,q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n}, \tag{2.11}$$

which, upon replacing γ by γq, yields

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n+1)/2}=(-\gamma q;q)_\infty\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n^2}}{(-\gamma q;q)_n(q;q)_n}. \tag{2.12}$$

By taking the quotient of the left-hand sides of (2.11) and (2.12), we find that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n+1)/2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n-1)/2}}=\cfrac{1}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n-1)/2}-\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n+1)/2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n+1)/2}}}$$

$$=\cfrac{1}{1+\cfrac{\displaystyle\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_{n-1}}\,\gamma^n q^{n(n-1)/2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\gamma^n q^{n(n+1)/2}}}$$

$$\cdot \frac{1}{1+\frac{\gamma(1-a)}{\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}}}. \tag{2.13}$$

It is easily observed that

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}} &= 1+\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}-\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}} \\
&= 1+\frac{\sum_{n=0}^{\infty}\frac{(aq;q)_{n-1}(-a)(1-q^n)}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}} \\
&= 1-\frac{a\gamma q}{\frac{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+3)/2}}},
\end{aligned} \tag{2.14}$$

which, when combined with (2.14), yields

$$\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}} = \frac{1}{1+}\,\frac{\gamma(1-a)}{1-}\,\frac{a\gamma q}{\frac{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+3)/2}}}. \tag{2.15}$$

Finally, by iterating the above process, we get the following result:

$$\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}} = \frac{1}{1+}\,\frac{\gamma(1-a)}{1-}\,\frac{a\gamma q}{1+}\,\frac{\gamma q(1-aq)}{1-}\,\frac{a\gamma q^3}{1+}\,\cdots. \tag{2.16}$$

Applying the q-identities (2.11), (2.12) and (2.16), we find that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n^2}}{(-\gamma q;q)_n(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n}}=\frac{(1+\gamma)}{1+}\,\frac{\gamma(1-a)}{1-}\,\frac{a\gamma q}{1+}\,\frac{\gamma q(1-aq)}{1-}\,\frac{a\gamma q^3}{1+}\cdots, \tag{2.17}$$

which, upon setting $\gamma=-q$, yields

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)}}{(q^2;q)_n(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{[(q;q)_n]^2}}=\frac{(1-q)}{1-}\,\frac{q(1-a)}{1+}\,\frac{aq^2}{1-}\,\frac{q^2(1-aq)}{1+}\,\frac{aq^4}{1-}\cdots. \tag{2.18}$$

In its *further* special case when $a=1$, (2.18) yields

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_{n+1}(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}=\frac{1}{(q;q)_\infty}. \tag{2.19}$$

For $\gamma=1$ and $a=-q$, we find from (2.17) that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q)_n(q;q)_n}}=\frac{2}{1+}\,\frac{(1+q)}{1+}\,\frac{q^2}{1+}\,\frac{q(1+q^2)}{1+}\,\frac{q^4}{1+}\cdots. \tag{2.20}$$

If, instead, we put $\gamma=1$ and $a=q$ in (2.17), we get

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-1;q)_n(q;q)_n}}=\frac{2}{1+}\,\frac{(1-q)}{1-}\,\frac{q^2}{1+}\,\frac{q(1-q^2)}{1-}\,\frac{q^4}{1+}\cdots. \tag{2.21}$$

We now recall a known result (Andrews & Berndt, 2005, p. 152, Entry (6.2.32)) *with* $a=-1$ as follows:

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n}=\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}q^{n(n+1)/2}=(q^2;q^2)_\infty, \tag{2.22}$$

which, in combination with (2.21), yields

$$\frac{2}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-1;q)_n(q;q)_n}=1+\frac{1-q}{1-}\,\frac{q^2}{1+}\,\frac{q(1-q^2)}{1-}\,\frac{q^4}{1+}\cdots. \tag{2.23}$$

III. Let us consider the following known q-identity (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.9)) *and* (McLaughlin *et al.*, 2008, p. 40, Eq. (6.1.6))):

$$\sum_{n=0}^{\infty} \frac{q^{3n(n-1)/2}\,\gamma^n}{(\gamma;q^2)_n (q;q)_n} = \frac{1}{(\gamma;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}\gamma^n}{(q^2;q^2)_n}, \tag{2.24}$$

which, upon replacing γ by γq^2, yields

$$\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}\,\gamma^n}{(\gamma q^2;q^2)_n (q;q)_n} = \frac{1}{(\gamma q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\gamma^n}{(q^2;q^2)_n}. \tag{2.25}$$

For the quotient of of the right-hand sides of (2.24) and (2.25), we have

$$\begin{aligned}\frac{(1-\gamma)\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\gamma^n}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^n}{(q^2;q^2)_n}} &= \frac{(1-\gamma)}{1+\dfrac{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n-1)}}{(q^2;q^2)_n}(1-q^{2n})}{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n+1)}}{(q^2;q^2)_n}}} \\ &= \frac{(1-\gamma)}{1+\dfrac{\displaystyle\sum_{n=1}^{\infty} \frac{\gamma^n q^{n(2n-1)}}{(q^2;q^2)_{n-1}}}{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n-1)}}{(q^2;q^2)_n}}} = \frac{(1-\gamma)}{1+\dfrac{\gamma q}{\dfrac{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n+1)}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n+3)}}{(q^2;q^2)_n}}}}.\end{aligned} \tag{2.26}$$

Proceeding in the above way, we obtain

$$\frac{(1-\gamma)\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\gamma^n}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}\gamma^n}{(q^2;q^2)_n}} = \frac{(1-\gamma)}{1+}\,\frac{\gamma q}{1+}\,\frac{\gamma q^3}{1+}\,\frac{\gamma q^5}{1+}\,\frac{\gamma q^7}{1+}\,\cdots. \tag{2.27}$$

Finally, by applying (2.24), (2.25) and (2.27), we get

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}\gamma^n}{(\gamma q^2;q^2)_n (q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{3n(n-1)/2}\gamma^n}{(\gamma;q^2)_n (q^2;q^2)_n}} = \frac{1-\gamma}{1+}\,\frac{\gamma q}{1+}\,\frac{\gamma q^3}{1+}\,\frac{\gamma q^5}{1+}\,\frac{\gamma q^7}{1+}\,\cdots. \tag{2.28}$$

In its special case when $\gamma = -1$, we find from (2.28) that

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(-1;q^2)_n (q^2;q^2)_n}} = \frac{2}{1-}\,\frac{q}{1-}\,\frac{q^3}{1-}\,\frac{q^5}{1-}\,\frac{q^7}{1-}\cdots. \tag{2.29}$$

Many other similar results involving q-series and associated continued fractions can also be derived analogously.

3. Concluding Remarks and Observations

While q-identities of Type 1 include such important and widely-investigated results as the celebrated Rogers-Ramanujan identities, we have successfully derived several families of q-identities of Type 2 involving q-series and associated continued fractions. We have also considered some other closely-related q-identities of Types 1 and 2.

Such q-series identities of Type 2 as (for example) (2.1), (2.2), (2.11) and (2.24), upon which our present investigation depends remarkably heavily, are derivable as special or limit cases of relatively more familiar known q-identities (see, for details, (Bowman & McLaughlin, 2006, pp. 4–7) and (McLaughlin *et al.*, 2008, p. 42)).

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References

Andrews, G. E. and B. C. Berndt (2005). *Ramanujan's Lost Notebook, Part I*. Springer, Berlin, Heidelberg and New York.

Bailey, W. N. (1947). Some identities in combinatory analysis. *Proc. London Math. Soc.* (*Ser.* 2) **49**, 421–435.

Bailey, W. N. (1949). Identities of the Rogers-Ramanujan type. *Proc. London Math. Soc.* (*Ser.* 2) **50**, 1–10.

Bowman, D. and J. McLaughlin (2006). Some more identities of the Rogers-Ramanujan type. Preprint 2006 [arXiv:math/0607202v2 [math.NT] 8 Jul 2006].

Cao, J. and H. M. Srivastava (2013). Some q-generating functions of the Carlitz and Srivastava-Agarwal types associated with the generalized Hahn polynomials and the generalized Rogers-Szegö polynomials. *Appl. Math. Comput.* **219**, 8398–8406.

Choi, J. and H. M. Srivastava (2014). q-extensions of a multivariable and multiparameter generalization of the Gottlieb polynomials in several variables. *Tokyo J. Math.* **37**, 111–125.

Jackson, F. H. (1928). Examples of a generalization of Euler's transformation for power series. *Messenger Math.* **57**, 169–187.

McLaughlin, J., A. V. Sills and P. Zimmer (2008). Rogers-Ramanujan-Slater type identities. *Electron. J. Combin.* **15**, 1–59. Article ID DS15.

McLaughlin, J. and A. V. Sills (2009). Some more identities of the Rogers-Ramanujan type. *Ramanujan J.* **18**, 307–325.

Ramanujan, S. (1988). *The 'Lost' Notebook and Other Unpublished Papers (With an Introduction by G. E. Andrews)*. Springer-Verlag, Berlin, Heidelberg and New York; Narosa Publishing House, New Delhi.

Rogers, L. J. (1894). Second memoir on the expansion of certain infinite products. *Proc. London Math. Soc.* (*Ser.* 1) **25**, 318–343.

Rogers, L. J. (1917). On two theorems of combinatory analysis and some allied identities. *Proc. London Math. Soc.* (*Ser.* 2) **16**, 315–336.

Schur, I. (1973). Ein bertrag zur additiven zahlentheorie and zur theorie der ketlenbruchen. In: *Gesammelte Abhandlungen, Band* **II**. Springer-Verlag, Berlin, Heidelberg and New York. pp. 117–136. [Originally published in *Sitzungsherichte der Preussischen Akademie der Wissenschaften*: *Physikalischo-Methematische Klasse* (1917), 302–321].

Slater, L. J. (1952). Further identities of the Rogers-Ramanujan type. *Proc. London Math. Soc.* (*Ser.* 2) **54**, 147–167.

Slater, L. J. (1966). *Generalized Hypergeometric Functions*. Cambridge University Press, Cambridge, London and New York.

Srivastava, H. M. (2011). Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inform. Sci.* **5**, 390–444.

Srivastava, H. M. and J. Choi (2012). *Zeta and q-Zeta Functions andAssociated Series and Integrals*. Elsevier Science Publishers, Amsterdam, London and New York.

Srivastava, H. M. and P. W. Karlsson (1985). *Multiple Gaussian Hypergeometric Series*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.