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New Subclass of p - valent Harmonic Meromorphic Functions

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Abstract

In this paper, we have introduced a new subclass of p-valent harmonic meromorphic and orientation preserving functions in the exterior of the unit disc. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for the functions belonging to this class are obtained.

Keywords: Harmonic functions, p-valent functions, meromorphic functions, convex combination, distortion bounds.
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1. Introduction

Let C be the field of complex numbers. A continuous function $f(z) = u + iv$ is a complex valued harmonic function in a domain $D \subseteq C$, if both u and v are real harmonic in D. Hengartner and Schober [5], among others, investigated the class of functions of the form $f(z) = h(z) + \overline{g(z)}$, which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$ so that $f(\infty) = \infty$. It is known that f(z) admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1.1}$$

where

$$h(z) = az + \sum_{n=0}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=0}^{\infty} b_n z^{-n} \tag{1.2}$$

For $0 \leq |\beta| \leq |\alpha|$ and $a(z) = \dfrac{\overline{f_{\bar{z}}}}{f_z}$ is analytic and satisfies $|a(z)| < 1$ for $z \in \widetilde{U}$. Since the affine transformation

$$\frac{\overline{\alpha} f - \overline{\beta f} - \overline{\alpha} a_0 + \overline{\beta a_0}}{|\alpha|^2 - |\beta|^2}$$

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is again in the class studied by Hengartner and Schober see (Hengartner & Schober, 1987). Recently, Jahangiri (Jahangiri, 2000) assumed $\alpha = 1, \beta = 0$ and removed the logarithmic singularity by letting $A = 0$ in (1.1) and focused on the study of the family of harmonic meromorphic functions.

For fixed positive integer p, consider the family $\Sigma_H(p)$ consisting of functions

$$f(z) = h(z) + \overline{g(z)} \tag{1.3}$$

which are $p-$valent harmonic meromorphic functions in $\widetilde{U}$, where

$$h(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)},$$
$$g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}, \quad |b_p| < 1 \tag{1.4}$$

we call $h(z)$ the analytic part and $g(z)$ is co-analytic part of $f(z)$. For $0 \leq \gamma < 1$, $k \geq 1$ and $0 \leq \alpha \leq 2\pi$, we define a new subclass as follows: Let $\Sigma_H(p, \gamma, k)$ consist of functions $f(z)$ satisfying the conditions

$$Re\left\{(1 + ke^{i\alpha})\frac{zf'(z)}{z'f(z)} - pke^{i\alpha}\right\} \geq p\gamma, \tag{1.5}$$

where $z' = \dfrac{\partial}{\partial\theta} z$ with $z = re^{i\theta}$, $r > 1$ and θ is real.

Further, let $\Sigma_{\overline{H}}(p, \gamma, k)$ denote the subclass of $\Sigma_H(p, \gamma, k)$ consisting of functions $f(z) = h(z) + \overline{g(z)}$ such that $h(z)$ and $g(z)$ are of the form

$$h(z) = z^p + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{-(n+p-1)},$$
$$g(z) = -\sum_{n=1}^{\infty} |b_{n+p-1}| z^{-(n+p-1)}, \quad |b_p| < 1 \tag{1.6}$$

Note that, various other subclasses of harmonic $p-$valent meromorphic functions have been studied rather extensively by Ahuja and Jahangiri (Ahuja & Jahangiri, 2003) and Murugusundaramoorthy (Murugusundaramoorthy, 2003), we also note that, $\Sigma_H(1, \gamma, 1)$, the class of harmonic meromorphic functions, was studied by Rosy (T. Rosy & Jahangiri, 2001). Among other things, Ahuja and Jahangiri (Ahuja & Jahangiri, 2003), proved that if, $f(z) = h(z) + \overline{g(z)}$ is given by (1.4) and if,

$$\sum_{n=1}^{\infty} (n + p - 1)(|a_{n+p-1}| + |b_{n+p-1}|) \leq p, \tag{1.7}$$

then $f(z)$ is harmonic, sense -preserving and $p-$valent in $\widetilde{U}$ and $f \in \Sigma_H(p)$.

In the present paper, we have obtained coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations for functions in the class $\Sigma_{\overline{H}}(p, \gamma, k)$.

2. Coefficient Bounds

First we state and prove the coefficient bound for the class $\Sigma_H(p,\gamma,k)$.

Theorem 2.1. *Let $f(z) = h(z) + \overline{g(z)}$ with $h(z)$ and $g(z)$ given by* (1.4) *. If*

$$\sum_{n=1}^{\infty}\left[(n+p-1)(1+k)+p(k+\gamma)\right]\left|a_{n+p-1}\right| + \left[(n+p-1)(1+k)-p(k+\gamma)\right]\left|b_{n+p-1}\right| \leq p(1-\gamma), \tag{2.1}$$

then $f(z)$ is harmonic, orientation preserving and $p-$valent in $\widetilde{U}$ and $f \in \Sigma_H(p,\gamma,k)$.

Proof. Suppose that (2.1) holds. Then we have

$$Re\frac{(1+ke^{i\alpha})(zh'(z)-\overline{zg'(z)})-pke^{i\alpha}(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}} = \frac{A(z)}{B(z)} \geq p\gamma, \tag{2.2}$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \gamma < 1$, $k \geq 1$, $0 \leq \alpha \leq 2\pi$,
here, we let

$$A(z) = (1+ke^{i\alpha})(zh'(z)-\overline{zg'(z)})-pke^{i\alpha}(h(z)+\overline{g(z)}) \tag{2.3}$$

and

$$B(z) = (h(z)+\overline{g(z)}). \tag{2.4}$$

Using the fact that $Rew \geq p\gamma$, if and only if $|p-\gamma+\omega| \geq |p+\gamma-\omega|$, it suffices to show that

$$|A(z)+p(1-\gamma)B(z)| - |A(z)-p(1+\gamma)B(z)| \geq 0. \tag{2.5}$$

Substituting the expressions for $A(z)$ and $B(z)$ in (2.5), we obtain

$$\begin{aligned}
|A(z)+p(1-\gamma)B(z)| - |A(z)-p(1+\gamma)B(z)| &= \left|p(1-\gamma)h(z)+(1+ke^{i\alpha})zh'(z))-pke^{i\alpha}h(z)\right| \\
&+\left|\overline{p(1-\gamma)g(z)-(1+ke^{i\alpha})zg'(z)-pke^{i\alpha}g(z)}\right| - |\, p(1+\gamma)h(z)-(1+ke^{i\alpha})zh'(z)+pke^{i\alpha}h(z) \\
&+\overline{p(1+\gamma)g(z)+(1+ke^{i\alpha})zg'h(z)+pke^{i\alpha}g(z)}\,| \\
&= \left|p(2-\gamma)z^p - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)+p(ke^{i\alpha}-p-\gamma)]\right| a_{n+p-1}z^{-(n+p-1)} \\
&- \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)+p(1-ke^{i\alpha}-p-\gamma)]\left|b_{n+p-1}\right| z^{-(n+p-1)}| \\
&\left|\gamma p z^p - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)+p(ke^{i\alpha}+1+\gamma)]\left|a_{n+p-1}\right| z^{-(n+p-1)}\right.
\end{aligned}$$

$$\left. - \sum_{n=1}^{\infty} [(1+ke^{i\alpha})(n+p-1) - p(ke^{i\alpha}+1+\gamma)] \left|b_{n+p-1}\right| z^{-(n+p-1)} \right|$$

$$2p(1-\gamma)|z|^p - \sum_{n=1}^{\infty} 2(n+p-1)(1+k) + 2p(k+\gamma)\left|a_{n+p-1}\right| |z|^{-(n+p-1)}$$

$$- \sum_{n=1}^{\infty} 2(n+p-1)(1+k) - 2p(k+\gamma)\left|b_{n+p-1}\right| |z|^{-(n+p-1)}$$

$$2|z|^p \left\{ p(1-\gamma) - \sum_{n=1}^{\infty} (n+p-1)(1+k) + p(k+\gamma)\left|a_{n+p-1}\right| |z|^{-(n+p-2)} \right\}$$

$$+ \sum_{n=1}^{\infty} (n+p-1)(1+k) - p(k+\gamma)\left|b_{n+p-1}\right| |z|^{-(n+p-2)}$$

$$\geq 2\{p(1-\gamma) - \sum_{n=1}^{\infty} (n+p-1)(1+k) + p(k+\gamma)\left|a_{n+p-1}\right|$$

$$+ \sum_{n=1}^{\infty} (n+p-1)(1+k) - p(k+\gamma)\left|b_{n+p-1}\right|\} \geq 0,$$

by (2.1). □

Remark 2.2. *It is natural to ask if the condition* (2.1) *is also necessary for functions* $f \in \Sigma_H(p,\gamma,k)$.

In the next theorem we show that the answer to that question which is in affirmative.

Theorem 2.3. *Let* $f(z) = h(z) + \overline{g(z)}$ *be such that* $h(z)$ *and* $g(z)$ *given by* (1.6). *Then* $f(z) \in \sum_{\overline{H}}(p,\gamma,k)$, *if and only if the inequality* (2.1) *holds for the coefficients of* $f(z) = h(z) + \overline{g(z)}$.

Proof. In view of Theorem I, we only need to show that $f(z) \notin \sum_{\overline{H}}(p,\gamma,k)$, if the condition (2.1) does not hold. We note that for $f(z) \in \sum_H(p,\gamma,k)$, we have

$$Re\left\{ \frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}} \right\} \geq p\gamma.$$

This is equivalent to

$$Re\left\{ \frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}} \right\} - p\gamma =$$

$$Re\left\{ \frac{1}{z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}} \Big[p(1-\gamma)z^p \right.$$

$$- \sum_{n=1}^{\infty} [(1+ke^{i\alpha})(n+p-1) + pke^{i\alpha} + p\gamma] a_{n+p-1} z^{-(n+p-1)}$$

$$\left. - \sum_{n=1}^{\infty} [(1+ke^{i\alpha})(n+p-1) - pke^{i\alpha} - p\gamma] b_{n+p-1} z^{-(n+p-1)} \Big] \right\} \geq 0$$

The above condition must hold for all values of z such that $|z| = r < 1$. Upon choosing the values of z on the positive real axis, we must have

$$Re\left\{\frac{1}{1+\sum_{n=1}^{\infty}(a_{n+p-1}-b_{n+p-1})r^{-(n-1)}}\left[p(1-\gamma)-\sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)+pke^{i\alpha}+p\gamma]a_{n+p-1}r^{-(n-1)} - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)-pke^{i\alpha}-p\gamma]b_{n+p-1}r^{-(n-1)}\right]\right\} \geq 0.$$

If the condition (2.1) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0 > 1$, for which the quotient in (2.5) is negative. This contradicts the conditions for $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ and this completes the proof. □

3. Distortion Bounds and Extreme Points

The determination of the extreme points of a compact family of harmonic univalent functions enables us to solve many extremal problems for the family. The fundamental reason for considering extreme points for starlike and convex functions is to more easily categorize extremal properties under continuous linear functionals acting on these classes. In this section, we shall obtain distortion bounds for functions in $\Sigma_{\overline{H}}(p, \gamma, k)$ and also determine the extreme points for the class $\Sigma_{\overline{H}}(p, \gamma, k)$.

Theorem 3.1. *If $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ then $r^p - p(1-\gamma)r^{-p} \leq |f(z)| \leq r^p + p(1-\gamma)r^{-p}, |z| = r < 1$.*

Proof. We only prove the inequality on the right. The argument for the inequality on the left is similar. Let $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Taking the absolute value of $f(z)$, we obtain

$$|f(z)| \leq \left|z^p + \sum_{n=1}^{\infty} a_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}z^{-(n+p-1)}\right| \leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-(n+p-1)}$$

$$\leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-p} \leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-(n+p-1)}$$

$$\leq r^p + \textstyle\sum_{n=1}^{\infty}[(n+p-1)(1+k)+p(k+\gamma)a_{n+p-1}] + \sum_{n=1}^{\infty}[(n+p-1)(1+k)-p(k+\sum)b_{n+p-1}]$$

$$\leq r^p + (p-\gamma)r^{-p}$$

by (2.1). Our next result shows how $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ looks like. We precisely proved. □

Theorem 3.2. *$f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$, if and only if $f(z)$ can be expressed as*

$$f(z) = \sum_{n=1}^{\infty}(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}) \tag{3.1}$$

where $z \in \widetilde{U}$,

$$h_{p-1}(z) = z^p,$$
$$h_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)+p(k+\gamma)} z^{(n+p-1)} \quad (n = 1, 2, 3, ...).$$

$$g_{p-1}(z) = z^p,$$
$$g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)-p(k+\gamma)} z^{-(n+p-1)} \quad (n = 1, 2, 3, ...)$$

$$\sum_{n=1}^{\infty}(x_{n+p-1} + y_{n+p-1}) = 1, x_{n+p-1} \geq 0 \ \ and \ \ y_{n+p-1} \geq 0.$$

Proof. For the functions $f(z)$ given by (3.1), we may write,

$$f(z) = \sum_{n=1}^{\infty}\left(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}\right)$$

$$= x_{p-1}h_{p-1} + y_{p-1}g_{p-1} + \sum_{n=1}^{\infty} x_{n+p-1}(z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)+p(k+\gamma)})z^{(n+p-1)}$$

$$+y_{n+p-1}z^p - \frac{p(1-\gamma)}{(n+p-1)(1+k)-p(k+\gamma)} z^{-(n+p-1)}.$$

Then,

$$= \sum_{n=1}^{\infty}\Bigg[((1+k)(n+p-1)+p(\gamma+k))\left(\frac{p(1-\gamma)}{(1+k)(n+p-1)+p(\gamma+k)}x_{n+p-1}\right)$$
$$+((1+k)(n+p-1)-p(\gamma+k))\left(\frac{p(1-\gamma)}{(1+k)(n+p-1)-p(\gamma+k)}y_{n+p-1}\right).$$
$$= p(1-\gamma)\sum_{n=1}^{\infty} x_{n+p-1} + y_{n+p-1} \leq p(1-\gamma),$$

and so $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Conversely, suppose that $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Set

$$x_n = \frac{(1+k)(n+p-1)+p(\gamma+k)}{p(1-\gamma)}|a_{n+p-1}|$$

and

$$y_n = \frac{(1+k)(n+p-1)+p(\gamma+k)}{p(1-\gamma)}|b_{n+p-1}|, (n = 1, 2, 3, ...)$$

Then note that by Theorem 2, $0 \leq x_{p-1} \leq 1$.

$$y_{p-1} = 1 - x_{p-1} - \sum_{n=1}^{\infty}(x_{n+p-1} + y_{n+p-1}),$$

we obtain

$$f(z) = \sum_{n=1}^{\infty}(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1})$$

as required. □

4. Convolution and Convex Linear Combination

In this section, we show that the class $\Sigma_{\overline{H}}(p, \gamma, k)$ is invariant under convolution and convex combinations of its members. For harmonic functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}(\overline{z})^{-(n+p-1)}$$

and

$$F(z) = z^p + \sum_{n=1}^{\infty} A_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} B_{n+p-1}(\overline{z})^{-(n+p-1)}$$

we define the convolution of $f(z)$ and $F(z)$ as

$$(f * F)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1}A_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}B_{n+p-1}(\overline{z})^{-(n+p-1)} \tag{4.1}$$

Using this definition, we show in the next theorem that the class $\Sigma_{\overline{H}}(p, \gamma, k)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \beta \leq \gamma \leq 1$, let $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ and $F(z) \in \Sigma_{\overline{H}}(p, \beta, k)$. Then*

$$f(z) * F(z) \in \Sigma_{\overline{H}}(p, \gamma, k) \subset \Sigma_{\overline{H}}(p, \beta, k). \tag{4.2}$$

Proof. Let

$$f(z) = z^p + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{-(n+p-1)} - \sum_{n=1}^{\infty} |b_{n+p-1}|(\overline{z})^{-(n+p-1)}$$

$$F(z) = z^p + \sum_{n=1}^{\infty} A_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} B_{n+p-1}(\overline{z})^{-(n+p-1)}$$

Note that $A_{n+p-1} \leq 1$ and $B_{n+p-1} \leq 1$. Obviously, the coefficients of f and F must satisfy conditions similar to the inequality (2.1). So for the coefficients of $f * F$ we can write,

$$\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)|a_{n+p-1}A_{n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{n+p-1}B_{n+p-1}|$$

$$\leq (1+k)(n+p-1)+p(\gamma+k)|a_{n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{n+p-1}|.$$

This right hand side of the above inequality is bounded by 2 because $f(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. By the same token, we can conclude that $f(z) * F(z) \in \Sigma_{\overline{H}}(p,\gamma,k) \subset \Sigma_{\overline{H}}(p,\beta,k)$. Our next result shows that $\Sigma_{\overline{H}}(p,\gamma,k)$ is closed under convex combination of its members. □

Theorem 4.2. *The family $\Sigma_{\overline{H}}(p,\gamma,k)$ is closed under convex combination*

Proof. For $i = 1,2,3,..$, let $f_i(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$ where $f_i(z)$ is given by

$$f_i(z) = z^p + \sum_{n=1}^{\infty}|a_{i,n+p-1}|(\overline{z})^{(n+p-1)} + \sum_{n=1}^{\infty}|b_{i,n+p-1}|(\overline{z})^{-(n+p-1)}.$$

Then by (2.1),

$$\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)|a_{i,n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{i,n+p-1}| \leq p(1-\gamma) \quad (4.3)$$

for $\sum_{n=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i(z)$ may be written as

$$\sum_{n=1}^{\infty} t_i f_i(z) = z^p + \sum_{n=1}^{\infty}(t_i|a_{i,n+p-1}|)z^{-(n+p-1)} + \sum_{n=1}^{\infty}(t_i|b_{i,n+p-1}|)(\overline{z})^{-(n+p-1)}.$$

Then by (4.2),

$$\sum_{n=1}^{\infty}[(1+k)(n+p-1)+p(\gamma+k)|\sum_{n=1}^{\infty}(t_i|a_{i,n+p-1}|)|+(1+k)(n+p-1)$$
$$-p(\gamma+k)|\sum_{n=1}^{\infty}(t_i|b_{i,n+p-1}|)|$$

$$\sum_{n=1}^{\infty} t_i\Big\{\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)a_{i,n+p-1}+(1+k)(n+p-1)$$
$$-p(\gamma+k)b_{i,n+p-1}\Big\} \leq \sum_{n=1}^{\infty} t_i p(1-\gamma) = (1-\gamma).$$

Since this is the condition required by (2.1), we conclude that $\sum_{n=1}^{\infty} t_i f_i(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. This completes the proof of Theorem (2.1). □

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