Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 6 (1) (2016) 69–76

Some Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows in Banach Spaces

Diana Borlea[a,*]

[a]*Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, V. Pârvan Blvd. No. 4, RO-300223 Timişoara, Romania*

Abstract

The exponential dichotomy is one of the most important asymptotic properties for the solutions of evolution equations, studied in the last years from various perspectives. In this paper we study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. Several illustrative examples motivate the approach.

Keywords: Skew-evolution semiflow, invariant projection, uniform exponential dichotomy.
2010 MSC: 34D05, 93D20.

1. Introduction

The property of exponential dichotomy is a mathematical domain with a substantial recent development as it plays an important role in describing several types of evolution equations. The literature dedicated to this asymptotic behavior begins with the results published in Perron (1930). The ideas were continued by in Massera & Schäffer (1966), with extensions in the infinite dimensional case accomplished in Daleckiĭ & Kreĭn (1974) and in Pazy (1983), respectively in Sacker & Sell (1994). Diverse and important concepts of dichotomy were introduced and studied, for example, in Appell *et al.* (1993), Babuţia & Megan (2015), Chow & Leiva (1995), Coppel (1978), Megan & Stoica (2010), Sasu & Sasu (2006) or Stoica & Borlea (2012).

The notion of skew-evolution semiflow that we sudy in this paper and which was introduced in Megan & Stoica (2008) generalizes the skew-product semiflows and the evolution operators. Several asymptotic properties for skew-evolution semiflows are defined and characterized see Viet Hai (2010), Viet Hai (2011), Stoica & Borlea (2014), Stoica & Megan (2010) or Yue *et al.* (2014).

*Corresponding author
Email address: dianab268@yahoo.com (Diana Borlea)

In this paper we intend to study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. The definitions of various types of dichotomy are illustrated by examples. We also aim to give connections between them, emphasized by counterexamples.

2. Preliminaries

Let (X, d) be a metric space, V a Banach space and $\mathcal{B}(V)$ the space of all V-valued bounded operators defined on V. Denote $Y = X \times V$ and $T = \{(t, t_0) \in \mathbb{R}_+^2 : t \geqslant t_0\}$.

Definition 2.1. A mapping
$varphi : T \times X \to X$ is said to be *evolution semiflow* on X if the following properties are satisfied:

(es1) $\phi(t, t, x) = x$, $(\forall)(t, x) \in \mathbb{R}_+ \times X$;

(es2) $\phi(t, s, \phi(s, t_0, x)) = \phi(t, t_0, x)$, $(\forall)(t, s), (s, t_0) \in T, x \in X$.

Definition 2.2. A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow ϕ if:

(ec1) $\Phi(t, t, x) = I$, $(\forall)t \geqslant 0, x \in X$ (I - identity operator).

(ec2) $\Phi(t, s, \phi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x)$, $(\forall)(t, s), (s, t_0) \in T$, $(\forall)x \in X$.

Let Φ be an evolution cocycle over an evolution semiflow φ. The mapping $C = (\varphi, \Phi)$, defined by:

$$C : T \times Y \to Y,\ C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$$

is called *skew-evolution semiflow* on Y.

Example 2.1. We will denote $C = C(\mathbb{R}, \mathbb{R})$ the set of continous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with uniform convergence topology on compact subsets of $\mathbb{R}$. The set C is metrizable with the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)},\ \text{unde } d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

For every $n \in \mathbb{N}^*$ we consider a decreasing function

$$x_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right),\ \lim_{t \to \infty} x_n(t) = \frac{1}{2n+1}.$$

We will denote

$$x_n^s(t) = x_n(t + s),\ \forall t, s \geqslant 0.$$

Let be X the closure in C of the set $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$. The application

$$\varphi : T \times X \to X,\ \varphi(t, s, x) = x_{t-s},\ \text{unde } x_{t-s}(\tau) = x(t - s + \tau),\ \forall \tau \geqslant 0,$$

is a evolution semiflow on X. Let consider the Banach space $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|$. Then, the application

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, e^{\alpha_2 \int_s^t x(\tau - s)d\tau} v_2\right),$$

where $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ is fixed, is a cocycle aplication of evolution over the semiflow φ, and $C = (\varphi, \Phi)$ is a evolution cocycle on Y.

Let us remind the definition of an evolution operator, followed by examples that punctuate the fact that it is generalized by an skew-evolution semiflows.

Definition 2.3. A mapping $E : T \to \mathcal{B}(V)$ is called *evolution operator* on V if following properties hold:

(e_1) $E(t, t) = I,\ \forall t \in \mathbb{R}_+$;

(e_2) $E(t, s)E(s, t_0) = E(t, t_0),\ \forall (t, s), (s, t_0) \in T$.

Example 2.2. One can naturally associate to every evolution operator E the mapping

$$\Phi_E : T \times X \to \mathcal{B}(V),\ \Phi_E(t, s, x) = E(t, s),$$

which is an evolution cocycle on V over every evolution semiflow φ. Therefore, the evolution operators are particular cases of evolution cocycles.

Example 2.3. Let $X = \mathbb{R}_+$. The mapping

$$\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+,\ \varphi(t, s, x) = t - s + x$$

is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : T \to \mathcal{B}(V)$ we obtain that

$$\Phi_E : T \times \mathbb{R}_+ \to \mathcal{B}(V),\ \Phi_E(t, s, x) = E(t - s + x, x)$$

is an evolution cocycle on V over the evolution semiflow φ. It follows that an evolution operator on V is generating a skew-evolution semiflow on Y.

3. Sequences of Invariant Projections for a Cocycle

Definition 3.1. A continuous map $P : X \to \mathcal{B}(V)$ which satisfies the following relation:

$$P(x)P(x) = P(x), (\forall)x \in X$$

is called projection on V.

Definition 3.2. A projection P on V is called *invariant* for a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

$$P(\varphi(t, s, x))\, \Phi(t, s, x) = \Phi(t, s, x)P(x),$$

for all $(t, s) \in T$ and $x \in X$.

Remark. If P is a projection on V, than the map

$$Q : X \to \mathcal{B}(V),\ Q(x) = I - P(x)$$

is also a projection on V, called complementary projection of P.

Remark. If the projection P is invariant for C then Q is also invariant for C.

Definition 3.3. We will name (C, P) a dichotomy pair where C is a skew-evolution semiflow and P is invariant or C.

4. Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows

Definition 4.1. Let (C, P) be a dichotomy pair. We say that (C, P) is *uniformly strongly exponentially dichotomic* (u.s.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(used1) $\|\Phi(t, s, x)P(x)\| \leqslant Ne^{-\nu(t-s)}$

(used2) $N\|\Phi(t, s, x)Q(x)\| \geqslant e^{\nu(t-s)}$

for all $(t, s) \in T$ and $x \in X$.

Definition 4.2. We say that (C, P) is *uniformly exponentially dichotomic* (u.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(ued1) $\|\Phi(t, s, x)P(x)v\| \leqslant Ne^{-\nu(t-s)}\|P(x)v\|$

(ued2) $N\|\Phi(t, s, x)Q(x)v\| \geqslant e^{\nu(t-s)}\|Q(x)v\|$

for all $(t, x) \in T \times X$ and for all $v \in V$.

Definition 4.3. We say that (C, P) is *uniformly weakly exponentially dichotomic* (u.w.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(uwed1) $\|\Phi(t, s, x)P(x)\| \leqslant Ne^{-\nu(t-s)}\|P(x)\|$

(uwed2) $N\|\Phi(t, s, x)Q(x)\| \geqslant e^{\nu(t-s)}\|Q(x)\|$

for all $(t, x) \in T_x X$ and for all $v \in V$.

Proposition 1. *If (C, P) is (s.u.e.d) then*

$$\sup_{x \in X} \|P(x)\| < +\infty. \tag{4.1}$$

Proof. Consider in (used1) $t = s$. Then we have

$$\|\Phi(t, t, x)P(x)\| = \|P(x)\| = \|P(x)\| \leqslant N \tag{4.2}$$

for all $x \in X$. □

Proposition 2. *If (C, P) is (u.s.e.d) then (C, P) is (u.w.e.d).*

Proof. If (C, P) is (u.s.e.d) then by (used1), for $x \in X$, we have that $\|P(x)\| \leqslant N$ and hence

$$\|Q(x)\| = \|I - P(x)\| \leqslant 1 + \|P(x)\| \leqslant 2N.$$

We have from (used1) and (used2) that:

$$\begin{aligned} \|\Phi(t, s, x)P(x)\| &\leqslant Ne^{-\nu(t-s)} \cdot 1 \leqslant Ne^{-\nu(t-s)}\|P(x)\| && (4.3)\\ &\leqslant 2N^2 e^{-\nu(t-s)}\|P(x)\|. && (4.4) \end{aligned}$$

$$2N^2\|\Phi(t, s, x)Q(x)\| \geqslant 2Ne^{\nu(t-s)} \geqslant e^{\nu(t-s)}\|Q(x)\|, \tag{4.5}$$

hence (C, P) is (u.w.e.d) □

Proposition 3. *If (C, P) is (u.e.d) then (C, P) is also (u.w.e.d)*

Proof. It follows immediately by taking the supremum over all $v \in V$ with $\|v\| = 1$. □

Definition 4.4. We say that C has a uniform exponential growth (u.e.g) if there exist $M \geqslant 1$, $\omega > 0$ such that

$$\|\Phi(t, s, x)\| \leqslant Me^{\omega(t-s)},$$

for all $(t, s) \in T$ and $x \in X$.

Theorem 4.1. *Assume that a dichotomy pair (C, P) is (u.w.e.d) and C has a uniform exponential growth. Then:*

$$\sup_{x\in X} \|P(x)\| < +\infty.$$

Proof. Let N, ν given by the (u.w.e.d) property of (C, P) and M, ω given by the (u.e.g) of C. Consider $s \geqslant 0$ fixed, $t \geqslant s$ and $x \in X$.

$$\begin{aligned} \left[\frac{1}{2N}e^{\nu(t-s)} - Ne^{-\nu(t-s)}\right]\|P(x)\| &\leqslant \frac{1}{N}e^{\nu(t-s)}\|Q(x)\| - Ne^{-\nu(t-s)}\|P(x)\| \\ &\leqslant \|\Phi(t, s, x)Q(x)\| - \| - \Phi(t, s, x)P(x)\| \\ &\leqslant \|\Phi(t, s, x)\| \leqslant Me^{\omega(t-s)}. \end{aligned}$$

Let $t_0 > 0$ be such that

$$\lambda_0 := \frac{1}{2N}e^{\nu t_0} - Ne^{-\nu t_0} > 0.$$

From the above estimation is follows that for $t = t_0 + s$,

$$\|P(x)\| \leqslant \frac{Me^{\omega t_0}}{\lambda_0}, \ (\forall)x \in X.$$

□

from where the conclusion follows.

Remark. In the following section we will see that for a dichotoomic pair (C, P):

1. (u.s.e.d) does not imply (u.e.d)
2. (u.e.d) does not imply (u.s.e.d)
3. (u.w.e.d) does not imply (u.e.d)
4. (u.w.e.d) does not imply (u.s.e.d)

5. Examples and Counterexamples

Example 5.1. Define, on $\mathbb{R}^3$, the family of projections

$$P(x)(v_1, v_2, v_3) = (v_1, 0, 0)$$

and the evolution cocycle on $\mathbb{R}^3$:

$$\Phi(t, s, x)(v_1, v_2, v_3) = \begin{cases} (v_1, v_2, v_3), \ t = s \\ (e^{s-t}v_1, e^{t-s}v_2, 0), \ t > s, \end{cases}$$

with the following norm:

$$\|x\| = |x_1| + |x_2| + |x_3|, \ x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We have that for all $(t, s) \in T, x \in X$ and $v \in \mathbb{R}^3$

$$\|\Phi(t, s, x)P(x)v\| = e^{s-t}v_1 = e^{s-t}\|P(x)v\|$$

from where we get that

$$\|\Phi(t, s, x)P(x)\| \leqslant e^{s-t}\|P(x)\|$$

and

$$\|\Phi(t, s, x)Q(x)v\| = \begin{cases} \|Q(x)v\|, \ t = s \\ \|\,(0, e^{t-s}v_2, 0)\,\|, \ t > s \end{cases} \leqslant e^{t-s}\|Q(x)v\|$$

hence

$$\|\Phi(t, s, x)Q(x)\| \leqslant \|Q(x)\|.$$

Choose $(0, 1, 0) \in \mathbb{R}^3$. Then

$$\|\Phi(t, s, x)Q(x)(0, 1, 0)\| = e^{t-s}\|Q(x)(0, 1, 0)\|$$

from where we finally obtain that:

$$\|\Phi(t, s, x)Q(x)\| = e^{t-s}\|Q(x)\|,$$

hence (C, P) is (u.w.e.d). Assume by a contradiction that (C, P) is (u.e.d). Then there exists, $N \geqslant 1$, $\nu > 0$ such that

$$N\|\Phi(t, s, x)Q(x)(v_1, v_2, v_3)\| \geqslant e^{\nu(t-s)}\|Q(x)(v_1, v_2, v_3)\|. \tag{5.1}$$

Put $t > s$ and $(v_1, v_2, v_3) = (0, 0, 1)$. Then $\|Q(x)(v_1, v_2, v_3)\| = 1$ and

$$e^{\nu(t-s)} \leqslant \|\Phi(t, s, x)(v_1, v_2, v_3)\| = \|\Phi(t, s, x)(0, 0, 1)\| = 0,$$

which is a contradiction.

Example 5.2 (u.e.d does not imply u.s.e.d)**.** On $V = \mathbb{R}^2$ and $(X, d) = (\mathbb{R}_+, d)$ endowed with the max - norm. Consider,

$$P(x) : \mathbb{R}^2 \to \mathbb{R}^2,\ P(x)(v_1, v_2) = (v_1 + xv_2, 0)$$

it follows that

$$\|P(x)\| = 1 + x, (\forall)x \geqslant 0 \tag{5.2}$$

Define the skew - evolutiv cocycle

$$\Phi(t, s, x) = e^{s-t}P(x) + e^{t-s}Q(x).$$

We have that

$$\begin{aligned} &\|\Phi(t, s, x)P(x)\| = e^{s-t}\|P(x)\| \text{ and} \\ &\|\Phi(t, s, x)Q(x)\| \geqslant e^{t-s}\|Q(x)\| \end{aligned} \tag{5.3}$$

Hence (C, P) is (u.e.d). It can not be (u.s.e.d) because of (5.2).

Remark. From the above example, by taking the sup norm in (5.3) over $\|v\| = 1$, we get that (C, P) is also (u.w.e.d). Hence (C, P) is (u.w.e.d) but not (u.s.e.d).

Remark. The connection between the three concepts studied in this paper is summarized in the below diagram

$$(u.s.e.d) \nRightarrow (u.e.d) \Rightarrow (u.w.e.d) \Leftarrow (u.s.e.d)$$

$$(u.s.e.d) \nLeftarrow (u.e.d) \nLeftarrow (u.w.e.d) \nRightarrow (u.s.e.d).$$

Acknowledgements

The author would like to gratefully acknowledge helpful support and comments from Professor Emeritus Mihail Megan, the Head of the Research Seminar on *Mathematical Analysis and Applications in Control Theory* at the West University of Timişoara, Romania.

References

Appell, J., V. Lakshmikantham, N. van Minh and P. Zabreiko (1993). A general model of evolutionary processes. Exponential dichotomy – I, II. *Nonlinear Anal.* **21**(3), 207–218, 219–225.

Babuţia, M.G. and M. Megan (2015). Exponential dichotomy concepts for evolution operators on the half-line. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **7**(2), 209–226.

Chow, S.N. and H. Leiva (1995). Existence and roughness of the exponential dichotomy for linear skew-product semiflows in banach spaces. *J. Differential Equations* **120**, 429–477.

Coppel, W.A. (1978). Dichotomies in stability theory. *Lecture Notes in Math.* ***629***.

Daleckiĭ, J.L. and M.G. Kreĭn (1974). *Stability of Solutions of Differential Equations in Banach Spaces*. Translations of Mathematical Monographs **43** Amer. Math. Soc., Providence, Rhode Island.

Massera, J.L. and J.J. Schäffer (1966). *Linear Differential Equations and Function Spaces*. Pure Appl. Math. **21** Academic Press.

Megan, M. and C. Stoica (2008). Exponential instability of skew-evolution semiflows in banach spaces. *Studia Univ. Babeş-Bolyai Math.* **53**(1), 17–24.

Megan, M. and C. Stoica (2010). Concepts of dichotomy for skew-evolution semiflows in banach spaces. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **2**(2), 125–140.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer Verlag, New York.

Perron, O. (1930). Die stabitätsfrage bei differentialgleichungen. *Math. Z.* **32**, 703–728.

Sacker, R.J. and G.R. Sell (1994). Dichotomies for linear evolutionary equations in Banach spaces. *J. Differential Equations* **113**(1), 17–67.

Sasu, B. and A.L. Sasu (2006). Exponential dichotomy and (l^p, l^q)-admissibility on the half-line. *J. Math. Anal. Appl.* **316**, 397–408.

Stoica, C. and D. Borlea (2012). On h-dichotomy for skew-evolution semiflows in Banach spaces. *Theory and Applications of Mathematics & Computer Science* **2**(1), 29–36.

Stoica, C. and D. Borlea (2014). Exponential stability versus polynomial stability for skew-evolution semiflows in infinite dimensional spaces. *Theory and Applications of Mathematics & Computer Science* **4**(2), 221–229.

Stoica, C. and M. Megan (2010). On uniform exponential stability for skew-evolution semiflows on Banach spaces. *Nonlinear Anal.* **72**(3–4), 1305–1313.

Viet Hai, P. (2010). Continuous and discrete characterizations for the uniform exponential stability of linear skew-evolution semiflows. *Nonlinear Anal.* **72**(12), 4390–4396.

Viet Hai, P. (2011). Discrete and continuous versions of barbashin-type theorem of linear skew-evolution semiflows. *Appl. Anal.* **90**(11–12), 1897–1907.

Yue, T., X.Q. Song and D.Q. Li (2014). On weak exponential expansiveness of skew-evolution semiflows in Banach spaces. *J. Inequal. Appl.* **DOI: 10.1186/1029-242X-2014-165**, 1–6.