Theory and Applications of
Mathematics & Computer Science
(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 6 (2) (2016) 103–109

Coefficient Estimates for New Subclasses of m-Fold Symmetric Bi-univalent Functions

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Abstract

In this paper, we introduce and investigate two subclasses $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disc $\mathbb{U}$. For functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

Keywords: Univalent functions, Bi-univalent functions, Coefficient estimates, m-fold symmetric bi-univalent functions.
2010 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized by the conditions $f(0) = 0$, $f'(0) = 1$ and having the following form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see Duren (1983)).

The Koebe One Quarter Theorem (e.g.,see (Duren, 1983)) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad\qquad (z \in \mathbb{U})$$

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and

$$f\left(f^{-1}(w)\right)=w \qquad \left(|w|<r_0(f),\ r_0(f)\geqslant \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.$$

A function $f\in\mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see (Srivastava *et al.*, 2010) (see also (Brannan & Taha, 1988), (Lewin, 1967), (Taha, 1981)).

In fact, the aforecited work of Srivastava et al. (Srivastava *et al.*, 2010) essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Ali et al. (Ali *et al.*, 2012), Srivastava et al. (Srivastava *et al.*, 2015*b*)(see also (Akın & Sümer-Eker, 2014), (Deniz, 2013), (Frasin & Aouf, 2011), (Srivastava, 2012), Xu et al (Xu *et al.*, 2012*a*), (Xu *et al.*, 2012*b*) and the references cited in each of them).

Let $m\in\mathbb{N}=\{1,2,...\}$. A domain E is said to be *m-fold symmetric* if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself (e.g.,see (Goodman, 1983)). It follows that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric in $\mathbb{U}$ if for every z in $\mathbb{U}$

$$f(e^{2\pi i/m}z)=e^{2\pi i/m}f(z).$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f\in\mathcal{S}_m$ is characterized by having a power series of the form

$$f(z)=z+\sum_{k=1}^{\infty}a_{mk+1}z^{mk+1} \qquad (z\in\mathbb{U},\ m\in\mathbb{N}). \tag{1.2}$$

Each bi-univalent function generates an m-fold symmetric bi-univalent function for each integer $m\in\mathbb{N}$. The normalized form of f is given as in (1.2) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al.(Srivastava *et al.*, 2014), is given as follows

$$g(w)=w-a_{m+1}w^{m+1}+\left[(m+1)a_{m+1}^2-a_{2m+1}\right]w^{2m+1} \tag{1.3}$$

$$-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]w^{3m+1}+\cdots$$

where $f^{-1}=g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$.

Recently, certain subclasses of m-fold bi-univalent functions class Σ_m similar to subclasses of Σ introduced and investigated by Sümer Eker (Sümer-Eker, 2016), Altınkaya and Yalçın (Altınkaya & Yalçın, 2015), Srivastava et al (Srivastava *et al.*, 2015*a*).

The aim of this paper is to introduce new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

2. Coefficient Estimates for the function class $\mathcal{A}_{\Sigma_m}(\lambda; \alpha)$

Definition 2.1. A function $f(z) \in \Sigma_m$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma_m}(\lambda; \alpha)$ $(0 < \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1)$ if the following conditions are satisfied:

$$\left| arg\left(\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \tag{2.2}$$

where the function g is given by (1.3).

Theorem 2.1. *Let $f \in \mathcal{A}_{\Sigma_m}(\lambda; \alpha)$ $(0 < \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1)$ be given by (1.2). Then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{2\alpha[1 + 2\lambda(m+1)] + (1-\alpha)[1 + \lambda(m+1)]^2}} \tag{2.3}$$

and

$$|a_{2m+1}| \leqslant \frac{\alpha(m+1)\left[1 + |\alpha - 1|\right]}{m^2[1 + 2\lambda(m+1)]}. \tag{2.4}$$

Proof. From (2.1) and (2.2) we have

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = [p(z)]^\alpha \tag{2.5}$$

and for its inverse map, $g = f^{-1}$, we have

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = [q(w)]^\alpha \tag{2.6}$$

where $p(z)$ and $q(w)$ are in familiar Caratheodory Class $\mathcal{P}$ (see for details (Duren, 1983)) and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$m[1 + \lambda(m+1)]a_{m+1} = \alpha p_m, \tag{2.9}$$

$$2m[1+\lambda(2m+1)]a_{2m+1} - m[1+\lambda(m+1)]a_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-m[1+\lambda(m+1)]a_{m+1} = \alpha q_m \tag{2.11}$$

and

$$m[(2m+1)+\lambda(m+1)(4m+1)]a_{m+1}^2 - 2m[1+\lambda(2m+1)]a_{2m+1} = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$2m^2[1+\lambda(m+1)]^2 a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also from (2.10), (2.12) and (2.14), we get

$$2m^2[1+2\lambda(m+1)]a_{m+1}^2 = \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2).$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{m^2\left[2\alpha[1+2\lambda(m+1)]+(1-\alpha)[1+\lambda(m+1)]^2\right]}. \tag{2.15}$$

Note that, according to the Caratheodory Lemma (see (Duren, 1983)), $|p_m| \leqslant 2$ and $|q_m| \leqslant 2$ for $m \in \mathbb{N}$. Now taking the absolute value of (2.15) and applying the Caratheodory Lemma for coefficients p_{2m} and q_{2m} we obtain

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{2\alpha[1+2\lambda(m+1)]+(1-\alpha)[1+\lambda(m+1)]^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted (2.3).

To find bounds on $|a_{2m+1}|$, we multiply $(2m+1)+\lambda(m+1)(4m+1)$ and $1+\lambda(m+1)$ to the relations (2.10) and (2.12) respectively and on adding them we obtain:

$$\begin{aligned} &4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]a_{2m+1} \\ &= \alpha\left\{[(2m+1)+\lambda(m+1)(4m+1)]\,p_{2m} + [1+\lambda(m+1)]\,q_{2m}\right\} \\ &+\frac{\alpha(\alpha-1)}{2}\left\{[(2m+1)+\lambda(m+1)(4m+1)]\,p_m^2 + [1+\lambda(m+1)]q_m^2\right\}. \end{aligned}$$

Now using $p_m^2 = q_m^2$ and the Caratheodory Lemma again for coefficients p_m, p_{2m} and q_{2m} we obtain

$$|a_{2m+1}| \leqslant \frac{\alpha(m+1)[1+|\alpha-1|]}{m^2[1+2\lambda(m+1)]}.$$

This completes the proof of the Theorem 2.1.

3. Coefficient Estimates for the function class $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$

Definition 3.1. A function $f(z) \in \Sigma_m$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$ $(0 \leqslant \lambda \leqslant 1, 0 \leqslant \beta < 1)$ if the following conditions are satisfied:

$$Re\left\{\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right\} > \beta \qquad (z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left\{\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)}\right\} > \beta \qquad (w \in \mathbb{U}) \tag{3.2}$$

where the function $g(w)$ is given by (1.3).

Theorem 3.1. *Let $f \in \mathcal{A}_{\Sigma_m}(\lambda;\beta)$ $(0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \beta < 1)$ be given by (1.2). Then*

$$|a_{m+1}| \leqslant \sqrt{\frac{2(1-\beta)}{m^2[1+2\lambda(m+1)]}} \tag{3.3}$$

and

$$|a_{2m+1}| \leqslant \frac{(1-\beta)(m+1)}{m^2[1+2\lambda(m+1)]}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$m[1+\lambda(m+1)]a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$2m[1+\lambda(2m+1)]a_{2m+1} - m[1+\lambda(m+1)]a_{m+1}^2 = (1-\beta)p_{2m}, \tag{3.8}$$

$$-m[1+\lambda(m+1)]a_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$m[(2m+1)+\lambda(m+1)(4m+1)]a_{m+1}^2 - 2m[1+\lambda(2m+1)]a_{2m+1} = (1-\beta)q_{2m}. \quad (3.10)$$

From (3.7) and (3.9) we get

$$p_m = -q_m \quad (3.11)$$

and

$$2m^2[1+\lambda(m+1)]^2 a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \quad (3.12)$$

Also from (3.8) and (3.10), we obtain

$$2m^2[1+2\lambda(m+1)]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \quad (3.13)$$

Thus we have

$$\begin{aligned} |a_{m+1}^2| &\leqslant \frac{(1-\beta)}{2m^2[1+2\lambda(m+1)]}(|p_{2m}|+|q_{2m}|) \\ &\leqslant \frac{2(1-\beta)}{m^2[1+2\lambda(m+1)]}, \end{aligned}$$

which is the bound on $|a_{m+1}|$ as given in the Theorem 3.1.

In order to find the bound on $|a_{2m+1}|$, we multiply $(2m+1)+\lambda(m+1)(4m+1)$ and $1+\lambda(m+1)$ to the relations (3.8) and (3.10) respectively and on adding them we obtain:

$$\begin{aligned} &4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]a_{2m+1} \\ &= (1-\beta)\{[(2m+1)+\lambda(m+1)(4m+1)]p_{2m} + [1+\lambda(m+1)]q_{2m}\} \end{aligned}$$

or equivalently

$$a_{2m+1} = \frac{(1-\beta)[(2m+1)+\lambda(m+1)(4m+1)]p_{2m} + [1+\lambda(m+1)]q_{2m}}{4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]}$$

Applying the Caratheodory Lemma for the coefficients p_{2m} and q_{2m}, we find

$$|a_{2m+1}| \leqslant \frac{(1-\beta)(m+1)}{m^2[1+2\lambda(m+1)]},$$

which is the bound on $|a_{2m+1}|$ as asserted in Theorem 3.1.

Remark. For 1-fold symmetric bi-univalent functions, if we put $\lambda = 0$ in our Theorems, we obtain the Theorem 2.1 and the Theorem 3.1 which were given by Brannan and Taha (Brannan & Taha, 1988).

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