Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 6 (2) (2016) 110 – 124

Subordination Properties of Certain Subclasses of Multivalent Functions Defined By Srivastava-Wright Operator

B. A. Frasin[a,*], H. Aaisha Farzana[b], B. Adolf Stephen[b]

[a]*Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq Jordan*
[b]*Department of Mathematics, Madras Christian College, Tambaram, Chennai - 600 059, India*

Abstract

Some subordination properties are investigated for functions belonging to each of the subclasses $\mathcal{V}(\lambda, A, B)$ and $\mathcal{W}(\lambda, A, B)$ of analytic p- valent functions involving the Srivastava-Wright operator in the open unit disk, $\mathbb{U}$ with suitable restrictions on the parameters λ, A and B. The authors also derive certain subordination results involving the Hadamard product (or convolution) of the associated functions. Relevant connections of the main results to various known results are established.

Keywords: Multivalent function, Srivastava-Wright Operator, Convex function, Differential subordination, Argument estimates.
2010 MSC: 30C45, 30C50, 30C55.

1. Introduction

Let $\mathcal{A}_k(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p, k \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and $p-$valent in the unit disc, $\mathbb{U} := \mathbb{U}(1)$, where $\mathbb{U}(r) = \{z \in \mathbb{C} : |z| < r\}$. Also, let $\mathcal{A}(p) = \mathcal{A}_{p+1}(p)$ and $\mathcal{A} = \mathcal{A}(1)$. For the functions $f \in \mathcal{A}_k(p)$ of the form (1.1) and $g \in \mathcal{A}_k(p)$ given by $g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n$, the *Hadamard product (or convolution) of f and g* is defined by

$$(f * g)(z) := z^p + \sum_{n=k}^{\infty} a_n b_n z^n, \; z \in \mathbb{U}.$$

*Corresponding author
Email addresses: bafrasin@yahoo.com (B. A. Frasin), h.aaisha@gmail.com (H. Aaisha Farzana), adolfmcc2003@yahoo.co.in (B. Adolf Stephen)

If f and g are two analytic functions in $\mathbb{U}$, we say that *f is subordinate to g*, written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function w,* which (by definition) is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.

If the function g is univalent in $\mathbb{U}$, then we have the following equivalence, (c.f (Miller & Mocanu, 1981, 2000)):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s (q, s \in \mathbb{N})$ be positive and real parameters such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i > 0.$$

The *Wright generalized hypergeometric function*

$${}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma\left(\alpha_i + nA_i\right)}{\prod_{i=1}^{s} \Gamma\left(\beta_i + nB_i\right)} \frac{z^n}{n!} \ (z \in \mathbb{U}).$$

If $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we have the following relationship:

$$\Omega_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] =_q F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the *generalized hypergeometric function* and

$$\Omega = \frac{\prod_{i=1}^{q} \Gamma\left(\beta_i\right)}{\prod_{i=1}^{s} \Gamma\left(\alpha_i\right)} \tag{1.2}$$

Now we define a function $\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$ by

$$\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \Omega\, z^p\, {}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$$

and also consider the following linear operator

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] : \mathcal{A}_k(p) \to \mathcal{A}_k(p)$$

defined using the convolution

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}] f(z) = \mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] * f(z).$$

We note that, for a function f of the form (1.1), we have

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}] f(z) = z^p + \sum_{n=k}^{\infty} \Omega \sigma_{n,p}(\alpha_1) a_n z^n, \tag{1.3}$$

where Ω is given by (1.2) and $\sigma_{n,p}(\alpha_1)$ is defined by

$$\sigma_{n,p}(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n-p))\ldots\Gamma(\alpha_q + A_q(n-p))}{\Gamma(\beta_1 + B_1(n-p))\ldots\Gamma(\beta_s + B_s(n-p))(n-p)!}. \tag{1.4}$$

If for convenience, we write

$$\theta_p^{q,s}(\alpha_1)f(z) = \theta_p^{q,s}[(\alpha_1, A_1)\ldots(\alpha_q, A_q); (\beta_1, B_1)\ldots(\beta_s, B_s)]f(z)$$

then we can easily verify from (1.3) that

$$zA_1\left(\theta_p^{q,s}(\alpha_1)f(z)\right)' = \tag{1.5}$$
$$\alpha_1\theta_p^{q,s}(\alpha_1+1)f(z) - (\alpha_1 - pA_1)\theta_p^{q,s}(\alpha_1)f(z)\ (A_1 > 0).$$

For $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we obtain $\theta_p^{q,s}[\alpha_1]f(z) = H_{p,q,s}f(z)$, which is known as the Dziok-Srivastava operator; it was introduced and studied by Dziok and Srivastava (Dziok & Srivastava, 1999, 2003). Also, for $f(z) \in \mathcal{A}$, the linear operator $\theta_1^{q,s}[\alpha_1]f(z) = \theta[\alpha_1]$ is popularly known in the current literature as the Srivastava-Wright operator; it was systematically and firmly investigated by Srivastava (Srivastava, 2007).(see also (Kiryakova, 2011; Dziok & Raina, 2004) and (Aouf *et al.*, 2010)).

Remark. For $f \in \mathcal{A}(p), A_i = 1(1 = 1, 2, \ldots, q), B_i = 1(i = 1, 2, \ldots, s), q = 2$ *and* $s = 1$ by specializing the parameters α_1, α_2 *and* β_1 the operator $\theta_p^{q,s}(\alpha_1)$ gets reduced to the following familiar operators:

(i) $\theta_p^{2,1}[a, 1; c]f(z) = L_p(a, c)f(z)$[see Saitoh (Saitoh, 1996)];

(ii) $\theta_p^{2,1}[\mu + p, 1; 1]f(z) = D^{\mu+p-1}f(z)\ (\mu > -p)$, where $D^{\mu+p-1}$ is the $\mu + p - 1$- the order *Ruscheweyh derivative* of a function $f \in \mathcal{A}(p)$. [see Kumar and Shukla (Kumar & Shukla, 1984*a*,*b*)]

(iii) $\theta_p^{2,1}[1 + p, 1; 1 + p - \mu]f(z)$, where the operator $\Omega_z^{\mu,p}$ is defined by [see Srivastava and Aouf (Srivastava & Aouf, 1992)];

$$\Omega_z^{\mu,p}f(z) = \frac{\Gamma(1 + p - \mu)}{\Gamma(1 + p)}z^{\mu}D_z^{\mu}f(z)\ (0 \le \mu < 1; p \in \mathbb{N}),$$

where D_z^{μ} is the *fractional derivative operator*.

(iv) $\theta_p^{2,1}[\nu + p, 1; \nu + p + 1]f(z) = J_{\nu,p}(f)(z)$, where $J_{\nu,p}$ is the generalized *Bernadi-Libera-Livingston-integral operator* (see (Bernardi, 1996; Libera, 1969; Livingston, 1966));

(v) $\theta_p^{2,1}[\lambda + p, a; c]f(z) = I_p^{\lambda}(a, c)f(z)\ (a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > -p)$, where $I_p^{\lambda}(a, c)$ is the *Cho-Kwon-Srivastava* operator (Cho *et al.*, 2004);

Definition 1.1. For the fixed parameters A and B, with $0 \le B < 1, -1 \le A < B$ and $0 \le \lambda < p, p \in \mathbb{N}$ and for a analytic $p-$ valent function of the form (1.1) we define the following subclasses:

$$\mathcal{V}(\lambda, A, B) = \left\{ f \in \mathcal{A}_k(p) : \frac{1}{p-\lambda}\left(\frac{z\left[\theta_p^{q,s}(\alpha_1)f(z)\right]'}{\theta_p^{q,s}(\alpha_1)f(z)} - \lambda\right) \prec \frac{1+Az}{1+Bz} \right\} \tag{1.6}$$

and

$$\mathcal{W}(\lambda, A, B) = \left\{ f \in \mathcal{A}_k(p) : \frac{1}{p-\lambda}\left(1 + \frac{z[\theta_p^{q,s}(\alpha_1) f(z)]''}{[\theta_p^{q,s}(\alpha_1) f(z)]'} - \lambda\right) \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.7}$$

The subclass $\mathcal{V}(\lambda, A, B)$ was discussed by Aouf et al., (Aouf *et al.*, 2010) for multivalent analytic functions with negative coefficients, also coefficients estimates, distortion theorem, the radii of $p-$valent starlikeness and $p-$valent convexity and modified Hadamard products were investigated. In (Murugusundaramoorthy & Aouf, 2013) Murugusundaramoorthy and Aouf obtained similar results for the meromorphic equivalent of the class $\mathcal{W}(\lambda, A, B)$. Sarkar et al., (Sarkar *et al.*, 2013) presented certain inclusion and convolution results involving the operator $\theta_p^{q,s}(\alpha_1)$ for functions belonging to certain favoured classes of analytic p-valent functions.
Motivated by the aforementioned works, in the present study we obtain certain strict subordination relationship involving the subclasses $\mathcal{V}(\lambda, A, B)$ and $\mathcal{W}(\lambda, A, B)$. Some subordination properties involving the linear operator defined in (1.3) are also considered. An argument estimate result is also obtained.

2. Preliminaries

Let $\mathcal{P}_m$ denote the class of function of the form

$$f(z) = 1 + a_m z^m + a_{m+1} z^{m+1} + \dots \tag{2.1}$$

that are analytic in the unit disc,$\mathbb{U}$. In proving our main results, we need each of the following definitions and lemmas.

Definition 2.1. (Wilf, 1961)
A sequence $\{b_n\}_{n\in\mathbb{N}}$ of complex numbers is said to be a *subordination factor sequence* if for each function $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{U}$, from the class of convex (univalent) functions in $\mathbb{U}$, denoted by S^c, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (\text{where} \quad a_1 = 1).$$

Lemma 2.1. *(Wilf, 1961) A sequence $\{b_n\}$ is a subordinating factor sequence if and only if*

$$Re\left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0, \; z \in \mathbb{U}. \tag{2.2}$$

Lemma 2.2. *(Miller & Mocanu, 1981, 2000) Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function ϕ given by (2.1). If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (Re\gamma \geq 0, \; \gamma \in \mathbb{C}^*), \tag{2.3}$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{m} z^{-\frac{\gamma}{m}} \int_0^z t^{\frac{\gamma}{m}-1} h(t)\, dt \prec h(z)$$

and ψ *is the best dominant.*

Lemma 2.3. *(Nunokawa, 1993)*

Let the function p *be analytic in* $\mathbb{U}$*, such that* $p(0) = 1$ *and* $p(z) \neq 0$ *for all* $z \in \mathbb{U}$*. If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi\delta}{2}, \quad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\delta,$$

where

$$k \geq \frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = \frac{\pi\delta}{2}$$

and

$$k \leq -\frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi\delta}{2},$$

where

$$p(z_0)^{1/\delta} = \pm ic, \quad \textit{and} \quad c > 0.$$

Lemma 2.4. *(Whittaker & Watson, 1927)*

For the complex numbers a, b *and* c*, with* $c \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$*, the following identities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z), \; z \in \mathbb{U}, \tag{2.4}$$

$$\textit{for } Re c > Re b > 0, \tag{2.5}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \; z \in \mathbb{U}, \tag{2.6}$$

and

$$(b+1)\, {}_2F_1(1, b; b+1; z) = (b+1) + bz\, {}_2F_1(1, b+1; b+2; z), \; z \in \mathbb{U}. \tag{2.7}$$

3. Coefficient estimates and subordination results for the function classes $\mathcal{W}(\lambda, A, B)$ and $\mathcal{V}(\lambda, A, B)$

Unless otherwise mentioned, we shall assume throughout the sequel that $0 \leq \lambda < p, p \in \mathbb{N}$ *and* $0 \leq B < 1$. First, we will give sufficient conditions for a function to be in the classes $\mathcal{W}(\lambda, A, B)$.

Lemma 3.1. *A sufficient condition for an analytic p-valent function f of the form (1.1), to be in the class $\mathcal{W}(\lambda, A, B)$ is*

$$\sum_{n=k}^{\infty} \gamma_{n,p}|a_n| \leq p(B-A)(p-\lambda) \tag{3.1}$$

where

$$\gamma_{n,p} = \Omega\sigma_{n,p}(\alpha_1)n[(n-p)(1+B)-(A-B)(p-\lambda)], \ (n \geq k). \tag{3.2}$$

Proof. An analytic $p-$valent function f of the form (1.1) belongs to the class $\mathcal{W}(\lambda, A, B)$, if and only if there exists a *Schwarz function* w, such that

$$\frac{1}{p-\lambda}\left(1+\frac{z[\theta_p^{q,s}(\alpha_1)f(z)]''}{[\theta_p^{q,s}(\alpha_1)f(z)]'}-\lambda\right) = \frac{1+Aw(z)}{1+Bw(z)}, \ z \in \mathbb{U}.$$

Since $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$, the above relation is equivalent to

$$\left|\frac{[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'}{([\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]')B - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'}\right| < 1.$$

Thus it is sufficient to show that

$$\begin{gathered}\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'\right| \\ -\left|([\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]')B - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'\right| < 0, \ z \in \mathbb{U}.\end{gathered}$$

Indeed, letting $|z| = r$ $(0 < r < 1)$ and using (3.1), we have

$$\begin{gathered}\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'\right| - \\ \left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]' - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'\right| \\ \leq \sum_{n=k}^{\infty} n(n-p)\Omega\sigma_{n,p}(\alpha_1)|a_n|r^n - (B-A)p(p-\lambda)\, r^{p-1}\end{gathered}$$

$$+\sum_{n=k}^{\infty} n[(n-p)B-(A-B)(p-\lambda)]\Omega\sigma_{n,p}(\alpha_1)|a_n|r^n = r^{p-1}\Big(\sum_{n=k}^{\infty}\gamma_{n,p}|a_n|r^{n-p+1} - (B-A)p(p-\lambda)\Big) < 0.$$

Hence $f \in \mathcal{W}(\lambda, A, B)$. □

Similarly, we have the following Lemma which gives sufficient condition for a function to be in the class $\mathcal{V}(\lambda, A, B)$.

Lemma 3.2. *A sufficient condition for an analytic p-valent function f of the form (1.1), to be in the class $\mathcal{V}(\lambda, A, B)$ is*

$$\sum_{n=k}^{\infty} \delta^{*}_{n,p}|a_n| \leq (B-A)(p-\lambda) \tag{3.3}$$

where

$$\delta^{*}_{n,p} = \Omega\sigma_{n,p}(\alpha_1)[(n-p)(1+B)-(A-B)(p-\lambda)], \ (n \geq k). \tag{3.4}$$

Our next result provides a sharp subordination result involving the functions of the class $\mathcal{W}(\lambda, A, B)$.

Theorem 3.1. *Let the sequence $\{\gamma_{n,p}\}_{n\in\mathbb{N}}$ defined in (3.2) be a nondecreasing sequence. If a function f of the form (1.1) belong to the class $\mathcal{W}(\lambda, A, B)$. and $g \in \mathcal{S}^c$, then*

$$\left(\epsilon\left(z^{1-p}\right) * g\right)(z) \prec g(z), \tag{3.5}$$

and

$$Re\left(z^{1-p}f(z)\right) > -\frac{1}{2\epsilon}, \ z \in \mathbb{U}, \tag{3.6}$$

$$\textit{whenever} \quad \epsilon = \frac{\gamma_{k,p}}{2\left[(B-A)p(p-\lambda)\right]+\gamma_{k,p}}.$$

Moreover, if $(k-p)$ is even, then the number ϵ cannot be replaced by a larger number.

Proof. Supposing that the function $g \in \mathcal{S}^c$ is of the form

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \ z \in \mathbb{U} \quad (\text{where} \quad b_1 = 1),$$

then

$$\sum_{n=1}^{\infty} d_n b_n z^n = \left(\epsilon\left(z^{1-p}f\right) * g\right)(z) \prec g(z),$$

where

$$d_n = \begin{cases} \epsilon, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad 2 \leq n \leq k-p, \\ \epsilon a_{n+p-1}, & \text{if} \quad n > k-p. \end{cases}$$

Now, using the Definition 2.1, the subordination result in (3.5) holds if $\{d_n\}$ is a subordinating factor sequence. Since $\{\gamma_{n,p}\}_{n\in\mathbb{N}}$ is a nondecreasing sequence we have,

$$\operatorname{Re}\left(1+2\sum_{n=1}^{\infty} d_n z^n\right) = \operatorname{Re}\left(1+\frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}}z+\sum_{n=k}^{\infty}\frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}}a_n z^{n-p}\right) \geq 1-\frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}}r-\frac{r}{p(p-\lambda)(B-A)+\gamma_{k,p}}\sum_{n=k}^{\infty}\delta_{n,p}|a_n|,\ |z|=r<1. \tag{3.7}$$

Thus, by using Lemma 3.1 in (3.7) we obtain

$$\operatorname{Re}\left(1+2\sum_{n=1}^{\infty} c_n z^n\right) \geq 1-\frac{\gamma_{k,p}}{p(B-A)(p-\lambda)+\gamma_{k,p}}r-\frac{r}{p(B-A)(p-\lambda)+\gamma_{k,p}}(B-A)p(p-\lambda)>0,\ z\in\mathbb{U},$$

which proves the inequality (2.2), hence also the subordination result asserted by (3.5). The inequality (3.6) asserted by Theorem 3.1 would follow from (3.5) upon setting

$$g(z)=\frac{z}{1-z}=\sum_{n=1}^{\infty} z^n,\ z\in\mathbb{U}.$$

We also observe that whenever the functions of the form

$$f_{n,p}(z)=z^p+\frac{(B-A)p(p-\lambda)}{\gamma_{n,p}}z^n,\ z\in\mathbb{U}\ (n\geq k),$$

belongs the class $\mathcal{W}(\lambda,A,B)$ and if $(k-p)$ is a even number, then

$$z^{1-p}f_{k,p}(z)\Big|_{z=-1}=-\frac{1}{2\epsilon},$$

and the constant ϵ is the best estimate. □

Using the same techniques as in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Let the sequence* $\{\delta^*_{n,p}\}_{n\in\mathbb{N}}$ *defined by* (3.4) *be a nondecreasing sequence. If the function* g *of the form* (1.1) *belongs to the class* $\mathcal{V}(\lambda,A,B)$ *and* $h\in\mathcal{S}^c$, *then*

$$\left(\mu\left(z^{1-p}f\right)*h\right)(z)\prec h(z), \tag{3.8}$$

and

$$Re\left(z^{1-p}f(z)\right) > -\frac{1}{2\mu},\ z \in \mathbb{U}, \tag{3.9}$$

where

$$\mu = \frac{\delta^*_{k,p}}{2\left[(B-A)(p-\lambda)\right] + \delta^*_{k,p}}.$$

Moreover, if $(k-p)$ *is even, then the number* μ *cannot be replaced by a larger number.*

4. Subordination Properties of the operator $\theta_p^{q,s}(\alpha_1)$

In this section we obtain certain subordination properties involving the operator $\theta_p^{q,s}(\alpha_1)$.

Theorem 4.1. *For* $f \in \mathcal{A}_k(p)$ *let the operator* $\mathcal{Q}$ *be defined by*

$$\mathcal{Q}f(z) := \left[1 - \tau - \tau\frac{(\alpha_1 - pA_1)}{A_1}\theta_p^{q,s}(\alpha_1)f(z)\right] + \frac{\tau\alpha_1}{A_1}\left[\theta_p^{q,s}(\alpha_1+1)f(z)\right], \tag{4.1}$$

for $A_1 \neq 0$ *and* $\tau > 0$.

(i) If

$$\frac{\mathcal{Q}^{(j)}f(z)(p-j)!}{z^{p-j}p!} \prec (1-\tau+\tau p)\frac{1+Az}{1+Bz} \quad (0 \leq j \leq p)\,, \tag{4.2}$$

, then

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)(p-j)!\right]^{(j)}}{z^{p-j}p!} \prec \widetilde{g}(z) \prec \frac{1+Az}{1+Bz}, \tag{4.3}$$

where for m positive, $\widetilde{g}$ *is given by*

$$\widetilde{g}(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1, 1; \dfrac{1-\tau+\tau p}{\tau m} + 1; \dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1 + \dfrac{Az(1-\tau+\tau p)}{1-\tau+\tau(m+p)}, & \text{if} \quad B = 0, \end{cases}$$

and $\widetilde{g}$ *is the best dominant of* (4.3).

(ii)

$$Re\Big(\frac{\mathcal{Q}^{(j)}f(z)}{z^{p-j}}\Big) > \frac{p!}{(p-j)!}\sigma,\ z \in \mathbb{U} \tag{4.4}$$

where

$$\sigma = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1, 1; \dfrac{1-\tau+\tau p}{\tau m} + 1; \dfrac{B}{B-1}\right), & \text{if} \quad B \neq 0, \\ 1 - \dfrac{A(1-\tau+\tau p)}{1-\tau+\tau(p+m)}, & \text{if} \quad B = 0. \end{cases}$$

The inequality (4.4) *is the best possible.*

Proof. From (1.5) and (4.1) we easily obtain

$$Q^{(j)}f(z) = (1-\tau+\tau j)\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)} + \tau z\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j+1)}, \ z \in \mathbb{U}. \tag{4.5}$$

Letting

$$g(z) := \frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}(p-j)!}{z^{p-j}p!}.$$

with $f \in \mathcal{A}_k(p)$, then g is analytic in $\mathbb{U}$ and has the form (2.1). Also, note that

$$(1-\tau+\tau p)\left[g(z) + \frac{\tau}{1-\tau+\tau p}zg'(z)\right] = \frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!}. \tag{4.6}$$

Then, by (4.2) we have

$$g(z) + \frac{\tau}{1-\tau+\tau p}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 2.2 for $\gamma = \dfrac{1-\tau+\tau p}{\tau}$ and whenever $\gamma > 0$, by a changing of variables followed by the use of the identities (2.5), (2.6) and (2.7), we deduce that

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}(p-j)!}{z^{p-j}p!} \prec \widetilde{g}(z) = \frac{(1-\tau+\tau p)}{\tau m}z^{-\frac{(1-\tau+\tau p)}{\tau m}}\int\limits_0^z t^{\frac{(1-\tau+\tau p)}{\tau m}-1}\frac{1+At}{1+Bt}dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\tau+\tau p}{\tau m}+1;\dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1 + \dfrac{A(1-\tau+\tau p)}{1-\tau+\tau(p+m)}z, & \text{if} \quad B = 0, \end{cases}$$

which proves the assertion (4.3) of our Theorem.

Next, in order to prove the assertion (4.4), it sufficies to show that

$$\inf\{\operatorname{Re}\widetilde{g}(z) : z \in \mathbb{U}\} = \widetilde{g}(-1). \tag{4.7}$$

Indeed, for $|z| \le r < 1$ we have

$$\operatorname{Re}\frac{1+Az}{1+Bz} \ge \frac{1-Ar}{1-Br},$$

and setting

$$\chi(s,z) = \frac{1+Asz}{1+Bsz} \quad \text{and} \quad d\mu(s) = \frac{1-\tau+\tau p}{\tau m}s^{\frac{1-\tau+\tau p}{\tau m}-1}ds \quad (0 \le s \le 1)$$

which is a positive measure on the closed interval $[0,1]$ whenever $\tau > 0$, we get

$$\widetilde{g}(z) = \int_0^1 \chi(s,z)\,d\mu(s),$$

and

$$\mathrm{Re}\,\widetilde{g}(z) \geq \int_0^1 \frac{1 - Asr}{1 - Bsr}\, d\mu(s) = \widetilde{g}(-r),\ |z| \leq r < 1.$$

Letting $r \to 1^-$ in the above inequality we obtain the assertion (4.7) of our Theorem. The estimate in (4.4) is the best possible since the function $\widetilde{g}$ is the best dominant of (4.3). □

Taking $q = 2$ and $s = 1$,for $A_i = B_i = 1, \alpha_1 = 1, \alpha_2 = \beta_1$ and $A = 1 - \dfrac{2\alpha(p-j)!}{(1-\tau+\tau p)p!}$ and $B = -1$ in Theorem 4.1 we get the following result:

Corollary 4.1. *Let $Qf(z) = (1-\tau)f(z) + \tau z f'(z)$, where $f \in \mathcal{A}_k(p)$. For $\tau > 0$*

$$Re\frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!} > \alpha,\ z \in \mathbb{U} \quad \left(0 \leq \alpha < \frac{(1-\tau+\tau p)p!}{(p-j)!},\ 0 \leq j \leq p\right),$$

implies that

$$Re\frac{f^{(j)}(z)}{z^{p-j}} > \frac{\alpha}{1-\tau+\tau p} +$$
$$\left[\frac{p!}{(p-j)!} - \frac{\alpha}{1-\tau+\tau p}\right]\left[{}_2F_1\left(1, 1; \frac{1-\tau+\tau p}{\tau m} + 1; \frac{1}{2}\right) - 1\right],\ z \in \mathbb{U}.$$

The above inequality is the best possible.

Theorem 4.2. *For $f \in \mathcal{A}_k(p)$ let the operator Q be given by (4.1), and let $\tau > 0$.*

(i) If

$$Re\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} > \rho,\ z \in \mathbb{U} \quad \left(\rho < \frac{p!}{(p-j)!}\right),$$

then

$$Re\frac{Q^{(j)}f(z)}{z^{p-j}} > \rho(1-\tau+\tau p),\ |z| < R,$$

where

$$R = \left[\sqrt{1 + \left(\frac{\tau m}{1-\tau+\tau p}\right)^2} - \frac{\tau m}{1-\tau+\tau p}\right]^{\frac{1}{m}}. \tag{4.8}$$

(ii) If

$$Re\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} < \rho,\ z \in \mathbb{U} \quad \left(\rho > \frac{p!}{(p-j)!},\right),$$

then

$$Re\frac{Q^{(j)}f(z)}{z^{p-j}} < \rho(1-\tau+\tau p),\ |z| < R.$$

The bound R is the best possible.

Proof. (i) Defining the function Φ by

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} =: \rho + \left[\frac{p!}{(p-j)!} - \rho\right]\Phi(z), \tag{4.9}$$

then Φ is an analytic function of the form (2.1) with positive real part in $\mathbb{U}$. Differentiating (4.9) with respect to z and using (4.5) we have

$$\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p) = \left[\frac{p!}{(p-j)!} - \rho\right]\left[(1-\tau+\tau p)\Phi(z) + \tau z\Phi'(z)\right]. \tag{4.10}$$

Now, by applying in (4.10) the following well-known estimate (MacGregor, 1963)

$$\frac{|z\Phi'(z)|}{\operatorname{Re}\Phi(z)} \leq \frac{2mr^m}{1-r^{2m}}, \quad |z| = r < 1, \tag{4.11}$$

we have

$$\operatorname{Re}\left[\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p)\right] \geq \tag{4.12}$$
$$\operatorname{Re}\Phi(z)\left[\frac{p!}{(p-j)!} - \rho\right]\left[(1-\tau+\tau p) - \frac{2\tau mr^m}{1-r^{2m}}\right], \quad |z| = r < 1.$$

Now, it is easy to see that the right hand side of (4.12) is positive whenever $r < R$, where R is given by (4.8). In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}_k(p)$ defined by

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} = \rho + \left[\frac{p!}{(p-j)!} - \rho\right]\frac{1+z^m}{1-z^m}.$$

Then,

$$\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p) =$$
$$\frac{\frac{p!}{(p-j)!} - \rho}{(1-z^m)^2}\left[(1-\tau+\tau p)\left(1-z^{2m}\right) + 2\tau mz^m\right] = 0,$$

for $z = R\exp^{\frac{i\pi}{m}}$, and the first part of the Theorem is proved.
Similarly, we can prove part (ii) of the Theorem.

□

5. An argument estimate

In this section we obtain an argument estimate involving the operator $\theta_p^{q,s}(\alpha_1)$ and connected with the linear operator Q.

Theorem 5.1. *For $f \in \mathcal{A}_k(p)$, let the operator Q be defined by (4.1), and let $0 \le \tau < \dfrac{1}{1-p}$. If*

$$\left|\arg \frac{Q^{(j)} f(z)}{z^{p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathbb{U} \quad \big(\delta > 0,\ 0 \le j \le p\big), \tag{5.1}$$

then

$$\left|\arg \frac{\left[\theta_p^{q,s}(\alpha_1) f(z)\right]^{(j)}}{z^{p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathbb{U}.$$

Proof. For $f \in \mathcal{A}_k(p)$, if we let

$$q(z) := \frac{\left[\theta_p^{q,s}(\alpha_1) f(z)\right]^{(j)}}{z^{p-j}} \frac{(p-j)!}{p!},$$

then q is of the form (2.1) and it is analytic in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg q(z)| < \frac{\pi\delta}{2},\ |z| < |z_0| \quad \text{and} \quad |\arg q(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0)\,,$$

then, accorollaryding to Lemma 2.3 we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\delta \quad \text{and} \quad q(z_0)^{1/\delta} = \pm ic \quad (c > 0).$$

Also, from the equality (4.5) we get

$$\frac{Q^{(j)} f(z_0)}{z_0^{p-j}} = \frac{p!}{(p-j)!}(1-\tau+\tau p)q(z_0)\left[1 + \frac{\tau}{1-\tau+\tau p}\frac{z_0 q'(z_0)}{q(z_0)}\right].$$

If $\arg q(z_0) = \dfrac{\pi\delta}{2}$, then

$$\arg \frac{Q^{(j)} f(z_0)}{z_0^{p-j}} = \frac{\pi\delta}{2} + \arg\left(1 + \frac{\tau}{1-\tau+\tau p} ik\delta\right) = \frac{\pi\delta}{2} + \tan^{-1}\left(\frac{\tau}{1-\tau+\tau p} k\delta\right) \ge \frac{\pi\delta}{2},$$

whenever $k \ge \dfrac{1}{2}\left(c + \dfrac{1}{c}\right)$ and $0 \le \tau < \dfrac{1}{1-p}$, and this last inequality contradicts the assumption (5.1).

Similarly, if $\arg q(z_0) = -\dfrac{\pi\delta}{2}$, then we obtain

$$\arg \frac{Q^{(j)} f(z_0)}{z_0^{p-j}} \le -\frac{\pi\delta}{2},$$

which also contradicts the assumption (5.1).

Consequently, the function q need to satisfy the inequality $|\arg q(z)| < \dfrac{\pi\delta}{2}$, $z \in \mathbb{U}$, i.e. the conclusion of our theorem. □

Acknowledgement

The work of the second author was supported by the *Department of Science and Technology*, India with reference to the sanction order no. SR/DST-WOS A/MS-10/2013(G). The work of the third author was supported by the grant given under *UGC Minor Research Project F.No: 5599/15(MRP-SEM/UGC SERO).*

References

Aouf, M. K., A. Shamandy, A. O. Mostafa and S. M. Madian (2010). Certain class of p-valent functions associated with the Wright generalized hypergeometric functions. *Demonstratio Mathematica* **XLIII**(1), 39–57.

Bernardi, S. D. (1996). Convex and univalent functions. *Transactions of the American Mathematical Society.* **135**, 429–446.

Cho, N. E., O. H. Kwon and H. M. Srivastava (2004). Inclusion and argument properties for certain subclasses of multivalent functions associated with a family of linear operators. *Journal of Mathematical Analysis and Application* **292**, 470–483.

Dziok, J. and H. M. Srivastava (1999). Classes of analytic functions with the generalized hyper-geometric function. *Applied Mathematics and Computation* **103**, 1–13.

Dziok, J. and H.M. Srivastava (2003). Certain subclasses of analytic functions associated with the generalized hypergeometric function. *Integral Transforms and Special Functions* **14**, 7–18.

Dziok, J. and R. K. Raina (2004). Families of analytic functions associated with the Wright generalized hypergeometric function. *Demonstratio Mathematica* **37**, 533–542.

Kiryakova, V. (2011). Criteria for univalence of the Dziok-Srivastava and the Srivastava-Wright operators in the class A. *Applied Mathematics and Computation* **218**, 883–892.

Kumar, V. and S. L. Shukla (1984*a*). Multivalent functions defined by Ruscheweyh derivatives. *Indian Journal of Pure and Applied Mathematics* **15**, 1216–1227.

Kumar, V. and S. L. Shukla (1984*b*). Multivalent functions defined by Ruscheweyh derivatives II. *Indian Journal of Pure and Applied Mathematics* **15**, 1228–1238.

Libera, R. J. (1969). Some classes of regular univalent functions. *Proceedings of the American Mathematical Society* **16**, 755–758.

Livingston, A. E. (1966). On the radius of univalence of certain analytic functions. *Proceedings of the American Mathematical Society* **17**, 352–357.

MacGregor, T. H. (1963). Radius of univalence of certain analytic functions. *Proceedings of American Mathematical Society* **14**, 514–520.

Miller, S. S. and P. T. Mocanu (1981). Differential subordinations and univalent functions. *Michigan Mathematical Journal* **28**, 157–171.

Miller, S. S. and P. T. Mocanu (2000). *Differential Subordination. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*. Vol. 225. Marcel Dekker.

Murugusundaramoorthy, G. and M. K. Aouf (2013). Families of meromorphic multivalent functions associated with the Dziok- Raina operator. *International Journal of Analysis and Applications* **2**(1), 1–18.

Nunokawa, M. (1993). On the order of strongly starlikeness. *Proceedings of the Japan Academy* **69**(Ser. A), 234–237.

Saitoh, H. (1996). A linear operator and its applications of first order differential subordinations. *Mathematica Japonica* **44**, 31–38.

Sarkar, N., P. Goswami, J. Dziok and J. Sokol (2013). Subordinations for multivalent analytic functions associated with Wright generalized hypergeometic function. *Tamkang Journal of Mathematics* **44**(1), 61–71.

Srivastava, H. M. (2007). Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators. *Applicable Analysis and Discrete Mathematics* **1**(1), 56–71.

Srivastava, H. M. and M. K. Aouf (1992). A certain fractional derivative operator and its application to a new class of analytic and multivalent functions with negative coefficients I. *Journal of Mathematical Analysis and Applications* **171**, 673–688.

Whittaker, E. T. and G. N. Watson (1927). *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, Fourth Edition*. Cambridge University Press, Cambridge.

Wilf, H. S. (1961). Subordinating factor sequence for convex maps of the unit circle. *Proceedings of American Mathematical Society* **12**, 689–693.