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On a Class of Harmonic Univalent Functions Defined by Using a New Differential Operator

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Abstract

In this paper, a new class of complex-valued harmonic univalent functions defined by using a new differential operator is introduced. We investigate coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

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1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics (e.g. see Choquet (Choquet, 1945), Dorff (Dorff, 2003), Duren (Duren, 2004)). A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense- preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|$ in D; see (Clunie & Sheil-Small, 1984).

Denote by SH the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

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Therefore

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1.$$

Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999), Jahangiri (Jahangiri, 1999) studied the harmonic univalent functions.

The differential operator $D^n_{\alpha,\mu}(\lambda, w)$ $(n \in \mathbb{N}_0)$ was introduced by Bucur et al. (Bucur *et al.*, 2015). For $f = h + \overline{g}$ given by (1.1), we define the following differential operator:

$$D^n_{\alpha,\mu}(\lambda, w)f(z) = D^n_{\alpha,\mu}(\lambda, w)h(z) + (-1)^n \overline{D^n_{\alpha,\mu}(\lambda, w)g(z)},$$

where

$$D^n_{\alpha,\mu}(\lambda, w)h(z) = z + \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n a_k z^k$$

and

$$D^n_{\alpha,\mu}(\lambda, w)g(z) = \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n b_k z^k,$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^\lambda$, with $D^n_{\alpha,\mu}(\lambda, w)f(0) = 0$.

Motivated by the differential operator $D^n_{\alpha,\mu}(\lambda, w)$, we define generalization of the differential operator for a function $f = h + \overline{g}$ given by (1.1).

$$D^0_{\alpha,\mu}(\lambda, w)f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D^1_{\alpha,\mu}(\lambda, w)f(z) = (\alpha - \mu w^\lambda)(h(z) + \overline{g(z)}) + (\mu w^\lambda - \alpha + 1)(zh'(z) - \overline{zg'(z)},$$

$$\vdots$$

$$D^n_{\alpha,\mu}(\lambda, w)f(z) = D\left(D^{n-1}_{\alpha,\mu}(\lambda, w)f(z)\right). \tag{1.2}$$

If f is given by (1.1), then from (1.2), we see that

$$D^n_{\alpha,\mu}(\lambda, w)f(z) = z + \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n \overline{b_k z^k}. \tag{1.3}$$

When , $w = \alpha = 0$, we get modified Salagean differential operator (Salagean, 1983).

Denote by $SH(\lambda, w, n, \alpha, \beta)$ the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\Re\left(\frac{D^{n+1}_{\alpha,\mu}(\lambda, w)f(z)}{D^n_{\alpha,\mu}(\lambda, w)f(z)}\right) \geq \beta; \quad (0 \leq \beta < 1), \tag{1.4}$$

where $D^n_{\alpha,\mu}(\lambda, w)f(z)$ is defined by (1.3).

We let the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{1.5}$$

By suitably specializing the parameters, the classes $SH(\lambda, w, n, \alpha, \beta)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH(0, 0, 0, 0, 0) = SH^*(0)$ (Avcı (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999)),

(ii) $SH(0, 0, 0, 0, \beta) = SH^*(\beta)$ (Jahangiri (Jahangiri, 1999)),
$SH(0, 0, 0, 0, \beta) = \overline{S}_H(1, 0, \beta)$ (Yalçın (Yalçın, 2005)),

(iii) $SH(0, 0, 1, 0, 0) = KH(0)$ (Avcı (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999)),

(iv) $SH(0, 0, 1, 0, \beta) = KH(\beta)$ (Jahangiri (Jahangiri, 1999)),
$SH(0, 0, 1, 0, \beta) = \overline{S}_H(2, 1, \beta)$ (Yalçın (Yalçın, 2005)),

(v) $SH(0, 0, n, 0, \beta) = H(n, \beta)$ (Jahangiri et al. (Jahangiri *et al.*, 2002)),
$SH(0, 0, n, 0, \beta) = \overline{S}_H(n + 1, n, \beta)$ (Yalçın (Yalçın, 2005)),

The object of the present paper is to give sufficient condition for functions $f = h+\overline{g}$ where h and g are given by (1.1) to be in the class $SH(\lambda, w, n, \alpha)$; and it is shown that this coefficient condition is also necessary for functions belonging to the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$. Also, we obtain coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

2. Coefficient Bounds

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n |a_k| + \sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n |b_k| \leq 1-\beta, \tag{2.1}$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^{\lambda}$, $n \in \mathbb{N}_0$, $0 \leq \beta < 1$. Then f is sense-preserving, harmonic univalent in U and $f \in SH(\lambda, w, n, \alpha, \beta)$.

Proof. If $z_1 \neq z_2$,

$$\begin{aligned} \left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum\limits_{k=1}^{\infty} b_k \left(z_1^k - z_2^k\right)}{(z_1 - z_2) + \sum\limits_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)}\right| \\ &> 1 - \frac{\sum\limits_{k=1}^{\infty} k\,|b_k|}{1 - \sum\limits_{k=2}^{\infty} k\,|a_k|} \end{aligned}$$

$$\geq \quad 1 - \frac{\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} |a_k|}$$

$$\geq \quad 0,$$

which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} > 1 - \sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} |a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} |b_k| > \sum_{k=1}^{\infty} k\,|b_k|\,|z|^{k-1} \\ &\geq |g'(z)|\,. \end{aligned}$$

Using the fact that $\Re(w) \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$\left|(1-\beta)D^n_{\alpha,\mu}(\lambda,w) + D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| - \left|(1+\beta)D^n_{\alpha,\mu}(\lambda,w) - D^{n+1}_{\alpha,\mu}(\lambda,w)\right| \geq 0. \tag{2.2}$$

Substituting for $D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)$ and $D^n_{\alpha,\mu}(\lambda,w)f(z)$ in (2.2), we obtain

$$\begin{aligned} &\left|(1-\beta)D^n_{\alpha,\mu}(\lambda,w) + D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| - \left|(1+\beta)D^n_{\alpha,\mu}(\lambda,w)f(z) - D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| \\ \geq\ & 2(1-\beta)\,|z| - \sum_{k=2}^{\infty}\left[(k+1-\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n |a_k|\,|z|^k \\ & - \sum_{k=1}^{\infty}\left[(k-1+\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n |b_k|\,|z|^k \\ & - \sum_{k=2}^{\infty}\left[(k-1-\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n |a_k|\,|z|^k \\ & - \sum_{k=1}^{\infty}\left[(k+1+\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n |b_k|\,|z|^k \\ \geq\ & 2(1-\beta)\,|z|\left(1 - \sum_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}|a_k| \right. \\ & \left. - \sum_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}|b_k|\right). \end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is completed. □

Theorem 2.2. *Let $f_n = h + \overline{g}_n$ be given by (1.5). Then $f_n \in \overline{SH}(\lambda, n, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k + \sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \leq 1-\beta, \qquad (2.3)$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^{\lambda}$, $n \in \mathbb{N}_0$, $0 \leq \beta < 1$.

Proof. The "if" part follows from Theorem 2.1 upon noting that $\overline{SH}(\lambda, w, n, \alpha, \beta) \subset SH(\lambda, w, n, \alpha, \beta)$. For the "only if" part, we show that $f \notin \overline{SH}(\lambda, w, n, \alpha, \beta)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (1.5), to be in $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is that the condition (1.4) to be satisfied. This is equivalent to

$$\Re\left\{\frac{(1-\beta)z - \sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k}{z - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k + \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}\right.$$

$$\left.\frac{-\sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}{z - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k + \sum_{k=1}^{\infty}(k+1)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}\right\} \geq 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\beta) - \sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1}}{1 - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1} + \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}}$$

$$\frac{-\sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}}{1 - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1} + \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}} \geq 0. \qquad (2.4)$$

If the condition (2.3) does not hold, then the numerator in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.4) is negative. This contradicts the required condition for $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ and so the proof is complete. □

3. Distortion Inequalities and Extreme Points

Theorem 3.1. *Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1+b_1)\, r + \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} - \frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^{n}}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} b_1\right) r^2,$$

and

$$|f_n(z)| \geq (1-b_1)\, r - \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} - \frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^{n}}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} b_1\right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^k \\
&\leq (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^2 \\
&= (1+b_1)\,r + \frac{(1-\beta)\,r^2}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}\sum_{k=2}^{\infty}\frac{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}{(1-\beta)}[a_k+b_k] \\
&\leq (1+b_1)\,r + \frac{(1-\beta)\,r^2}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n} \\
&\quad \times\sum_{k=2}^{\infty}\left(\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}a_k\right. \\
&\quad \left.+\frac{(k+\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}b_k\right) \\
&\leq (1+b_1)\,r + \frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}\left(1-\frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^n}{1-\beta}b_1\right)r^2 \\
&\leq (1+b_1)\,r + \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}-\frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^n}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}b_1\right)r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.1. □

Corollary 3.1. *Let f_n of the form (1.5) be so that $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then*

$$\left\{w : |w| < \frac{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n-1+\beta}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}\right.$$

$$\left.-\frac{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n-(1+\beta)\left[2(\mu w^{\lambda}-\alpha)+1\right]^n}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^n}\right\} \subset f_n(U).$$

Theorem 3.2. *Let f_n be given by (1.5). Then $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right),$$

where

$$h_1(z) = z, \qquad h_k(z) = z - \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda-\alpha)+k\right]^n} z^k; \quad (k \geq 2),$$

$$g_{n_k}(z) = z + (-1)^n \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda-\alpha)+k\right]^n} \overline{z}^k; \quad (k \geq 2),$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0.$$

In particular, the extreme points of $\overline{SH}(\lambda, w, n, \alpha, \beta)$ *are* $\{h_k\}$ *and* $\{g_{n_k}\}$*.*

Proof. For functions f_n of the form (1.5) we may write

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k)\, z - \sum_{k=2}^{\infty} \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n} X_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} \left(\frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n} X_k \right) \\ &+ \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} \left(\frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n} Y_k \right) \\ = &\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1, \text{ and so } f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta). \end{aligned}$$

Conversely, if $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$, then

$$a_k \leq \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}$$

and

$$b_k \leq \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n}.$$

Setting

$$X_k = \frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} a_k; \; (k \geq 2),$$

$$Y_k = \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_k; \quad (k \geq 1),$$

and

$$X_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\right)$$

where $X_1 \geq 0$. Then

$$f_n(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z)$$

as required. □

4. Inclusion Results

Theorem 4.1. *The class $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is closed under convex combinations.*

Proof. Let $f_{n_i} \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ for $i = 1, 2, ...,$ where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \overline{z}^k.$$

Then by (2.3),

$$\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} a_{k_i} + \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_{k_i} \leq 1. \tag{4.1}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \overline{z}^k.$$

Then by (4.1),

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) \\
&+\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\nu} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \\
= &\sum_{i=1}^{\infty} t_i \Bigg(\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} a_{k_i} \\
&+\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_{k_i}\Bigg) \leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. □

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